

ACTIONS OF MONOIDAL CATEGORIES AND REPRESENTATIONS OF CARTAN TYPE LIE ALGEBRAS

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(Received 4 February 2021; revised 19 January 2022; accepted 24 January 2022; first published online 22 February 2022)

Abstract Using crossed homomorphisms, we show that the category of weak representations (respectively admissible representations) of Lie–Rinehart algebras (respectively Leibniz pairs) is a left module category over the monoidal category of representations of Lie algebras. In particular, the corresponding bifunctor of monoidal categories is established to give new weak representations (respectively admissible representations) of Lie–Rinehart algebras (respectively Leibniz pairs). This generalises and unifies various existing constructions of representations of many Lie algebras by using this new bifunctor. We construct some crossed homomorphisms in different situations and use our actions of monoidal categories to recover some known constructions of representations of various Lie algebras and to obtain new representations for generalised Witt algebras and their Lie subalgebras. The cohomology theory of crossed homomorphisms between Lie algebras is introduced and used to study linear deformations of crossed homomorphisms.

Key words and phrases: crossed homomorphism; deformation; cohomology; Lie–Rinehart algebra; Leibniz pair; action of monoidal categories; Lie algebra of Cartan type

2020 Mathematics Subject Classification: 17B10; 17B40; 17B56; 17B66; 53D17; 81R10

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1. Introduction

This article aims to give a conceptual approach to unify various constructions of representations of certain Lie algebras and construct new representations of some Lie algebras using crossed homomorphisms, Lie–Rinehart algebras and Leibniz pairs.

1.1. Representations of Cartan-type Lie algebras

The representation theory of Lie algebras is of great importance due to its own overall completeness and applications in mathematics and mathematical physics. The Cartan-type Lie algebras, originally introduced and studied by Cartan, consist of four classes of infinite-dimensional simple Lie algebras of vector fields with formal power series coefficients: the Witt algebras, the divergence-free algebras, the Hamiltonian algebras and the contact algebras. The representation theory of Cartan-type Lie algebras was first studied by Rudakov [40, 41]. He showed that irreducible continuous representations can be described explicitly as induced representations or quotients of induced representations. Later, Shen [42] studied graded modules of graded Lie algebras of Cartan type (W_n^+ , S_n^+ , and H_n^+) with polynomial coefficients of positive characteristic. Larsson constructed a class of representations for the Witt algebras W_n with Laurent polynomial coefficients [24]. More precisely, Shen’s modules, called mixed product, were constructed by certain monomorphism, while Larsson’s modules, named conformal fields, came from a physics background. We call the methods of constructing these modules Shen–Larsson functors. Many other authors have contributed to the theory along these approaches in the last

few decades. In particular, irreducible modules with finite-dimensional weight spaces over the Virasoro algebra (universal central extension of the Lie algebra W_1 of vector fields on a circle) was classified by Mathieu in [33], while Billig and Futorny recently gave the classification of irreducible modules over the Witt algebras W_n ($n \geq 2$) with finite-dimensional weight spaces [2]. Note that intrinsically there is a functor from the category of finite-dimensional irreducible representations of finite-dimensional simple Lie algebras to the category of representations of Cartan-type Lie algebras among these works. There should be some essential part that applies to all of those constructions (even more) of complicated modules over some classes of Lie algebras (not only Cartan-type Lie algebras) as a whole regardless of any specific feature exhibited in each particular case. From this point of view, it is no surprise that earlier results in this direction due to many authors are fragments of the general theory. We find a unifying conceptual approach generalising Shen–Larsson functors. This is one of the main purposes of the article.

1.2. Representations of Lie–Rinehart algebras and Leibniz pairs

Note that the abovementioned Cartan-type Lie algebras are either Lie–Rinehart algebras or Leibniz pairs.

Lie–Rinehart algebras, originally studied by Rinehart in [39] in 1963, arose from a wide variety of constructions in differential geometry and they have been introduced repeatedly into many areas under different terminologies; for example, Lie pseudoalgebras. Lie–Rinehart algebras are the underlying structures of Lie algebroids. See [32] and references therein for more details. A Lie–Rinehart algebra is a quadruple $(A, \mathcal{L}, [\cdot, \cdot]_{\mathcal{L}}, \alpha)$, where A is a commutative associative algebra, \mathcal{L} is an A -module, $[\cdot, \cdot]_{\mathcal{L}}$ is a Lie bracket on \mathcal{L} and $\alpha : \mathcal{L} \rightarrow \text{Der}_{\mathbb{K}}(A)$ is an A -module homomorphism with some compatibility conditions involving the Lie brackets. Lie–Rinehart algebras have been further investigated in many aspects [6, 19, 20, 21, 31, 34]. In particular, Rinehart constructed the universal enveloping algebra of a Lie–Rinehart algebra [39]. Huebschmann gave an alternative construction of the universal enveloping algebra $\mathcal{U}(A, \mathcal{L})$ of a Lie–Rinehart algebra $(A, \mathcal{L}, [\cdot, \cdot]_{\mathcal{L}}, \alpha)$ via the smash product, namely, $\mathcal{U}(A, \mathcal{L}) = (A \# U(\mathcal{L})) / J$, where J is a certain two-sided ideal in $A \# U(\mathcal{L})$, and showed that there is a one-to-one correspondence between representations of a Lie–Rinehart algebra and representations of its universal enveloping algebra [19]. Representations of Lie–Rinehart algebras are deeply related to the theory of \mathcal{D} -modules [38], which are modules over the algebra \mathcal{D} of linear differential operators on a manifold. Since the algebra \mathcal{D} is the universal enveloping algebra of the Lie–Rinehart algebra of vector fields, a \mathcal{D} -module is the same as a module with a representation of the Lie–Rinehart algebra of vector fields. We introduce the notion of a weak representation of a Lie–Rinehart algebra. The adjoint action is naturally a weak representation of a Lie–Rinehart algebra on itself. There is a one-to-one correspondence between weak representations of a Lie–Rinehart algebra and representations of the smash product $A \# U(\mathcal{L})$.

The notion of a Leibniz pair was originally introduced by Flato–Gerstenhaber–Voronov in [11], which consists of a \mathbb{K} -Lie algebra $(\mathcal{S}, [\cdot, \cdot]_{\mathcal{S}})$ and a \mathbb{K} -Lie algebra homomorphism $\beta : \mathcal{S} \rightarrow \text{Der}_{\mathbb{K}}(A)$. In this article, we only consider the case that A is a commutative

associative algebra. A Leibniz pair was also studied by Winter [47] and called a Lie algebr. Leibniz pairs were further studied in [18, 22]. A Lie–Rinehart algebra $(A, \mathcal{L}, [\cdot, \cdot]_{\mathcal{L}}, \alpha)$ naturally gives rise to a Leibniz pair by forgetting the A -module structure on \mathcal{L} . We introduce the notion of an admissible representation of a Leibniz pair. If $\text{WRep}_{\mathbb{K}}(\mathcal{L})$ denotes the category of weak representations of a Lie–Rinehart algebra \mathcal{L} and $\text{ARep}_{\mathbb{K}}(\mathcal{S})$ denotes the category of admissible representations of a Leibniz pair \mathcal{S} , then we have the following category equivalence:

$$\text{WRep}_{\mathbb{K}}(\mathcal{L}) \simeq \text{ARep}_{\mathbb{K}}(\mathcal{L}),$$

where the right-hand side \mathcal{L} is considered as the underlying Leibniz pair of a Lie–Rinehart algebra. On the other hand, a Leibniz pair also gives rise to a Lie–Rinehart algebra $\mathcal{S} \otimes_{\mathbb{K}} A$, known as the action Lie–Rinehart algebra. We show that an admissible representation of a Leibniz pair can be naturally extended to a representation of the corresponding action Lie–Rinehart algebra. We have the following category equivalence:

$$\text{ARep}_{\mathbb{K}}(\mathcal{S}) \simeq \text{Rep}(\mathcal{S} \otimes_{\mathbb{K}} A),$$

where $\text{Rep}(\mathcal{S} \otimes_{\mathbb{K}} A)$ denotes the category of representations of the Lie–Rinehart algebra $\mathcal{S} \otimes_{\mathbb{K}} A$. See Remark 3.34 for more details about this equivalence.

1.3. Crossed homomorphisms

The concept of a crossed homomorphism of Lie algebras was introduced in [30] in the study of nonabelian extensions of Lie algebras in 1966. A special class of crossed homomorphisms was recently called a differential operator of weight 1 in [14, 15]. A flat connection 1-form of a trivial principle bundle is naturally a crossed homomorphism. To the best of our knowledge, this concept has not been investigated for many years, and we will use it in this article. More precisely, by using crossed homomorphisms, we show that the category of weak representations (respectively admissible representations) of Lie–Rinehart algebras (respectively Leibniz pairs) is a left module category over the monoidal category of representations of Lie algebras. In particular, we obtain bifunctors among categories of certain representations:

$$F_H : \text{Rep}_{\mathbb{K}}(\mathfrak{g}) \times \text{WRep}_{\mathbb{K}}(\mathcal{L}) \rightarrow \text{WRep}_{\mathbb{K}}(\mathcal{L}), \quad \mathcal{F}_H : \text{Rep}_{\mathbb{K}}(\mathfrak{h}) \times \text{ARep}_{\mathbb{K}}(\mathcal{S}) \rightarrow \text{ARep}_{\mathbb{K}}(\mathcal{S}),$$

which we call the actions of monoidal categories, generalising Shen–Larsson constructions of representations for Cartan-type Lie algebras. Our construction sheds light on some difficult classification problems in representation theory of Lie algebras.

We observe the importance of crossed homomorphisms in our above construction. To better understand crossed homomorphisms and our actions of monoidal categories, we also study deformations and cohomologies of crossed homomorphisms. The deformation of algebraic structures began with the seminal work of Gerstenhaber [12, 13] for associative algebras, followed by its extension to Lie algebras by Nijenhuis and Richardson [36]. A suitable deformation theory of an algebraic structure needs to follow a certain general principle: on one hand, for a given object with the algebraic structure, there should be a differential graded Lie algebra whose Maurer–Cartan elements characterise

deformations of this object. On the other hand, there should be a suitable cohomology so that the infinitesimal of a formal deformation can be identified with a cohomology class. We successfully construct a differential graded Lie algebra such that crossed homomorphisms are characterised as Maurer–Cartan elements. The cohomology groups of crossed homomorphisms are also defined to control their linear deformations.

1.4. Outline of the article

In Section 2, we recall the concept of crossed homomorphisms between Lie algebras and show that there is a one-to-one correspondence between crossed homomorphisms and certain Lie algebra homomorphisms (Theorem 2.7). This fact is the key ingredient in our later construction of the left module category.

In Section 3, we introduce the new concepts: weak representations (respectively admissible representations) of Lie–Rinehart algebras (respectively Leibniz pairs). Using crossed homomorphisms, we show that the category of weak representations (respectively admissible representations) of Lie–Rinehart algebras (respectively Leibniz pairs) is a left module category over the monoidal category of representations of Lie algebras. In particular, the corresponding bifunctor \mathcal{F}_H , which we call the action of monoidal categories, is established to give new representations of Lie–Rinehart algebras (respectively Leibniz pairs). See Theorems 3.26 and 3.36. This generalises and unifies various existing constructions of representations of many Lie algebras by using this new bifunctor.

In Section 4, to show the power of our action of monoidal categories \mathcal{F}_H established in Section 3, we construct some examples of crossed homomorphisms in different situations using our action of monoidal categories to recover some known representations of various Lie algebras (see Subsections 4.1–4.3) and to obtain new representations of generalised Witt algebras and their Lie subalgebras (see Corollaries 4.13, 4.15, 4.16). Certainly, our action of monoidal categories will be used to other situations to give new simple representations of suitable Lie algebras.

In Section 5, we characterise crossed homomorphisms as Maurer–Cartan elements in a suitable differential graded Lie algebra and introduce the cohomology theory of crossed homomorphisms. We use the cohomology theory of crossed homomorphisms that we established to study linear deformations of crossed homomorphisms and to prove that the linear deformation $H_t := H + td_{\rho_H}(-Hx)$ is trivial for any Nijenhuis element x (Theorem 5.14).

We conclude our article in Section 6 by asking three questions.

As usual, we denote by \mathbb{Z} , \mathbb{Z}_+ and \mathbb{C} the sets of all integers, positive integers and complex numbers. All vector spaces are over an algebraically closed field \mathbb{K} of characteristic 0.

2. Crossed homomorphisms between Lie algebras

Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ and $(\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}})$ be Lie algebras. We will denote by $\text{Der}(\mathfrak{g})$ and $\text{Der}(\mathfrak{h})$ the Lie algebras of derivations on \mathfrak{g} and \mathfrak{h} , respectively. A Lie algebra homomorphism $\rho : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{h})$ will be called an **action** of \mathfrak{g} on \mathfrak{h} in the sequel.

Definition 2.1 ([30]). Let $\rho : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{h})$ be an action of $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ on $(\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}})$. A linear map $H : \mathfrak{g} \rightarrow \mathfrak{h}$ is called a **crossed homomorphism with respect to the action** ρ if

$$H[x, y]_{\mathfrak{g}} = \rho(x)(Hy) - \rho(y)(Hx) + [Hx, Hy]_{\mathfrak{h}}, \quad \forall x, y \in \mathfrak{g}. \tag{1}$$

Remark 2.2. A crossed homomorphism from \mathfrak{g} to \mathfrak{g} with respect to the adjoint action is also called a **differential operator of weight 1**. See [14, 15] for more details.

Example 2.3. Let P be a trivial G -principle bundle over a differential manifold M , where G is a Lie group. Let $\omega \in \Omega^1(M, \mathfrak{g})$ be a connection 1-form, where \mathfrak{g} is the Lie algebra of G . Then ω is flat if and only if $d\omega + \frac{1}{2}[\omega, \omega]_{\mathfrak{g}} = 0$, which is equivalent to

$$X\omega(Y) - Y\omega(X) - \omega([X, Y]) + [\omega(X), \omega(Y)]_{\mathfrak{g}} = 0, \quad \forall X, Y \in \mathfrak{X}(M).$$

Therefore, a flat connection 1-form – that is, $\omega \in \Omega^1(M, \mathfrak{g}) = \text{Hom}(\mathfrak{X}(M), \mathfrak{g} \otimes C^\infty(M))$ satisfying the above equality – is a crossed homomorphism from the Lie algebra of vector fields $\mathfrak{X}(M)$ to the Lie algebra $\mathfrak{g} \otimes C^\infty(M)$ with respect to the action ρ given by

$$\rho(X)(u \otimes f) = u \otimes X(f), \quad \forall X \in \mathfrak{X}(M), u \in \mathfrak{g}, f \in C^\infty(M).$$

Example 2.4. If the action ρ of \mathfrak{g} on \mathfrak{h} is zero, then any crossed homomorphism from \mathfrak{g} to \mathfrak{h} is nothing but a Lie algebra homomorphism. If \mathfrak{h} is commutative, then any crossed homomorphism from \mathfrak{g} to \mathfrak{h} is simply a derivation from \mathfrak{g} to \mathfrak{h} with respect to the representation $(\mathfrak{h}; \rho)$.

Definition 2.5. Let H and H' be crossed homomorphisms from \mathfrak{g} to \mathfrak{h} with respect to the action ρ . A **homomorphism** from H' to H consists of two Lie algebra homomorphisms $\phi_{\mathfrak{g}} : \mathfrak{g} \rightarrow \mathfrak{g}$ and $\phi_{\mathfrak{h}} : \mathfrak{h} \rightarrow \mathfrak{h}$ such that

$$H \circ \phi_{\mathfrak{g}} = \phi_{\mathfrak{h}} \circ H', \tag{2}$$

$$\phi_{\mathfrak{h}}(\rho(x)u) = \rho(\phi_{\mathfrak{g}}(x))(\phi_{\mathfrak{h}}(u)), \quad \forall x \in \mathfrak{g}, u \in \mathfrak{h}. \tag{3}$$

In particular, if $\phi_{\mathfrak{g}}$ and $\phi_{\mathfrak{h}}$ are invertible, then $(\phi_{\mathfrak{g}}, \phi_{\mathfrak{h}})$ is called an **isomorphism** from H' to H .

The following result can be also found in [30].

Lemma 2.6. Let H be a crossed homomorphism from \mathfrak{g} to \mathfrak{h} with respect to the action ρ . Define $\rho_H : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{h})$ by

$$\rho_H(x)u := \rho(x)u + [Hx, u]_{\mathfrak{h}}, \quad \forall x \in \mathfrak{g}, u \in \mathfrak{h}. \tag{4}$$

Then ρ_H is also an action of \mathfrak{g} on \mathfrak{h} ; that is, $\rho_H : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{h})$ is a Lie algebra homomorphism.

We use $\mathfrak{g} \ltimes_{\rho_H} \mathfrak{h}$ and $\mathfrak{g} \ltimes_{\rho} \mathfrak{h}$ to denote the two semidirect products of \mathfrak{g} and \mathfrak{h} with respect to the actions ρ_H and ρ , respectively. More precisely, we have

$$\begin{aligned} [(x, u), (y, v)]_{\rho_H} &= [x, y]_{\mathfrak{g}} + \rho_H(x)v - \rho_H(y)u + [u, v]_{\mathfrak{h}}, \\ [(x, u), (y, v)]_{\rho} &= [x, y]_{\mathfrak{g}} + \rho(x)v - \rho(y)u + [u, v]_{\mathfrak{h}}. \end{aligned}$$

Theorem 2.7. Let $H : \mathfrak{g} \rightarrow \mathfrak{h}$ be a linear map and $\rho : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{h})$ an action of \mathfrak{g} on \mathfrak{h} .

- (a) Suppose that ρ_H given by (4) is an action of \mathfrak{g} on \mathfrak{h} . Then the linear map $\hat{H} : \mathfrak{g} \times_{\rho_H} \mathfrak{h} \rightarrow \mathfrak{g} \times_{\rho} \mathfrak{h}$ defined by

$$\hat{H}(x, u) := (x, Hx + u), \quad \forall x \in \mathfrak{g}, u \in \mathfrak{h}, \tag{5}$$

is a Lie algebra isomorphism if and only if H is a crossed homomorphism from \mathfrak{g} to \mathfrak{h} with respect to the action ρ .

- (b) H is a crossed homomorphism from \mathfrak{g} to \mathfrak{h} with respect to the action ρ if and only if the map $\iota_H : \mathfrak{g} \rightarrow \mathfrak{g} \times_{\rho} \mathfrak{h}$ defined by

$$\iota_H(x) := (x, Hx), \quad \forall x \in \mathfrak{g} \tag{6}$$

is a Lie algebra homomorphism.

Proof. (a). Clearly, \hat{H} is an invertible linear map. For all $x, y \in \mathfrak{g}, u, v \in \mathfrak{h}$, we have

$$\begin{aligned} [\hat{H}(x, u), \hat{H}(y, v)]_{\rho} &= [(x, Hx + u), (y, Hy + v)]_{\rho} \\ &= ([x, y]_{\mathfrak{g}}, \rho(x)(Hy + v) - \rho(y)(Hx + u) + [Hx + u, Hy + v]_{\mathfrak{h}}) \\ &= ([x, y]_{\mathfrak{g}}, \rho(x)v - \rho(y)u + [Hx, v]_{\mathfrak{h}} - [Hy, u]_{\mathfrak{h}} + [u, v]_{\mathfrak{h}} + [Hx, Hy]_{\mathfrak{g}} \\ &\quad + \rho(x)(Hy) - \rho(y)(Hx)), \\ \hat{H}[(x, u), (y, v)]_{\rho_H} &= ([x, y]_{\mathfrak{g}}, H[x, y]_{\mathfrak{g}} + \rho_H(x)v - \rho_H(y)u + [u, v]_{\mathfrak{h}}) \\ &= ([x, y]_{\mathfrak{g}}, H[x, y]_{\mathfrak{g}} + \rho(x)v - \rho(y)u + [Hx, v]_{\mathfrak{h}} - [Hy, u]_{\mathfrak{h}} + [u, v]_{\mathfrak{h}}). \end{aligned}$$

Thus, $[\hat{H}(x, u), \hat{H}(y, v)]_{\rho} = \hat{H}[(x, u), (y, v)]_{\rho_H}$, if and only if (1) holds for H , which is equivalent to that H is a crossed homomorphism from \mathfrak{g} to \mathfrak{h} with respect to the action ρ .

- (b) follows from the proof of (a) by taking $u = v = 0$. □

Remark 2.8. In fact, crossed homomorphisms correspond to split nonabelian extensions of Lie algebras. More precisely, we consider the following nonabelian extension of Lie algebras:

$$0 \rightarrow \mathfrak{h} \rightarrow \mathfrak{g} \oplus \mathfrak{h} \rightarrow \mathfrak{g} \rightarrow 0.$$

A section $s : \mathfrak{g} \rightarrow \mathfrak{g} \oplus \mathfrak{h}$ must be of the form $s(x) = (x, Hx)$, $x \in \mathfrak{g}$. Statement (b) says that s is a Lie algebra homomorphism if and only if H is a crossed homomorphism. Such an extension is called a split nonabelian extension. See [30] for more details.

3. Action of monoidal categories arising from Lie–Rinehart algebras and Leibniz pairs

In this section, first we introduce the notion of a weak representation of a Lie–Rinehart algebra and show that the category of weak representations of Lie–Rinehart algebras is a left module category over the monoidal category of representations of Lie algebras by using crossed homomorphisms. Then we introduce the notion of an admissible representation of a Leibniz pair and obtain similar results. In particular, the corresponding bifunctors are called the actions of monoidal categories for Lie–Rinehart algebras and Leibniz pairs.

3.1. Weak representations of Lie–Rinehart algebras

Let A be a commutative associative algebra over \mathbb{K} . We denote by $\text{Der}_{\mathbb{K}}(A)$ the set of \mathbb{K} -linear derivations of A ; that is,

$$\text{Der}_{\mathbb{K}}(A) = \{D \in \text{End}_{\mathbb{K}}(A) : D(ab) = D(a)b + aD(b), \forall a, b \in A\}.$$

Definition 3.1 ([39]). A **Lie–Rinehart algebra** over A is a \mathbb{K} -Lie algebra $(\mathcal{L}, [\cdot, \cdot]_{\mathcal{L}})$ together with an A -module structure on \mathcal{L} and a map $\alpha : \mathcal{L} \rightarrow \text{Der}_{\mathbb{K}}(A)$ (called the anchor) which is simultaneously a \mathbb{K} -Lie algebra and an A -module homomorphism such that

$$[x, ay]_{\mathcal{L}} = a[x, y]_{\mathcal{L}} + \alpha(x)(a)y, \quad \forall x, y \in \mathcal{L}, a \in A.$$

We usually denote a Lie–Rinehart algebra over A by $(A, \mathcal{L}, [\cdot, \cdot]_{\mathcal{L}}, \alpha)$ or simply by \mathcal{L} .

Remark 3.2. It is clear that a Lie–Rinehart algebra with $\alpha = 0$ is exactly a Lie A -algebra.

Example 3.3. $(A, \text{Der}_{\mathbb{K}}(A), [\cdot, \cdot]_C, \alpha = \text{Id})$ is a Lie–Rinehart algebra, where $[\cdot, \cdot]_C$ is the commutator bracket.

Example 3.4. Let M be an A -module. Denote by $\mathfrak{gl}_A(M)$ the set of A -module homomorphisms from M to M . It is obvious that $(\mathfrak{gl}_A(M), [\cdot, \cdot]_C)$ is a Lie A -algebra.

Example 3.5. Let M be an A -module. A **first-order differential operator** on M is a pair (D, σ) , where $D : M \rightarrow M$ is a \mathbb{K} -linear map and $\sigma = \sigma_D \in \text{Der}_{\mathbb{K}}(A)$, satisfying the following compatibility condition:

$$D(am) = aD(m) + \sigma(a)m, \quad \forall a \in A, m \in M. \tag{7}$$

Denote by $\mathfrak{D}(M)$ the set of first-order differential operators on M . It is obvious that $\mathfrak{D}(M)$ is an A -module. Define a bracket operation $[\cdot, \cdot]_C$ on $\mathfrak{D}(M)$ by

$$[(D_1, \sigma_1), (D_2, \sigma_2)]_C := (D_1 \circ D_2 - D_2 \circ D_1, \sigma_1 \circ \sigma_2 - \sigma_2 \circ \sigma_1), \quad \forall (D_1, \sigma_1), (D_2, \sigma_2) \in \mathfrak{D}(M), \tag{8}$$

and an A -module homomorphism $\text{Pr} : \mathfrak{D}(M) \rightarrow \text{Der}_{\mathbb{K}}(A)$ by $\text{Pr}(D, \sigma) = \sigma$ for all $(D, \sigma) \in \mathfrak{D}(M)$. Then $(A, \mathfrak{D}(M), [\cdot, \cdot]_C, \alpha = \text{Pr})$ is a Lie–Rinehart algebra.

Remark 3.6. Let M be an A -module. It is straightforward to see that we have a semidirect product commutative associative algebra $A \ltimes M$, where the multiplication is given by

$$(a, m) \cdot (b, n) = (ab, an + bm), \quad \forall a, b \in A, m, n \in M.$$

Then (D, σ) is a first-order differential operator on M if and only if (σ, D) is a derivation on the commutative associative algebra $A \ltimes M$. This result is the algebraic counterpart of the fact that a first-order differential operator on a vector bundle E can be viewed as a linear vector field on the dual bundle E^* . In fact, functions on the vector bundle $E^* \rightarrow N$ are generated by $C^\infty(N)$ and $\Gamma(E)$, while the latter are fibre linear functions on E^* . Since a first-order differential operator maps a fibre linear function to a fibre linear function, it is viewed as a linear vector field on E^* .

Definition 3.7.

- (i) Let $(A, \mathcal{L}, [\cdot, \cdot]_{\mathcal{L}}, \alpha)$ and $(A, \mathcal{L}', [\cdot, \cdot]_{\mathcal{L}'}, \alpha')$ be Lie–Rinehart algebras. A **Lie–Rinehart weak homomorphism** is a \mathbb{K} -Lie algebra homomorphism $f : \mathcal{L} \rightarrow \mathcal{L}'$ such that $\alpha' \circ f = \alpha$.
- (ii) A Lie–Rinehart weak homomorphism f is called a **Lie–Rinehart homomorphism** if f is also an A -module homomorphism; that is, $f(ax) = af(x)$, for all $a \in A$ and $x \in \mathcal{L}$.

Note that zero map from \mathcal{L} to \mathcal{L}' is not a Lie–Rinehart weak homomorphism if $\alpha \neq 0$.

Proposition 3.8. *Let $f_1 : (A, \mathcal{L}_1, [\cdot, \cdot]_{\mathcal{L}_1}, \alpha_1) \rightarrow (A, \mathcal{L}_2, [\cdot, \cdot]_{\mathcal{L}_2}, \alpha_2)$ and $f_2 : (A, \mathcal{L}_2, [\cdot, \cdot]_{\mathcal{L}_2}, \alpha_2) \rightarrow (A, \mathcal{L}_3, [\cdot, \cdot]_{\mathcal{L}_3}, \alpha_3)$ be two Lie–Rinehart weak homomorphisms. Then $f_2 \circ f_1$ is a Lie–Rinehart weak homomorphism from $(A, \mathcal{L}_1, [\cdot, \cdot]_{\mathcal{L}_1}, \alpha_1)$ to $(A, \mathcal{L}_3, [\cdot, \cdot]_{\mathcal{L}_3}, \alpha_3)$.*

Proof. This is easy to see. □

We denote by $\text{WH}(\mathcal{L}, \mathcal{L}')$ the set of weak homomorphisms from the Lie–Rinehart algebra $(A, \mathcal{L}, [\cdot, \cdot]_{\mathcal{L}}, \alpha)$ to $(A, \mathcal{L}', [\cdot, \cdot]_{\mathcal{L}'}, \alpha')$. By Proposition 3.8, it is easy to see that $\text{WH}(\mathcal{L}, \mathcal{L})$ is a monoid.

Definition 3.9.

- (i) A **weak representation** of a Lie–Rinehart algebra $(A, \mathcal{L}, [\cdot, \cdot]_{\mathcal{L}}, \alpha)$ on an A -module M is a Lie–Rinehart weak homomorphism $\rho : \mathcal{L} \rightarrow \mathfrak{D}(M)$. We denote a weak representation by $(M; \rho)$.
- (ii) A weak representation $(M; \rho)$ is called a **representation** if ρ is also an A -module homomorphism; that is, $\rho : \mathcal{L} \rightarrow \mathfrak{D}(M)$ is a Lie–Rinehart homomorphism.

Remark 3.10. A weak representation of a Lie–Rinehart algebra $(A, \mathcal{L}, [\cdot, \cdot]_{\mathcal{L}}, \alpha)$ on an A -module M means a \mathbb{K} -Lie algebra homomorphism $\rho : \mathcal{L} \rightarrow \mathfrak{gl}_{\mathbb{K}}(M)$ such that

$$\rho(x)(au) = a\rho(x)(u) + \alpha(x)(a)u, \quad \forall x \in \mathcal{L}, a \in A, u \in M;$$

that is, $(D = \rho(x), \sigma = \alpha(x))$ is a first-order differential operator on M .

Remark 3.11. In [19], Huebschmann showed that there is a one-to-one correspondence between representations of a Lie–Rinehart algebra and representations of its universal enveloping algebra $U(A, \mathcal{L}) := (A \# U(\mathcal{L})) / J$, where J is a certain ideal of the smash product $A \# U(\mathcal{L})$. More explicitly, it is known that $U(\mathcal{L})$ is a Hopf algebra and A is a $U(\mathcal{L})$ -module algebra. Then the smash product $A \# U(\mathcal{L})$ (see [35]) is a \mathbb{K} -vector space $A \otimes U(\mathcal{L})$ with elements denoted by $a \# u$ and with product defined for all $a, b \in A$ and $u, v \in U(\mathcal{L})$ by

$$(a \# u)(b \# v) = \sum a\alpha(u_{(1)})b \# u_{(2)}v,$$

where we use the standard Sweedler notation $\Delta(u) = \sum u_{(1)} \otimes u_{(2)}$ for the coproduct Δ . The algebra $A \# U(\mathcal{L})$ is also called the Massey–Peterson algebra in [19]. It is not hard to see that there is a one-to-one correspondence between weak representations of a Lie–Rinehart algebra and representations of the smash product $A \# U(\mathcal{L})$.

Example 3.12. Let $(A, \mathcal{L}, [\cdot, \cdot]_{\mathcal{L}}, \alpha)$ be a Lie–Rinehart algebra. Define $\text{ad} : \mathcal{L} \rightarrow \mathfrak{D}(\mathcal{L})$ by

$$\text{ad}_x y = [x, y]_{\mathcal{L}}, \quad \sigma_{\text{ad}_x} = \alpha(x), \quad \forall x, y \in \mathcal{L}.$$

Then ad is a weak representation of \mathcal{L} on \mathcal{L} . Note that ad is generally not a representation of \mathcal{L} on itself.

Definition 3.13. Let $(A, \mathcal{L}, [\cdot, \cdot]_{\mathcal{L}}, \alpha)$ be a Lie–Rinehart algebra and $(M; \rho)$ and $(M'; \rho')$ be two weak representations of \mathcal{L} . An A -module homomorphism $\phi : M \rightarrow M'$ is said to be a **homomorphism of weak representations** if $\phi \circ \rho(x) = \rho'(x) \circ \phi$ for all $x \in \mathcal{L}$.

Proposition 3.14. Let $\phi : (M; \rho) \rightarrow (M'; \rho')$ and $\phi' : (M'; \rho') \rightarrow (M''; \rho'')$ be two homomorphisms of weak representations of \mathcal{L} . Then $\phi' \circ \phi$ is a homomorphism from $(M; \rho)$ to $(M''; \rho'')$.

Proof. This is easy to see. □

We usually denote by $M \xrightarrow{\phi} M'$ a homomorphism between the weak representations $(M; \rho)$ and $(M'; \rho')$ and denote by $\text{WRep}_{\mathbb{K}}(\mathcal{L})$ the category of weak representations of a Lie–Rinehart algebra $(A, \mathcal{L}, [\cdot, \cdot]_{\mathcal{L}}, \alpha)$ and $\text{Rep}_{\mathbb{K}}(\mathfrak{g})$ the category of representations of a \mathbb{K} -Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$. It is obvious that the category of representations of a Lie–Rinehart algebra $(A, \mathcal{L}, [\cdot, \cdot]_{\mathcal{L}}, \alpha)$, denoted by $\text{Rep}(\mathcal{L})$, is a full subcategory of the category $\text{WRep}_{\mathbb{K}}(\mathcal{L})$. Please note the subtle difference between the two categories $\text{Rep}_{\mathbb{K}}(\mathcal{L})$ and $\text{Rep}(\mathcal{L})$.

Definition 3.15 ([6]). Let $(A, \mathcal{L}, [\cdot, \cdot]_{\mathcal{L}}, \alpha)$ be a Lie–Rinehart algebra and $(\mathcal{G}, [\cdot, \cdot]_{\mathcal{G}})$ be a Lie A -algebra. We say that \mathcal{L} **acts on** \mathcal{G} if a \mathbb{K} -Lie algebra homomorphism $\rho : \mathcal{L} \rightarrow \text{Der}_{\mathbb{K}}(\mathcal{G})$ is given such that

$$\rho(ax) = a\rho(x), \quad \rho(x)(au) = a\rho(x)u + \alpha(x)(a)u, \quad \forall a \in A, x \in \mathcal{L}, u \in \mathcal{G}.$$

Let $(A, \mathcal{L}, [\cdot, \cdot]_{\mathcal{L}}, \alpha)$ be a Lie–Rinehart algebra and $(\mathcal{G}, [\cdot, \cdot]_{\mathcal{G}})$ be a Lie A -algebra on which \mathcal{L} acts via $\rho : \mathcal{L} \rightarrow \text{Der}_{\mathbb{K}}(\mathcal{G})$. On the A -module $\mathcal{L} \oplus \mathcal{G}$, define a bracket operation $[\cdot, \cdot]_{\rho}$ by

$$[(x, u), (y, v)]_{\rho} = ([x, y]_{\mathcal{L}}, \rho(x)v - \rho(y)u + [u, v]_{\mathcal{G}}), \quad \forall x, y \in \mathcal{L}, u, v \in \mathcal{G},$$

and define an A -module homomorphism $\tilde{\alpha} : \mathcal{L} \oplus \mathcal{G} \rightarrow \text{Der}_{\mathbb{K}}(A)$ by

$$\tilde{\alpha}(x, u) = \alpha(x), \quad \forall x \in \mathcal{L}, u \in \mathcal{G}.$$

Then $(A, \mathcal{L} \oplus \mathcal{G}, [\cdot, \cdot]_{\rho}, \tilde{\alpha})$ is a Lie–Rinehart algebra [6], which is called the **semi-direct product** of \mathcal{L} and \mathcal{G} and denoted by $\mathcal{L} \ltimes_{\rho} \mathcal{G}$.

Note that the Lie algebra $\mathcal{L} \ltimes_{\rho} \mathcal{G}$ acts on the Lie algebra $(\mathcal{G}, [\cdot, \cdot]_{\mathcal{G}})$ by

$$\tilde{\rho}(x, u)v = \rho(x)v, \quad \forall x \in \mathcal{L}, u, v \in \mathcal{G}. \tag{9}$$

Then using Theorem 2.7 (b), we can easily verify the following result.

Proposition 3.16. *With the above notations, the projection $\text{Pr} : \mathcal{L} \ltimes_{\rho} \mathcal{G} \rightarrow \mathcal{G}$ is a crossed homomorphism with respect to the action $\tilde{\rho}$.*

3.2. Left module categories over monoidal categories

Proposition 3.17. *Let $(A, \mathcal{L}, [\cdot, \cdot]_{\mathcal{L}}, \alpha)$ be a Lie–Rinehart algebra and ρ an action of \mathcal{L} on a Lie A -algebra $(\mathcal{G}, [\cdot, \cdot]_{\mathcal{G}})$. For a crossed homomorphism $H : \mathcal{L} \rightarrow \mathcal{G}$ between \mathbb{K} -Lie algebras, we define a \mathbb{K} -linear map $\iota_H : \mathcal{L} \rightarrow \mathcal{L} \times_{\rho} \mathcal{G}$ by*

$$\iota_H(x) = (x, Hx), \quad \forall x \in \mathcal{L}.$$

Then ι_H is a Lie–Rinehart injective weak homomorphism from \mathcal{L} to $\mathcal{L} \times_{\rho} \mathcal{G}$.

Proof. By Theorem 2.7 (b), we know that ι_H is a \mathbb{K} -Lie algebra monomorphism. Moreover, for all $x \in \mathcal{L}$, we have

$$\tilde{\alpha}(\iota_H(x)) = \tilde{\alpha}(x, Hx) = \alpha(x),$$

which implies that $\alpha = \tilde{\alpha} \circ \iota_H$. Thus, ι_H is a Lie–Rinehart injective weak homomorphism. □

Corollary 3.18. *Let $(M; \mu)$ be a Lie–Rinehart weak representation of $(A, \mathcal{L} \times_{\rho} \mathcal{G}, [\cdot, \cdot]_{\rho}, \tilde{\alpha})$ and H be a crossed homomorphism from \mathcal{L} to \mathcal{G} . Then $(M; \mu \circ \iota_H)$ is a Lie–Rinehart weak representation of $(A, \mathcal{L}, [\cdot, \cdot]_{\mathcal{L}}, \alpha)$.*

Proof. By Propositions 3.8 and 3.17, we deduce that $\mu \circ \iota_H : \mathcal{L} \xrightarrow{\iota_H} \mathcal{L} \times_{\rho} \mathcal{G} \xrightarrow{\mu} \mathfrak{D}(M)$ is a Lie–Rinehart weak homomorphism. □

Let $(A, \mathcal{L}, [\cdot, \cdot]_{\mathcal{L}}, \alpha)$ be a Lie–Rinehart algebra and $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ be a \mathbb{K} -Lie algebra. Then $\mathcal{G} = \mathfrak{g} \otimes_{\mathbb{K}} A$ is a Lie A -algebra, where the A -module structure and the Lie bracket $[\cdot, \cdot]_{\mathcal{G}}$ are given by

$$a(g \otimes b) = g \otimes ab, \quad [g \otimes a, h \otimes b]_{\mathcal{G}} = [g, h]_{\mathfrak{g}} \otimes ab, \quad \forall a, b \in A, g, h \in \mathfrak{g}.$$

Moreover, the Lie–Rinehart algebra $(A, \mathcal{L}, [\cdot, \cdot]_{\mathcal{L}}, \alpha)$ acts on the Lie A -algebra $\mathfrak{g} \otimes_{\mathbb{K}} A$ by α as follows:

$$\alpha(x)(g \otimes a) = g \otimes \alpha(x)(a), \quad \forall x \in \mathcal{L}, a \in A, g \in \mathfrak{g}. \tag{10}$$

Consequently, we have the semidirect product Lie–Rinehart algebra $(A, \mathcal{L} \times_{\alpha} (\mathfrak{g} \otimes_{\mathbb{K}} A), [\cdot, \cdot]_{\alpha}, \tilde{\alpha})$.

Let $(A, \mathcal{L}, [\cdot, \cdot]_{\mathcal{L}}, \alpha)$ be a Lie–Rinehart algebra and $(M; \rho)$ be a Lie–Rinehart weak representation of $(A, \mathcal{L}, [\cdot, \cdot]_{\mathcal{L}}, \alpha)$. Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ be a \mathbb{K} -Lie algebra and $(V; \theta)$ be a representation of \mathfrak{g} . Then $V \otimes_{\mathbb{K}} M$ has a natural A -module structure:

$$a(v \otimes m) = v \otimes am, \quad \forall a \in A, v \in V, m \in M.$$

We define a \mathbb{K} -linear map $\rho \boxplus \theta : \mathcal{L} \times_{\alpha} (\mathfrak{g} \otimes_{\mathbb{K}} A) \rightarrow \mathfrak{gl}_{\mathbb{K}}(V \otimes_{\mathbb{K}} M)$ by

$$(\rho \boxplus \theta)(x, g \otimes a)(v \otimes m) := v \otimes \rho(x)m + \theta(g)v \otimes am$$

for all $x \in \mathcal{L}, a \in A, g \in \mathfrak{g}, m \in M, v \in V$.

Lemma 3.19. *With the above notations, $(V \otimes_{\mathbb{K}} M; \rho \boxplus \theta)$ is a Lie–Rinehart weak representation of the Lie–Rinehart algebra $(A, \mathcal{L} \times_{\alpha} (\mathfrak{g} \otimes_{\mathbb{K}} A), [\cdot, \cdot]_{\alpha}, \tilde{\alpha})$.*

Proof. Since $\rho : \mathcal{L} \rightarrow \mathfrak{D}(M)$ and $\theta : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ are \mathbb{K} -Lie algebra homomorphisms, for all $a, b \in A, x, y \in \mathcal{L}, g, h \in \mathfrak{g}, m \in M, v \in V$, we have

$$\begin{aligned} & \left([(\rho \boxplus \theta)(x, g \otimes a), (\rho \boxplus \theta)(y, h \otimes b)]_{\mathcal{C}} - (\rho \boxplus \theta)([(x, g \otimes a), (y, h \otimes b)]_{\alpha}) \right) (v \otimes m) \\ &= (\rho \boxplus \theta)(x, g \otimes a) \left(v \otimes \rho(y)m + \theta(h)v \otimes bm \right) - (\rho \boxplus \theta)(y, h \otimes b) \left(v \otimes \rho(x)m + \theta(g)v \otimes am \right) \\ & \quad - (\rho \boxplus \theta) \left([x, y]_{\mathcal{L}}, h \otimes \alpha(x)(b) - g \otimes \alpha(y)(a) + [g, h]_{\mathfrak{g}} \otimes ab \right) (v \otimes m) \\ &= v \otimes \rho(x)(\rho(y)m) + \theta(g)v \otimes a(\rho(y)m) + \theta(h)v \otimes \rho(x)(bm) + \theta(g)(\theta(h)v) \otimes a(bm) \\ & \quad - v \otimes \rho(y)(\rho(x)m) - \theta(h)v \otimes b(\rho(x)m) - \theta(g)v \otimes \rho(y)(am) - \theta(h)(\theta(g)v) \otimes b(am) \\ & \quad - v \otimes \rho([x, y]_{\mathcal{L}})m - \theta(h)v \otimes \alpha(x)(b)m + \theta(g)v \otimes \alpha(y)(a)m - \theta([g, h]_{\mathfrak{g}})v \otimes (ab)m \\ &= 0. \end{aligned}$$

Therefore, we deduce that $\rho \boxplus \theta$ is a \mathbb{K} -Lie algebra homomorphism.

Furthermore, by $\rho(x) \in \mathfrak{D}(M)$, we have

$$\begin{aligned} (\rho \boxplus \theta)(x, g \otimes b) \left(a(v \otimes m) \right) &= (\rho \boxplus \theta)(x, g \otimes b)(v \otimes am) \\ &= v \otimes \rho(x)(am) + \theta(g)v \otimes a(bm) \\ &= v \otimes \left(a\rho(x)(m) + \alpha(x)(a)m \right) + \theta(g)v \otimes a(bm) \\ &= a \left((\rho \boxplus \theta)(x, g \otimes b)(v \otimes m) \right) + \alpha(x)(a)(v \otimes m), \end{aligned}$$

which implies that $(\rho \boxplus \theta)(x, g \otimes b) \in \mathfrak{D}(V \otimes_{\mathbb{K}} M)$ and $\tilde{\alpha} = \text{Pr} \circ (\rho \boxplus \theta)$.

Therefore, $\rho \boxplus \theta : \mathcal{L} \ltimes_{\alpha} (\mathfrak{g} \otimes_{\mathbb{K}} A) \rightarrow \mathfrak{D}(V \otimes_{\mathbb{K}} M)$ is a Lie–Rinehart weak homomorphism. □

Corollary 3.20. *Let $(M; \rho)$ be a Lie–Rinehart representation of $(A, \mathcal{L}, [\cdot, \cdot]_{\mathcal{L}}, \alpha)$ and $(V; \theta)$ be a representation of \mathfrak{g} . Then $(V \otimes_{\mathbb{K}} M; \rho \boxplus \theta)$ is a Lie–Rinehart representation of $\mathcal{L} \ltimes_{\alpha} (\mathfrak{g} \otimes_{\mathbb{K}} A)$.*

Proof. Since ρ is an A -module homomorphism, we have

$$\begin{aligned} & \left((\rho \boxplus \theta)(b(x, g \otimes a)) - b(\rho \boxplus \theta)(x, g \otimes a) \right) (v \otimes m) \\ &= v \otimes \rho(bx)m + \theta(g)v \otimes (ba)m - b \left(v \otimes \rho(x)m + \theta(g)v \otimes am \right) \\ &= 0, \quad \forall a, b \in A, x \in \mathcal{L}, g \in \mathfrak{g}, m \in M, v \in V. \end{aligned}$$

Thus, $\rho \boxplus \theta$ is also an A -module homomorphism. □

Before we give the main result of the article, we recall the notions of a monoidal category and a left module category over a monoidal category.

Definition 3.21 ([10]). A **monoidal category** is a 6-tuple $(\mathcal{C}, \otimes, a, \mathbf{1}, l, r)$ that consists of the following data:

- a category \mathcal{C} ;
- a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ called the monoidal product;

- a natural isomorphism $a : \otimes \circ (\otimes \times \text{Id}_{\mathcal{C}}) \rightarrow \otimes \circ (\text{Id}_{\mathcal{C}} \times \otimes)$ called the associativity isomorphism;
- an object $\mathbf{1} \in \text{Ob}(\mathcal{C})$ called the unit object;
- a natural isomorphism $l : \otimes \circ (\mathbf{1} \times \text{Id}_{\mathcal{C}}) \rightarrow \text{Id}_{\mathcal{C}}$ called the left unit isomorphism and a natural isomorphism $r : \otimes \circ (\text{Id}_{\mathcal{C}} \times \mathbf{1}) \rightarrow \text{Id}_{\mathcal{C}}$ called the right unit isomorphism.

These data satisfy the following two axioms:

(1) the **pentagon axiom**: the pentagon diagram

$$\begin{array}{ccc}
 & ((W \otimes X) \otimes Y) \otimes Z & \\
 \swarrow^{a_{W \otimes X, Y, Z}} & & \searrow^{a_{W, X, Y} \otimes \text{Id}_Z} \\
 (W \otimes X) \otimes (Y \otimes Z) & & (W \otimes (X \otimes Y)) \otimes Z \\
 \downarrow^{a_{W, X, Y \otimes Z}} & & \downarrow^{a_{W, X \otimes Y, Z}} \\
 W \otimes (X \otimes (Y \otimes Z)) & \xleftarrow{\text{Id}_W \otimes a_{X, Y, Z}} & W \otimes ((X \otimes Y) \otimes Z)
 \end{array}$$

commutes for all $W, X, Y, Z \in \text{Ob}(\mathcal{C})$.

(2) the **triangle axiom**: the triangle diagram

$$\begin{array}{ccc}
 (X \otimes \mathbf{1}) \otimes Y & \xrightarrow{a_{X, \mathbf{1}, Y}} & X \otimes (\mathbf{1} \otimes Y) \\
 \searrow^{r_X \otimes \text{Id}_Y} & & \swarrow^{\text{Id}_X \otimes l_Y} \\
 & X \otimes Y &
 \end{array}$$

commutes for all $X, Y \in \text{Ob}(\mathcal{C})$.

The monoidal category \mathcal{C} is **strict** if the associativity isomorphism, left unit isomorphism and right unit isomorphism a, l, r are all identities.

Example 3.22. Let \mathcal{C} be a category and $\mathcal{E}nd(\mathcal{C})$ be the category of endofunctors (the functors from \mathcal{C} into itself). Then $\mathcal{E}nd(\mathcal{C})$ is a strict monoidal category with the composition of functors as the monoidal product and the identity functor as the unit object of this category.

Example 3.23. The category of representations $\text{Rep}_{\mathbb{K}}(\mathfrak{g})$ of a \mathbb{K} -Lie algebra \mathfrak{g} is a monoidal category: the monoidal product of $(V_1; \theta_1)$ and $(V_2; \theta_2)$ is defined by

$$(V_1; \theta_1) \otimes (V_2; \theta_2) := (V_1 \otimes V_2; \theta_1 \otimes \text{Id}_{V_2} + \text{Id}_{V_1} \otimes \theta_2),$$

and the unit object $\mathbf{1}$ is the 1-dimensional trivial representation $(\mathbb{K}; 0)$ of \mathfrak{g} . Moreover, the associativity isomorphism

$$a_{(V_1; \theta_1), (V_2; \theta_2), (V_3; \theta_3)} : ((V_1; \theta_1) \otimes (V_2; \theta_2)) \otimes (V_3; \theta_3) \rightarrow (V_1; \theta_1) \otimes ((V_2; \theta_2) \otimes (V_3; \theta_3))$$

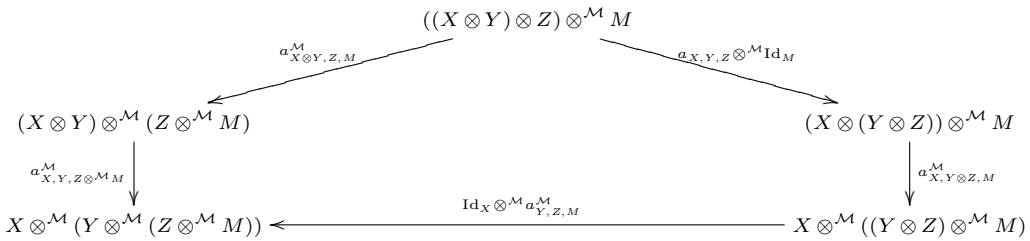
is defined by

$$a_{(V_1; \theta_1), (V_2; \theta_2), (V_3; \theta_3)} \left((v_1 \otimes v_2) \otimes v_3 \right) := v_1 \otimes (v_2 \otimes v_3), \quad \forall v_i \in V_i, \quad i = 1, 2, 3, \quad (11)$$

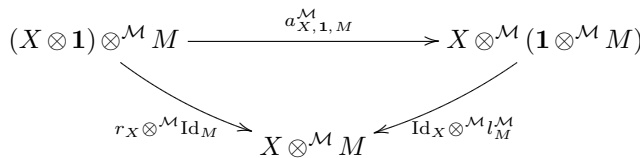
and the left unit isomorphism $l_{(V;\theta)}$ and the right unit isomorphism $r_{(V;\theta)}$ are defined by

$$l_{(V;\theta)}(k \otimes v) := kv, \quad r_{(V;\theta)}(v \otimes k) := kv, \quad \forall k \in \mathbb{K}, v \in V. \tag{12}$$

Definition 3.24 ([10]). Let $(\mathcal{C}, \otimes, a, \mathbf{1}, l, r)$ be a monoidal category. A **left module category** over \mathcal{C} is a category \mathcal{M} equipped with a bifunctor $\otimes^{\mathcal{M}} : \mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M}$, a natural isomorphism $a^{\mathcal{M}} : \otimes^{\mathcal{M}} \circ (\otimes \times \text{Id}_{\mathcal{M}}) \rightarrow \otimes^{\mathcal{M}} \circ (\text{Id}_{\mathcal{C}} \times \otimes^{\mathcal{M}})$ and a natural isomorphism $l^{\mathcal{M}} : \otimes^{\mathcal{M}} \circ (\mathbf{1} \times \text{Id}_{\mathcal{M}}) \rightarrow \text{Id}_{\mathcal{M}}$ such that the **pentagon diagram**



and the **triangle diagram**



commute for all $X, Y, Z \in \text{Ob}(\mathcal{C}), M \in \text{Ob}(\mathcal{M})$.

Example 3.25. Any monoidal category $(\mathcal{C}, \otimes, a, \mathbf{1}, l, r)$ is a left module category over itself. More precisely, we set $\otimes^{\mathcal{C}} = \otimes, a^{\mathcal{C}} = a, l^{\mathcal{C}} = l$. This left module category can be considered as a categorification of the regular representation of an associative algebra.

Let $(A, \mathcal{L}, [\cdot, \cdot]_{\mathcal{L}}, \alpha)$ be a Lie–Rinehart algebra and \mathfrak{g} be a \mathbb{K} -Lie algebra. Let H be a crossed homomorphism from the \mathbb{K} -Lie algebra \mathcal{L} to $\mathfrak{g} \otimes_{\mathbb{K}} A$ with respect to the action α given by (10). For all $x \in \mathcal{L}$, we set $Hx = \sum_i x_i^{\mathfrak{g}} \otimes x_i^A$ or $Hx = x_i^{\mathfrak{g}} \otimes x_i^A$ for simplicity.

By Corollary 3.18 and Lemma 3.19, our main theorem can be stated as follows.

Theorem 3.26. *Let $(A, \mathcal{L}, [\cdot, \cdot]_{\mathcal{L}}, \alpha)$ be a Lie–Rinehart algebra and $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ be a \mathbb{K} -Lie algebra. Then any crossed homomorphism $H : \mathcal{L} \rightarrow \mathfrak{g} \otimes_{\mathbb{K}} A$ induces a left module category structure of the category of weak representations $\text{WRep}_{\mathbb{K}}(\mathcal{L})$ over the monoidal category $\text{Rep}_{\mathbb{K}}(\mathfrak{g})$. More precisely, the left module structure is given by*

- the bifunctor $F_H : \text{Rep}_{\mathbb{K}}(\mathfrak{g}) \times \text{WRep}_{\mathbb{K}}(\mathcal{L}) \rightarrow \text{WRep}_{\mathbb{K}}(\mathcal{L})$, which is defined on the set of objects and on the set of morphisms respectively by

$$F_H \left((V; \theta), (M; \rho) \right) = (V \otimes_{\mathbb{K}} M; (\rho \boxplus \theta) \circ \iota_H), \tag{13}$$

$$F_H (V \xrightarrow{\psi} V', M \xrightarrow{\phi} M') = V \otimes M \xrightarrow{\psi \otimes \phi} V' \otimes M', \tag{14}$$

for $(V; \theta), (V'; \theta') \in \text{Rep}_{\mathbb{K}}(\mathfrak{g})$, $(M; \rho), (M'; \rho') \in \text{WRep}_{\mathbb{K}}(\mathcal{L})$, representation homomorphism $V \xrightarrow{\psi} V'$ of the \mathbb{K} -Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ and weak representation homomorphism $M \xrightarrow{\phi} M'$ of the Lie–Rinehart algebra $(A, \mathcal{L}, [\cdot, \cdot]_{\mathcal{L}}, \alpha)$;

- the natural isomorphism

$$\begin{aligned} a_{(V_1; \theta_1), (V_2; \theta_2), (M; \rho)} &: F_H((V_1; \theta_1) \otimes (V_2; \theta_2), (M; \rho)) \\ &\rightarrow F_H\left((V_1; \theta_1), F_H((V_2; \theta_2), (M; \rho))\right), \end{aligned}$$

which is defined by

$$a_{(V_1; \theta_1), (V_2; \theta_2), (M; \rho)}((v_1 \otimes v_2) \otimes m) = v_1 \otimes (v_2 \otimes m), \tag{15}$$

- the natural isomorphism $l_{(M; \rho)} : F_H((\mathbb{K}; 0), (M; \rho)) \rightarrow (M; \rho)$, which is defined by

$$l_{(M; \rho)}(k \otimes m) = km. \tag{16}$$

Proof. By Corollary 3.18 and Lemma 3.19, $(V \otimes_{\mathbb{K}} M; (\rho \boxplus \theta) \circ \iota_H)$ is a weak representation of \mathcal{L} . Thus, F_H is well-defined on the set of objects. To see that F_H is also well-defined on the set of morphisms, we need to show that the linear map $\psi \otimes \phi : V \otimes M \rightarrow V' \otimes M'$ is indeed a homomorphism from $(V \otimes_{\mathbb{K}} M; (\rho \boxplus \theta) \circ \iota_H)$ to $(V' \otimes_{\mathbb{K}} M'; (\rho' \boxplus \theta') \circ \iota_H)$. In fact, for all $a \in A$, $v \in V$, $m \in M$, we have

$$\begin{aligned} (\psi \otimes \phi)(a(v \otimes m)) - a((\psi \otimes \phi)(v \otimes m)) &= (\psi \otimes \phi)(v \otimes am) - a(\psi(v) \otimes \phi(m)) \\ &= \psi(v) \otimes \phi(am) - \psi(v) \otimes a\phi(m) \\ &= 0. \end{aligned}$$

For all $x \in \mathfrak{g}$, $v \in V$ and $m \in M$, we have

$$\begin{aligned} &(\psi \otimes \phi)\left(\left((\rho \boxplus \theta) \iota_H(x)\right)(v \otimes m)\right) - \left(\left(\rho' \boxplus \theta'\right) \iota_H(x)\right)\left((\psi \otimes \phi)(v \otimes m)\right) \\ &= (\psi \otimes \phi)\left(\left((\rho \boxplus \theta)(x, x_i^{\mathfrak{g}} \otimes x_i^A)\right)(v \otimes m)\right) - \left(\left(\rho' \boxplus \theta'\right)(x, x_i^{\mathfrak{g}} \otimes x_i^A)\right)\left(\psi(v) \otimes \phi(m)\right) \\ &= (\psi \otimes \phi)\left(v \otimes \rho(x)m + \theta(x_i^{\mathfrak{g}})v \otimes x_i^A m\right) - \left(\psi(v) \otimes \rho'(x)\phi(m) + \theta'(x_i^{\mathfrak{g}})\psi(v) \otimes x_i^A \phi(m)\right) \\ &= \left(\psi(v) \otimes \phi(\rho(x)m) + \psi(\theta(x_i^{\mathfrak{g}})v) \otimes \phi(x_i^A m)\right) \\ &\quad - \left(\psi(v) \otimes \rho'(x)\phi(m) + \theta'(x_i^{\mathfrak{g}})\psi(v) \otimes x_i^A \phi(m)\right) \\ &= 0. \end{aligned}$$

Thus, we obtain that $F_H(\psi, \phi) = \psi \otimes \phi$ is a homomorphism of the weak representations. Moreover, by straightforward computations, we deduce that F_H preserves identity morphisms and composite morphisms. Therefore, F_H is a bifunctor.

Let $(V_1; \theta_1)$ and $(V_2; \theta_2)$ be representations of the \mathbb{K} -Lie algebra \mathfrak{g} and $(M; \rho)$ be a weak representation of the Lie–Rinehart algebra $(A, \mathcal{L}, [\cdot, \cdot]_{\mathcal{L}}, \alpha)$. For all $b \in A$, $v_1 \in V_1$, $v_2 \in V_2$ and $m \in M$, we have

$$\begin{aligned} &a_{(V_1; \theta_1), (V_2; \theta_2), (M; \rho)}\left(b((v_1 \otimes v_2) \otimes m)\right) - ba_{(V_1; \theta_1), (V_2; \theta_2), (M; \rho)}\left((v_1 \otimes v_2) \otimes m\right) \\ &= a_{(V_1; \theta_1), (V_2; \theta_2), (M; \rho)}\left((v_1 \otimes v_2) \otimes bm\right) - b\left(v_1 \otimes (v_2 \otimes m)\right) \end{aligned}$$

$$\begin{aligned}
 &= v_1 \otimes (v_2 \otimes bm) - v_1 \otimes b(v_2 \otimes m) \\
 &= 0.
 \end{aligned}$$

For all $x \in \mathcal{L}$, $v_1 \in V_1$, $v_2 \in V_2$ and $m \in M$, we have

$$\begin{aligned}
 &a_{(V_1;\theta_1),(V_2;\theta_2),(M;\rho)} \left(\left(\left(\rho \boxplus (\theta_1 \otimes \text{Id}_{V_2} + \text{Id}_{V_1} \otimes \theta_2) \right) \right) \iota_H(x) \right) ((v_1 \otimes v_2) \otimes m) \\
 &- \left(\left(\left((\rho \boxplus \theta_2) \circ \iota_H \right) \boxplus \theta_1 \right) \iota_H(x) \right) a_{(V_1;\theta_1),(V_2;\theta_2),(M;\rho)}((v_1 \otimes v_2) \otimes m) \\
 = &a_{(V_1;\theta_1),(V_2;\theta_2),(M;\rho)} \left(\left(\left(\rho \boxplus (\theta_1 \otimes \text{Id}_{V_2} + \text{Id}_{V_1} \otimes \theta_2) \right) (x, x_i^{\mathfrak{g}} \otimes x_i^A) \right) ((v_1 \otimes v_2) \otimes m) \right) \\
 &- \left(\left(\left((\rho \boxplus \theta_2) \circ \iota_H \right) \boxplus \theta_1 \right) \iota_H(x) \right) (v_1 \otimes (v_2 \otimes m)) \\
 = &a_{(V_1;\theta_1),(V_2;\theta_2),(M;\rho)} \left((v_1 \otimes v_2) \otimes \rho(x)m + (\theta_1 \otimes \text{Id}_{V_2} + \text{Id}_{V_1} \otimes \theta_2)(x_i^{\mathfrak{g}})(v_1 \otimes v_2) \otimes x_i^A m \right) \\
 &- \left(\left(\left((\rho \boxplus \theta_2) \circ \iota_H \right) \boxplus \theta_1 \right) (x, x_i^{\mathfrak{g}} \otimes x_i^A) \right) (v_1 \otimes (v_2 \otimes m)) \\
 = &v_1 \otimes (v_2 \otimes \rho(x)m) + \theta_1(x_i^{\mathfrak{g}})v_1 \otimes (v_2 \otimes x_i^A m) + v_1 \otimes (\theta_2(x_i^{\mathfrak{g}})v_2 \otimes x_i^A m) \\
 &- \left(v_1 \otimes ((\rho \boxplus \theta_2) \iota_H(x)(v_2 \otimes m)) + \theta_1(x_i^{\mathfrak{g}})v_1 \otimes x_i^A(v_2 \otimes m) \right) \\
 = &v_1 \otimes (v_2 \otimes \rho(x)m) + \theta_1(x_i^{\mathfrak{g}})v_1 \otimes (v_2 \otimes x_i^A m) + v_1 \otimes (\theta_2(x_i^{\mathfrak{g}})v_2 \otimes x_i^A m) \\
 &- \left(v_1 \otimes (v_2 \otimes \rho(x)m + \theta_2(x_i^{\mathfrak{g}})v_2 \otimes x_i^A m) + \theta_1(x_i^{\mathfrak{g}})v_1 \otimes (v_2 \otimes x_i^A m) \right) \\
 = &0.
 \end{aligned}$$

Thus, we obtain that $a_{(V_1;\theta_1),(V_2;\theta_2),(M;\rho)}$ is a homomorphism of the weak representations. Moreover, by straightforward computations, we obtain that $a_{(V_1;\theta_1),(V_2;\theta_2),(M;\rho)}$ is a natural isomorphism and satisfies the pentagon diagram in Definition 3.24.

Let $(M; \rho)$ be a weak representation of the Lie–Rinehart algebra $(A, \mathcal{L}, [\cdot, \cdot]_{\mathcal{L}}, \alpha)$. We have

$$\begin{aligned}
 l_{(M;\rho)}(a(k \otimes m)) &= l_{(M;\rho)}(k \otimes am) = k(am) = a(km) \\
 &= al_{(M;\rho)}(k \otimes m), \quad \forall a \in A, k \in \mathbb{K}, m \in M.
 \end{aligned}$$

For all $x \in \mathcal{L}$, $k \in \mathbb{K}$ and $m \in M$, we have

$$\begin{aligned}
 &l_{(M;\rho)} \left(\left((\rho \boxplus \theta) \iota_H(x) \right) (k \otimes m) \right) - \rho(x) \left(l_{(M;\rho)}(k \otimes m) \right) \\
 = &l_{(M;\rho)} \left(\left((\rho \boxplus \theta)(x, x_i^{\mathfrak{g}} \otimes x_i^A) \right) (k \otimes m) \right) - \rho(x)(km) = l_{(M;\rho)} \left(k \otimes \rho(x)m \right) - \rho(x)(km) \\
 = &k(\rho(x)m) - \rho(x)(km) = 0.
 \end{aligned}$$

Thus, we deduce that $l_{(M;\rho)}$ is a homomorphism of weak representations. Moreover, by straightforward computations, we obtain that $l_{(M;\rho)}$ is a natural isomorphism and satisfies the triangle diagram in Definition 3.24. The proof is finished. □

Since $(A; \alpha)$ is a representation of a Lie–Rinehart algebra $(A, \mathcal{L}, [\cdot, \cdot]_{\mathcal{L}}, \alpha)$, which is known as the **natural representation**, we obtain the following result.

Corollary 3.27. *Let $(A, \mathcal{L}, [\cdot, \cdot]_{\mathcal{L}}, \alpha)$ be a Lie–Rinehart algebra, $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ a \mathbb{K} -Lie algebra and H a crossed homomorphism from \mathcal{L} to $\mathfrak{g} \otimes_{\mathbb{K}} A$. Then we have a functor*

$$F_H^A : \text{Rep}_{\mathbb{K}}(\mathfrak{g}) \rightarrow \text{WRep}_{\mathbb{K}}(\mathcal{L}),$$

$$(V; \theta) \mapsto (V \otimes_{\mathbb{K}} A; (\alpha \boxplus \theta) \circ \iota_H), \quad \forall (V; \theta) \in \text{Rep}_{\mathbb{K}}(\mathfrak{g}).$$

We can also have a very useful functor on $\text{WRep}_{\mathbb{K}}(\mathcal{L})$ as follows.

Corollary 3.28. *Let $(A, \mathcal{L}, [\cdot, \cdot]_{\mathcal{L}}, \alpha)$ be a Lie–Rinehart algebra, $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ a \mathbb{K} -Lie algebra, H a crossed homomorphism from \mathcal{L} to $\mathfrak{g} \otimes_{\mathbb{K}} A$ and $(V; \theta)$ a given representation of \mathfrak{g} . Then we have a functor*

$$F_H^{\theta} : \text{WRep}_{\mathbb{K}}(\mathcal{L}) \rightarrow \text{WRep}_{\mathbb{K}}(\mathcal{L}),$$

$$(M; \rho) \mapsto (V \otimes_{\mathbb{K}} M; (\rho \boxplus \theta) \circ \iota_H), \quad \forall (M; \rho) \in \text{WRep}_{\mathbb{K}}(\mathcal{L}).$$

A special but very interesting case of the above result is that $(V; \theta) = (\mathfrak{g}; \text{ad})$. In the next section we will show that Corollary 3.28 is a very efficient way to construct interesting modules from easy modules.

3.3. Admissible representations of Leibniz pairs

In this subsection, we introduce the notion of an admissible representation of a Leibniz pair. In the sequel, A is always a commutative associative algebra. The notion of a Leibniz pair was originally given in [11].

Definition 3.29 ([11]). A **Leibniz pair** consists of a \mathbb{K} -Lie algebra $(\mathcal{S}, [\cdot, \cdot]_{\mathcal{S}})$ and a \mathbb{K} -Lie algebra homomorphism $\beta : \mathcal{S} \rightarrow \text{Der}_{\mathbb{K}}(A)$.

We denote a Leibniz pair by $(A, \mathcal{S}, [\cdot, \cdot]_{\mathcal{S}}, \beta)$ or simply by \mathcal{S} .

Definition 3.30. An **admissible representation** of a Leibniz pair $(A, \mathcal{S}, [\cdot, \cdot]_{\mathcal{S}}, \beta)$ consists of an A -module M and a \mathbb{K} -Lie algebra homomorphism $\rho : \mathcal{S} \rightarrow \mathfrak{gl}_{\mathbb{K}}(M)$ such that

$$\rho(x)(am) = a\rho(x)m + \beta(x)(a)m, \quad \forall x \in \mathcal{S}, a \in A, m \in M. \tag{17}$$

Definition 3.31. Let $(A, \mathcal{S}, [\cdot, \cdot]_{\mathcal{S}}, \beta)$ be a Leibniz pair and $(M; \rho)$ and $(M'; \rho')$ two admissible representations of \mathcal{S} . An A -module homomorphism $\phi : M \rightarrow M'$ is said to be a **homomorphism of admissible representations** if $\phi \circ \rho(x) = \rho'(x) \circ \phi$ for all $x \in \mathcal{S}$.

Admissible representations of Leibniz pairs are like weak representations of Lie–Rinehart algebras. We use $\text{ARep}_{\mathbb{K}}(\mathcal{S})$ to denote the category of admissible representations of \mathcal{S} .

It is straightforward to obtain the following result.

Lemma 3.32. *Let $(A, \mathcal{S}, [\cdot, \cdot]_{\mathcal{S}}, \beta)$ be a Leibniz pair, M an A -module and $\rho : \mathcal{S} \rightarrow \mathfrak{gl}_{\mathbb{K}}(M)$ a \mathbb{K} -linear map. Then $(M; \rho)$ is an admissible representation of \mathcal{S} if and only if $(A \rtimes M, \mathcal{S} \oplus M, [\cdot, \cdot]_{\rho}, \hat{\beta})$ is a Leibniz pair, where $A \rtimes M$ is the commutative associative algebra given*

in Remark 3.6, $[\cdot, \cdot]_\rho$ is the semidirect product Lie bracket and $\hat{\beta} : \mathcal{S} \oplus M \rightarrow \text{Der}_{\mathbb{K}}(A \ltimes M)$ is defined by

$$\hat{\beta}(x, m)(a, n) := (\beta(x)a, \rho(x)n), \quad \forall x \in \mathcal{S}, a \in A, m, n \in M.$$

It is obvious that any Lie–Rinehart algebra is a Leibniz pair. A weak representation of a Lie–Rinehart algebra is naturally an admissible representation of the underlying Leibniz pair. We have the following category equivalence:

$$\text{WRep}_{\mathbb{K}}(\mathcal{L}) \rightleftarrows \text{ARep}_{\mathbb{K}}(\mathcal{L}),$$

where the right-hand side \mathcal{L} is considered as a Leibniz pair.

Conversely, given a Leibniz pair $(A, \mathcal{S}, [\cdot, \cdot]_{\mathcal{S}}, \beta)$, we also have an action Lie–Rinehart algebra $(A, \mathcal{S} \otimes_{\mathbb{K}} A, [\cdot, \cdot], \alpha)$, where the A -module structure and the \mathbb{K} -Lie bracket $[\cdot, \cdot]$ are given by

$$a(x \otimes b) = x \otimes ab, \quad [x \otimes a, y \otimes b] = [x, y]_{\mathcal{S}} \otimes ab + y \otimes (a\beta(x)b) - x \otimes (b\beta(y)a),$$

and an A -module homomorphism $\alpha : \mathcal{S} \otimes_{\mathbb{K}} A \rightarrow \text{Der}_{\mathbb{K}}(A)$ is defined by $\alpha(x \otimes a) := a\beta(x)$ for all $a, b \in A, x, y \in \mathcal{S}$. Furthermore, we obtain the following result.

Proposition 3.33. *Let $(M; \rho)$ be an admissible representation of a Leibniz pair $(A, \mathcal{S}, [\cdot, \cdot]_{\mathcal{S}}, \beta)$. Define $\bar{\rho} : \mathcal{S} \otimes_{\mathbb{K}} A \rightarrow \mathfrak{gl}_{\mathbb{K}}(M)$ by*

$$\bar{\rho}(x \otimes a) := a\rho(x), \quad \forall x \in \mathcal{S}, a \in A.$$

Then $(M; \bar{\rho})$ is a representation of the Lie–Rinehart algebra $(A, \mathcal{S} \otimes_{\mathbb{K}} A, [\cdot, \cdot], \alpha)$.

Proof. First, it is obvious that $\bar{\rho}$ is an A -module homomorphism from $\mathcal{S} \otimes_{\mathbb{K}} A$ to $\mathfrak{gl}_{\mathbb{K}}(M)$. Then it is straightforward to deduce that $\bar{\rho}$ is a \mathbb{K} -Lie algebra homomorphism. Finally, by (17), we deduce that

$$\bar{\rho}(x \otimes a)(bm) = a\rho(x)(bm) = a(b\rho(x)m + \beta(x)(b)m) = b\bar{\rho}(x \otimes a)m + \alpha(x \otimes a)(b)m.$$

Thus, $(M; \bar{\rho})$ is a representation of the Lie–Rinehart algebra $\mathcal{S} \otimes_{\mathbb{K}} A$. □

Remark 3.34. We have the following category equivalence if A is unital:

$$\text{ARep}_{\mathbb{K}}(\mathcal{S}) \rightleftarrows \text{Rep}(\mathcal{S} \otimes_{\mathbb{K}} A).$$

First, the construction of Proposition 3.33 can be easily enhanced to a functor. In fact, assume that $\phi : M \rightarrow M'$ is a homomorphism of admissible representations of a Leibniz pair $(A, \mathcal{S}, [\cdot, \cdot]_{\mathcal{S}}, \beta)$; then it is straightforward to deduce that

$$\phi \circ \bar{\rho}(x \otimes a) = \bar{\rho}'(x \otimes a) \circ \phi$$

for all $x \in \mathcal{S}, a \in A$. Thus, $\phi : M \rightarrow M'$ is also a homomorphism of representations of the Lie–Rinehart algebra $\mathcal{S} \otimes_{\mathbb{K}} A$. So we obtain a functor $P : \text{ARep}_{\mathbb{K}}(\mathcal{S}) \rightarrow \text{Rep}(\mathcal{S} \otimes_{\mathbb{K}} A)$, which is defined on the sets of objects and morphisms respectively by

$$\begin{aligned} P(M; \rho) &= (M; \bar{\rho}), \\ P(\phi : M \rightarrow M') &= (\phi : M \rightarrow M'). \end{aligned}$$

Conversely, let $(M; \rho)$ be a representation of the Lie–Rinehart algebra $\mathcal{S} \otimes_{\mathbb{K}} A$. Define $\tilde{\rho} : \mathcal{S} \rightarrow \mathfrak{gl}_{\mathbb{K}}(M)$ by

$$\tilde{\rho}(x) := \rho(x \otimes 1), \quad \forall x \in \mathcal{S}.$$

Similar to the above discussion, this can also be enhanced to a functor and give the equivalence between the categories $\text{ARep}_{\mathbb{K}}(\mathcal{S})$ and $\text{Rep}(\mathcal{S} \otimes_{\mathbb{K}} A)$.

Let $(A, \mathcal{S}, [\cdot, \cdot]_{\mathcal{S}}, \beta)$ be a Leibniz pair and \mathfrak{h} be a \mathbb{K} -Lie algebra. Then $(A, \mathcal{S} \oplus (\mathfrak{h} \otimes_{\mathbb{K}} A), [\cdot, \cdot], \tilde{\beta})$ is a Leibniz pair, where the \mathbb{K} -Lie algebra structure on $\mathcal{S} \oplus (\mathfrak{h} \otimes_{\mathbb{K}} A)$ is given by

$$\begin{aligned} & [(x, g \otimes a), (y, h \otimes b)] \\ &= ([x, y]_{\mathcal{S}}, h \otimes \beta(x)(b) - g \otimes \beta(y)(a) + [g, h]_{\mathfrak{h}} \otimes ab), \quad \forall x, y \in \mathcal{S}, g \otimes a, h \otimes b \in \mathfrak{h} \otimes_{\mathbb{K}} A, \end{aligned}$$

and $\tilde{\beta} : \mathcal{S} \oplus (\mathfrak{h} \otimes_{\mathbb{K}} A) \rightarrow \text{Der}_{\mathbb{K}}(A)$ is given by

$$\tilde{\beta}(x, g \otimes a) = \beta(x).$$

Denote this Leibniz pair by $\mathcal{S} \times_{\beta} (\mathfrak{h} \otimes_{\mathbb{K}} A)$.

Let $(M; \rho)$ be an admissible representation over \mathcal{S} and $(V; \theta)$ be a representation of a \mathbb{K} -Lie algebra \mathfrak{h} . Then $V \otimes_{\mathbb{K}} M$ has a natural A -module structure:

$$a(v \otimes m) = v \otimes am, \quad \forall a \in A, v \in V, m \in M.$$

We define a \mathbb{K} -linear map $\rho \boxplus \theta : \mathcal{S} \times_{\beta} (\mathfrak{h} \otimes_{\mathbb{K}} A) \rightarrow \mathfrak{gl}_{\mathbb{K}}(V \otimes_{\mathbb{K}} M)$ by

$$(\rho \boxplus \theta)(x, g \otimes a)(v \otimes m) := v \otimes \rho(x)m + \theta(g)v \otimes am, \quad \forall x \in \mathcal{S}, a \in A, g \in \mathfrak{h}, m \in M, v \in V.$$

Then it is straightforward to verify the following result.

Lemma 3.35. *With the above notations, $(V \otimes_{\mathbb{K}} M; \rho \boxplus \theta)$ is an admissible representation of the Leibniz pair $\mathcal{S} \times_{\beta} (\mathfrak{h} \otimes_{\mathbb{K}} A)$.*

Let H be a crossed homomorphism from the \mathbb{K} -Lie algebra \mathcal{S} to $\mathfrak{h} \otimes_{\mathbb{K}} A$. Then we have the Lie algebra homomorphism

$$\begin{aligned} \iota_H : \mathcal{S} &\rightarrow \mathcal{S} \times_{\beta} (\mathfrak{h} \otimes_{\mathbb{K}} A) \\ \iota_H(x) &= (x, Hx), \quad \forall x \in \mathcal{S}. \end{aligned}$$

Similar to Theorem 3.26, we have the following result.

Theorem 3.36. *Any crossed homomorphism $H : \mathcal{S} \rightarrow \mathfrak{h} \otimes_{\mathbb{K}} A$ induces a left module category structure of the category of admissible representations $\text{ARep}_{\mathbb{K}}(\mathcal{S})$ over the monoidal category $\text{Rep}_{\mathbb{K}}(\mathfrak{h})$,*

$$\begin{aligned} \mathcal{F}_H : \text{Rep}_{\mathbb{K}}(\mathfrak{h}) \times \text{ARep}_{\mathbb{K}}(\mathcal{S}) &\rightarrow \text{ARep}_{\mathbb{K}}(\mathcal{S}) \\ \mathcal{F}_H\left((V; \theta), (M; \rho)\right) &= (V \otimes_{\mathbb{K}} M; (\rho \boxplus \theta) \circ \iota_H). \end{aligned}$$

Proof. We verify that the representation $(V \otimes_{\mathbb{K}} M; (\rho \boxplus \theta) \circ \iota_H)$ satisfies (17). For any $x \in \mathcal{S}, a \in A, v \in V, m \in M$. Suppose $H(x) = Hx = \sum_i x_i^{\mathfrak{h}} \otimes x_i^A$ or $Hx = x_i^{\mathfrak{h}} \otimes x_i^A$ for simplicity. Then

$$\begin{aligned} ((\rho \boxplus \theta) \circ \iota_H)(x)(a(v \otimes m)) &= (\rho \boxplus \theta)(x, x_i^{\mathfrak{h}} \otimes x_i^A)(v \otimes am) \\ &= v \otimes \rho(x)(am) + \theta(x_i^{\mathfrak{h}})v \otimes x_i^A am \\ &= a \left(v \otimes \rho(x)(m) + \theta(x_i^{\mathfrak{h}})v \otimes x_i^A m \right) + \beta(x)(a)(v \otimes m) \\ &= a((\rho \boxplus \theta) \circ \iota_H)(x)(v \otimes m) + \beta(x)(a)(v \otimes m). \end{aligned}$$

The proof is similar to Theorem 3.26, so the details will be omitted. □

Since $(A; \beta)$ is an admissible representation of a Leibniz pair $(A, \mathcal{S}, [\cdot, \cdot]_{\mathcal{S}}, \beta)$, we obtain the following result.

Corollary 3.37. *Let $(A, \mathcal{S}, [\cdot, \cdot]_{\mathcal{S}}, \beta)$ be a Leibniz pair, $(\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}})$ a \mathbb{K} -Lie algebra and H a crossed homomorphism from \mathcal{S} to $\mathfrak{h} \otimes_{\mathbb{K}} A$. Then we have a functor*

$$\begin{aligned} \mathcal{F}_H^A : \text{Rep}_{\mathbb{K}}(\mathfrak{h}) &\rightarrow \text{ARep}_{\mathbb{K}}(\mathcal{S}), \\ (V; \theta) &\mapsto (V \otimes_{\mathbb{K}} A; (\beta \boxplus \theta) \circ \iota_H), \quad \forall (V; \theta) \in \text{Rep}_{\mathbb{K}}(\mathfrak{h}). \end{aligned}$$

We can also have a very useful functor on $\text{WRep}_{\mathbb{K}}(\mathcal{L})$ as follows.

Corollary 3.38. *Let $(A, \mathcal{S}, [\cdot, \cdot]_{\mathcal{S}}, \beta)$ be a Leibniz pair, $(\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}})$ a \mathbb{K} -Lie algebra and H a crossed homomorphism from \mathcal{S} to $\mathfrak{h} \otimes_{\mathbb{K}} A$ and $(V; \theta)$ a given representation of \mathfrak{h} . Then we have a functor*

$$\begin{aligned} \mathcal{F}_H^{\theta} : \text{ARep}_{\mathbb{K}}(\mathcal{S}) &\rightarrow \text{ARep}_{\mathbb{K}}(\mathcal{S}), \\ (M; \rho) &\mapsto (V \otimes_{\mathbb{K}} M; (\rho \boxplus \theta) \circ \iota_H), \quad \forall (M; \rho) \in \text{ARep}_{\mathbb{K}}(\mathcal{S}). \end{aligned}$$

A special but very interesting case of the above result is that $(V; \theta) = (\mathfrak{h}; \text{ad})$.

According to Corollaries 3.27 and 3.37, the bifunctors F_H in Theorem 3.26 and \mathcal{F}_H given in Theorem 3.36 are the actions of monoidal categories.

4. Representations of Cartan-type Lie algebras

From the definition of a crossed homomorphism we see that it is generally hard to find nontrivial crossed homomorphisms. Next, we will show some examples of crossed homomorphisms and their tremendous power in obtaining new irreducible modules via results in the previous section.

4.1. Shen–Larsson functors of Witt type

For $n \geq 1$, recall the Witt algebra $\mathcal{W}_n = \text{Der}(A_n)$ over the Laurent polynomial algebra $A_n = \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, which can be interpreted as the Lie algebra of (complex-valued) polynomial vector fields on an n -dimensional torus. Let $\partial_i = \frac{\partial}{\partial x_i}$ be the partial derivation

with respect to the variable x_i for $i = 1, 2, \dots, n$, and denote $d_i = x_i \partial_i$ and $x^r = x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}$ for $r = (r_1, r_2, \dots, r_n)^T \in \mathbb{Z}^n$. Then

$$\mathcal{W}_n = \text{span}\{x^r d_i \mid r \in \mathbb{Z}^n, 1 \leq i \leq n\}$$

with the Lie bracket

$$[x^r d_i, x^s d_j]_{\mathcal{W}_n} = s_i x^{r+s} d_j - r_j x^{r+s} d_i, \quad \forall 1 \leq i, j \leq n, r, s \in \mathbb{Z}^n.$$

Obviously, $(A_n, \mathcal{W}_n, [\cdot, \cdot]_{\mathcal{W}_n}, \text{Id})$ is a Lie–Rinehart algebra. Certainly, (A_n, Id) is the natural representation of the Lie–Rinehart algebra $(A_n, \mathcal{W}_n, [\cdot, \cdot]_{\mathcal{W}_n}, \text{Id})$. Let $\mathfrak{g} = \mathfrak{gl}_n$ be the Lie algebra of all $n \times n$ complex matrices. Then $\mathcal{G} = \mathfrak{gl}_n \otimes A_n$ is a Lie A_n -algebra. For $1 \leq i, j \leq n$, we use E_{ij} to denote the $n \times n$ matrix with 1 at the (i, j) entry and zeros elsewhere.

Lemma 4.1. *The linear map $H : \mathcal{W}_n \rightarrow \mathfrak{gl}_n \otimes A_n$ defined by*

$$H(x^r d_j) = \sum_{i=1}^n r_i E_{ij} \otimes x^r, \quad \forall r \in \mathbb{Z}^n, 1 \leq j \leq n$$

is a crossed homomorphism from \mathcal{W}_n to $\mathfrak{gl}_n \otimes A_n$.

Proof. This follows from (2.5) in [16] (or (2.3) and Lemma 2.1 in [25]) and Theorem 2.7. □

By Lemma 4.1 and Corollary 3.27, we obtain the following result.

Corollary 4.2. *We have a functor $F_H^{A_n} : \text{Rep}_{\mathbb{C}}(\mathfrak{gl}_n) \rightarrow \text{WRep}_{\mathbb{C}}(\mathcal{W}_n)$ given by*

$$F_H^{A_n}(V; \theta) = (V \otimes_{\mathbb{C}} A_n; (\text{Id} \boxplus \theta) \circ \iota_H), \quad \forall (V; \theta) \in \text{Rep}_{\mathbb{C}}(\mathfrak{gl}_n).$$

Let $\mathcal{A}_n = \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}, \partial_1, \dots, \partial_n]$ be the Weyl algebra, which is the universal enveloping algebra of the Lie–Rinehart algebra $(A_n, \mathcal{W}_n, [\cdot, \cdot]_{\mathcal{W}_n}, \text{Id})$. Let $(P; \rho)$ be a representation of \mathcal{A}_n . It is obvious that $(P; \rho|_{\mathcal{W}_n})$ is a \mathcal{W}_n -module. By Lemma 4.1 and Corollary 3.28, we obtain the following result.

Corollary 4.3. *We have a functor $F_H^P : \text{Rep}_{\mathbb{C}}(\mathfrak{gl}_n) \rightarrow \text{WRep}_{\mathbb{C}}(\mathcal{W}_n)$ given by*

$$F_H^P(V; \theta) = (V \otimes_{\mathbb{C}} P; (\rho|_{\mathcal{W}_n} \boxplus \theta) \circ \iota_H), \quad \forall (V; \theta) \in \text{Rep}_{\mathbb{C}}(\mathfrak{gl}_n).$$

Remark 4.4. The functor F_H^P , introduced by Liu, Lu and Zhao in [25] is a generalisation of the Shen–Larsson functor of type $(\mathcal{W}_n, \mathfrak{gl}_n)$, which gives a class of new simple modules over \mathcal{W}_n . This class of simple \mathcal{W}_n -modules was used in the classification of simple \mathcal{W}_n -modules that are finitely generated as modules over its Cartan subalgebra (see [16]).

Next we take $\mathfrak{g} = \mathbb{C}$, the 1-dimensional trivial Lie algebra. Let $p = (p_1, p_2, \dots, p_n) \in \mathbb{C}[t_1^{\pm 1}] \times \mathbb{C}[t_2^{\pm 1}] \times \cdots \times \mathbb{C}[t_n^{\pm 1}]$, $q \in \mathbb{C}$. Similar to the automorphism σ_b in Section 2 of [45], we can easily see that the linear map

$$\begin{aligned} \mathcal{W}_n &\rightarrow \mathcal{W}_n \rtimes_{\text{Id}} A_n, \\ x^r d_i &\mapsto x^r (d_i + p_i) + q r_i x^r, \end{aligned}$$

is a Lie algebra homomorphism. By Theorem 2.7, we see that the linear map

$$H_{p,q} : \mathcal{W}_n \rightarrow \mathfrak{g} \otimes A_n \cong A_n, \\ x^r d_i \mapsto (p_i + qr_i)x^r, \quad \forall r \in \mathbb{Z}^n, 1 \leq i \leq n,$$

is a crossed homomorphism from \mathcal{W}_n to A_n . In fact, $H_{p,q} \in \text{Der}_{\mathbb{C}}(\mathcal{W}_n, A_n)$. By Lemma 4.1 and Corollary 3.28, we obtain the following result.

Corollary 4.5. *We have a functor $F_{p,q} : \text{WRep}_{\mathbb{C}}(\mathcal{W}_n) \rightarrow \text{WRep}_{\mathbb{C}}(\mathcal{W}_n)$ defined by*

$$F_{p,q}(M; \rho) = (M; \rho \circ \iota_{H_{p,q}}), \quad \forall (M; \rho) \in \text{WRep}_{\mathbb{C}}(\mathcal{W}_n).$$

Remark 4.6. By forgetting the A_n -module structure, the corresponding functor $F_{p,q}$ is just the twisting functor in the \mathcal{W}_n -module category introduced in [28, 29, 45], where a lot of new simple modules were obtained over the Virasoro algebra and \mathcal{W}_n .

4.2. Shen–Larsson functors of divergence zero type

In this section we assume that $n \geq 2$. Let us recall the divergence map $\text{div} : \mathcal{W}_n \rightarrow A_n$ with $x^r d_i \mapsto r_i x^r$, for all $r \in \mathbb{Z}^n$. It is well-known that

$$\mathcal{S}_n = \{w \in \mathcal{W}_n \mid \text{div}(w) = 0\}$$

is a Lie subalgebra of \mathcal{W}_n , called the Lie algebra of divergence zero vector fields on an n -dimensional torus. Let $d_{ij}(r) = r_j x^r d_i - r_i x^r d_j$. Then

$$\mathcal{S}_n = \text{span}_{\mathbb{C}}\{d_i, d_{ij}(r) \mid i, j = 1, 2, \dots, n\}$$

with the Lie bracket

$$[d_k, d_{ij}(r)]_{\mathcal{W}_n} = r_k d_{ij}(r), \\ [d_{ij}(r), d_{pq}(s)]_{\mathcal{W}_n} = r_j s_p d_{iq}(r+s) - r_j s_q d_{ip}(r+s) - r_i s_p d_{jq}(r+s) + r_i s_q d_{jp}(r+s),$$

for $r, s \in \mathbb{Z}^N, i, j, p, q = 1, \dots, n$.

Note that \mathcal{S}_n is not a Lie–Rinehart subalgebra since \mathcal{S}_n is not an A_n -module. It is straightforward to see that $(A_n, \mathcal{S}_n, [\cdot, \cdot]_{\mathcal{W}_n}, \text{Id})$ is a Leibniz pair.

Recall that \mathfrak{sl}_n is the Lie subalgebra of \mathfrak{gl}_n consisting of all traceless complex matrices. The restriction $H|_{\mathcal{S}_n}$ of the crossed homomorphism H in Lemma 4.1 is a crossed homomorphism from \mathcal{S}_n to $\mathfrak{sl}_n \otimes A_n$. By Corollary 3.37, we obtain the following result.

Corollary 4.7. *We have a functor $\mathcal{F}_H^{A_n} : \text{Rep}_{\mathbb{C}}(\mathfrak{sl}_n) \rightarrow \text{ARep}_{\mathbb{C}}(\mathcal{S}_n)$ defined by*

$$\mathcal{F}_H^{A_n}(V; \theta) = (V \otimes_{\mathbb{C}} A_n; (\text{Id} \boxplus \theta) \circ \iota_H), \quad \forall (V; \theta) \in \text{Rep}_{\mathbb{C}}(\mathfrak{sl}_n).$$

Let $(P; \rho)$ be a representation of \mathcal{A}_n . It follows that $(P; \rho|_{\mathcal{S}_n})$ is an admissible representation of \mathcal{S}_n since $\mathcal{S}_n \subset \mathcal{A}_n$. By Theorem 3.36, we obtain the following result.

Corollary 4.8. *We have a functor $\mathcal{F}_H^P : \text{Rep}_{\mathbb{C}}(\mathfrak{sl}_n) \rightarrow \text{ARep}_{\mathbb{C}}(\mathcal{S}_n)$ defined by*

$$\mathcal{F}_H^P(V; \theta) = (V \otimes_{\mathbb{C}} P; (\rho|_{\mathcal{S}_n} \boxplus \theta) \circ \iota_H), \quad \forall (V; \theta) \in \text{Rep}_{\mathbb{C}}(\mathfrak{sl}_n).$$

Remark 4.9. The functor \mathcal{F}_H^P was introduced in [8] and is a generalisation of the Shen–Larsson functor of type $(\mathcal{S}_n, \mathfrak{sl}_n)$, to give a class of new simple modules over \mathcal{S}_n .

4.3. Shen–Larsson functors of Hamiltonian type

For $r \in \mathbb{Z}^{2n}$, let

$$h(r) = \sum_{i=1}^n (r_{n+i}x^r \partial_i - r_i x^r \partial_{n+i}) \in \mathcal{W}_{2n}.$$

It is well-known that $\mathcal{H}_n = \text{Span}_{\mathbb{C}}\{h(r) \mid r \in \mathbb{Z}^{2n}\}$ is a Lie subalgebra of \mathcal{W}_{2n} , with

$$[h(r), h(s)]_{\mathcal{W}_{2n}} = \sum_{i=1}^n (r_{n+i}s_i - s_{n+i}r_i)h(r+s), \quad \forall r, s \in \mathbb{Z}^{2n}.$$

This Lie algebra \mathcal{H}_n is called the Lie algebra of Hamiltonian vector fields on a $2n$ -dimensional torus. Note that \mathcal{H}_n is not a Lie–Rinehart algebra since \mathcal{H}_n is not an A_{2n} -module. It is straightforward to see that $(A_n, \mathcal{H}_n, [\cdot, \cdot]_{\mathcal{W}_{2n}}, \text{Id})$ is a Leibniz pair.

Let \mathfrak{sp}_{2n} be the Lie subalgebra of \mathfrak{gl}_{2n} consisting of all symplectic matrices. The restriction $H|_{\mathcal{H}_n}$ of the crossed homomorphism H in Lemma 4.1 is a linear map $\mathcal{H}_n \rightarrow \mathfrak{sp}_{2n} \otimes A_{2n}$ given by

$$H(h(r)) = \begin{pmatrix} r_1 r_{n+1} & \cdots & r_1 r_{2n} & -r_1 r_1 & \cdots & -r_1 r_n \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ r_n r_{n+1} & \cdots & r_n r_{2n} & -r_n r_1 & \cdots & -r_n r_n \\ r_{n+1} r_{n+1} & \cdots & r_{n+1} r_{2n} & -r_{n+1} r_1 & \cdots & -r_{n+1} r_n \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ r_{2n} r_{n+1} & \cdots & r_{2n} r_{2n} & -r_{2n} r_1 & \cdots & -r_{2n} r_n \end{pmatrix} \otimes x^r \in \mathfrak{sp}_{2n} \otimes A_{2n},$$

which is certainly a crossed homomorphism from \mathcal{H}_n to $\mathfrak{sp}_{2n} \otimes A_{2n}$. By Corollary 3.37, we obtain the following result.

Corollary 4.10. *We have a functor $\mathcal{F}_H^{A_{2n}} : \text{Rep}_{\mathbb{C}}(\mathfrak{sp}_{2n}) \rightarrow \text{ARep}_{\mathbb{C}}(\mathcal{H}_n)$ defined by*

$$\mathcal{F}_H^{A_{2n}}(V; \theta) = (V \otimes_{\mathbb{C}} A_{2n}; (\text{Id} \boxplus \theta) \circ \iota_H), \quad \forall (V; \theta) \in \text{Rep}_{\mathbb{C}}(\mathfrak{sp}_{2n}).$$

Remark 4.11. The functors defined in Corollaries 4.2, 4.7 and 4.10 are the well-known Shen–Larsson functors of type $(\mathcal{W}_n, \mathfrak{gl}_n)$, type $(\mathcal{S}_n, \mathfrak{sl}_n)$ and type $(\mathcal{H}_n, \mathfrak{sp}_{2n})$, respectively. The functor of type $(\mathcal{W}_n, \mathfrak{gl}_n)$ was introduced by Shen [42] (over polynomial algebras) and Larsson [24] (over Laurent polynomial algebras) independently in different settings. The functors of type $(\mathcal{S}_n, \mathfrak{sl}_n)$ and $(\mathcal{H}_n, \mathfrak{sp}_{2n})$ were introduced by Shen over polynomial algebras [42] and further studied in [5, 44] over Laurent polynomial algebras. For any simple \mathfrak{gl}_n -module V the simplicity of the \mathcal{W}_n -module $F_H^{A_n}(V; \theta)$ was determined in [9, 17, 26]. In particular, simple \mathcal{W}_n -modules of this class (with V as simple finite-dimensional \mathfrak{gl}_n -modules) are all simple Harish–Chandra \mathcal{W}_n -modules [2]. Note that there are still no results for \mathcal{H}_n similar to those in [9, 17, 26].

4.4. Actions of monoidal categories for generalised Cartan type

Let A be a commutative associative \mathbb{C} -algebra and let Δ be a nonzero \mathbb{C} -vector space of commuting \mathbb{C} -derivations of A . Let us first recall the construction of the generalised Witt algebras from [37]. The tensor product $A\Delta := A \otimes_{\mathbb{C}} \Delta$ acts on A by

$$a \otimes \partial : x \mapsto a\partial(x), \quad a, x \in A, \partial \in \Delta.$$

Since A is commutative, this gives rise to a linear transformation $\alpha : A\Delta \rightarrow \text{Der}_{\mathbb{C}}(A)$. Define a bracket $[\cdot, \cdot]_{A\Delta}$ on $A\Delta$ by

$$[a\partial, b\delta]_{A\Delta} = a\partial(b)\delta - b\delta(a)\partial, \quad \forall a, b \in A, \partial, \delta \in \Delta,$$

which gives a Lie algebra structure on $A\Delta$. Then α is clearly an action of $A\Delta$ on the commutative Lie algebra A . Assume that $\dim_{\mathbb{C}} \Delta < \infty$. Then there are $\partial_1, \dots, \partial_n \in \Delta$ such that $A\Delta$ is a free A -module with basis $\{\partial_1, \dots, \partial_n\}$ (see [49]). We denote this Lie algebra by $\mathcal{W}_n(A, \Delta)$. Note that $(A, \mathcal{W}_n(A, \Delta), [\cdot, \cdot]_{A\Delta}, \alpha)$ is a Lie–Rinehart algebra.

Now we have a generalisation of Lemma 4.1.

Lemma 4.12. *The linear map $H : \mathcal{W}_n(A, \Delta) \rightarrow \mathfrak{gl}_n \otimes A$ defined by*

$$H \left(\sum_{i=1}^n a_i \partial_i \right) = \sum_{i=1}^n \sum_{j=1}^n E_{ij} \otimes \partial_i(a_j), \quad a_i \in A$$

is a crossed homomorphism from $\mathcal{W}_n(A, \Delta)$ to $\mathfrak{gl}_n \otimes A$.

Proof. It is straightforward but tedious to verify the above formula. We omit the details. □

Similar to Corollary 4.2, by Lemma 4.12 and Theorem 3.26 we obtain the following result.

Corollary 4.13. *We have a functor $F_H^A : \text{Rep}_{\mathbb{C}}(\mathfrak{gl}_n) \rightarrow \text{WRep}_{\mathbb{C}}(\mathcal{W}_n(A, \Delta))$ defined by*

$$F_H^A(V; \theta) = (V \otimes_{\mathbb{C}} A; (\alpha \boxplus \theta) \circ \iota_H), \quad \forall (V; \theta) \in \text{Rep}_{\mathbb{C}}(\mathfrak{gl}_n).$$

Remark 4.14.

- (1) If $A = \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and $\Delta = \text{Span}_{\mathbb{C}}\{x_1 \frac{\partial}{\partial x_1}, \dots, x_n \frac{\partial}{\partial x_n}\}$, $\mathcal{W}_n(A, \Delta)$ is the standard Witt algebra \mathcal{W}_n and the corresponding F_H^A is the Shen–Larsson functor of type $(\mathcal{W}_n, \mathfrak{gl}_n)$.
- (2) If $A = \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and $\bar{\Delta} = \text{Span}_{\mathbb{C}}\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$, $\mathcal{W}_n(A, \bar{\Delta})$ is also the standard Witt algebra \mathcal{W}_n . However, the corresponding Shen–Larsson functor \bar{F}_H^A is different from the standard F_H^A except on the category of finite-dimensional \mathfrak{gl}_n -modules. This was pointed out by Liu, Lu and Zhao in [25].
- (3) If A is taken to be a polynomial algebra with finitely many variables x_i together with some x_i^{-1} and Δ to be some mixed differential operators w.r.t. x_i , the Lie algebra $\mathcal{W}_n(A, \Delta)$ was introduced by Xu [48]. The corresponding Shen–Larsson functor F_H^A was introduced and studied by Zhao [50], generalising Rao’s results in [9].
- (4) Under certain finite conditions, the functor F_H^A is the Shen–Larsson functor $\mathcal{W}_n(A, \Delta)$ introduced and studied by Skryabin in [43].

- (5) Let A be the coordinate ring of an irreducible affine variety and Δ a certain subalgebra of $\text{Der}(A)$. The corresponding Shen–Larsson functor F_H^A was introduced and studied in [3, 4] to give new simple modules over $\mathcal{W}_n(A, \Delta)$.

Now let us define the divergence map $\text{div} : \mathcal{W}_n(A, \Delta) \rightarrow A$ to be the \mathbb{C} -linear extension of

$$\text{div}(a\partial) = \partial(a), \quad \forall a \in A, \partial \in \Delta.$$

Let $\mathcal{S}_n(A, \Delta) = \{w \in A\Delta \mid \text{div}(w) = 0\}$. Then $\mathcal{S}_n(A, \Delta)$ is a Lie subalgebra of $\mathcal{W}_n(A, \Delta)$; see [1] for more details. If $A = \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and $\Delta = \text{Span}_{\mathbb{C}}\{x_1 \frac{\partial}{\partial x_1}, \dots, x_n \frac{\partial}{\partial x_n}\}$, $\mathcal{S}_n(A, \Delta)$ is the Lie algebra \mathcal{S}_n of divergence zero vector fields on an n -dimensional torus.

Note that $\mathcal{S}_n(A, \Delta)$ is not a Lie–Rinehart subalgebra since $\mathcal{S}_n(A, \Delta)$ is not an A -module. It is straightforward to see that $(A, \mathcal{S}_n(A, \Delta), [\cdot, \cdot]_{A\Delta}, \text{Id})$ is a Leibniz pair.

It is clear that $H|_{\mathcal{S}_n(A, \Delta)}$ is a crossed homomorphism from $\mathcal{S}_n(A, \Delta)$ to $\mathfrak{sl}_n \otimes A$. Similar to Corollary 4.5, by Lemma 4.12 and Corollary 3.37 we obtain the following result.

Corollary 4.15. *We have a functor $\mathcal{F}_H^A : \text{Rep}_{\mathbb{C}}(\mathfrak{sl}_n) \rightarrow \text{ARep}_{\mathbb{C}}(\mathcal{S}_n(A, \Delta))$ defined by*

$$\mathcal{F}_H^A(V; \theta) = (V \otimes_{\mathbb{C}} A; (\alpha \boxplus \theta) \circ \iota_H), \quad \forall (V; \theta) \in \text{Rep}_{\mathbb{C}}(\mathfrak{sl}_n).$$

Now let us define a map $D : A \rightarrow \mathcal{W}_{2n}(A, \Delta)$ to be the linear extension of

$$D(a) = \sum_{i=1}^n (\partial_i(a)\partial_{n+i} - \partial_{n+i}(a)\partial_i), \quad \forall a \in A.$$

Let $\mathcal{H}_n(A, \Delta) = \{D(a) \mid a \in A\}$. Then $\mathcal{H}_n(A, \Delta)$ is a Lie subalgebra of $\mathcal{W}_{2n}(A, \Delta)$, with

$$[D(a), D(b)]_{A\Delta} = D\left(\sum_{i=1}^n (\partial_i(a)\partial_{n+i}(b) - \partial_{n+i}(a)\partial_i(b))\right), \quad \forall a, b \in A.$$

If $A = \mathbb{C}[x_1^{\pm 1}, \dots, x_{2n}^{\pm 1}]$ and $\Delta = \text{Span}_{\mathbb{C}}\{x_1 \frac{\partial}{\partial x_1}, \dots, x_{2n} \frac{\partial}{\partial x_{2n}}\}$, then $\mathcal{H}_n(A, \Delta)$ is the Lie algebra of Hamiltonian vector fields on a $2n$ -dimensional torus.

Note that $\mathcal{H}_n(A, \Delta)$ is not a Lie–Rinehart algebra since $\mathcal{H}_n(A, \Delta)$ is not an A -module. It is straightforward to see that $(A, \mathcal{H}_n(A, \Delta), [\cdot, \cdot]_{A\Delta}, \text{Id})$ is a Leibniz pair.

The restriction $H|_{\mathcal{H}_n(A, \Delta)}$ of the crossed homomorphism H in Lemma 4.12 is a crossed homomorphism from $\mathcal{H}_n(A, \Delta)$ to $\mathfrak{sp}_{2n} \otimes A$. Similar to Corollary 4.10, by Lemma 4.12 and Corollary 3.37 we obtain the following result.

Corollary 4.16. *We have a functor $\mathcal{F}_H^A : \text{Rep}_{\mathbb{C}}(\mathfrak{sp}_{2n}) \rightarrow \text{ARep}_{\mathbb{C}}(\mathcal{H}_n(A, \Delta))$ defined by*

$$\mathcal{F}_H^A(V; \theta) = (V \otimes_{\mathbb{C}} A; (\alpha \boxplus \theta) \circ \iota_H), \quad \forall (V; \theta) \in \text{Rep}_{\mathbb{C}}(\mathfrak{sp}_{2n}).$$

5. Deformation and cohomologies of crossed homomorphisms

In this section, first we give the Maurer–Cartan characterisation of crossed homomorphisms of Lie algebras. In particular, we give the differential graded Lie algebra that controls deformations of crossed homomorphisms. Then we define the cohomology groups of crossed homomorphisms, which can be applied to study linear deformations of crossed homomorphisms.

5.1. The differential graded Lie algebra controlling deformations

Definition 5.1. A differential graded Lie algebra $(\mathfrak{g}, [\cdot, \cdot], d)$ is a \mathbb{Z} -graded vector space $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ together with a bilinear bracket $[\cdot, \cdot]: \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ and a linear map $d: \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying the following conditions:

- $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ and $[a, b] = -(-1)^{\bar{a}\bar{b}}[b, a]$ for every a, b homogeneous.
- Every a, b, c homogeneous satisfies the Jacobi identity

$$[a, [b, c]] = [[a, b], c] + (-1)^{\bar{a}\bar{b}}[b, [a, c]].$$

- $d(\mathfrak{g}_i) \subset \mathfrak{g}_{i+1}$, $d \circ d = 0$ and $d[a, b] = [da, b] + (-1)^{\bar{a}}[a, db]$. The map d is called the differential of \mathfrak{g} .

We have used the notation $\bar{a} = i$ if $a \in \mathfrak{g}_i$.

Definition 5.2 ([27]). Let $(\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k, [\cdot, \cdot], d)$ be a differential graded Lie algebra. A degree 1 element $\theta \in \mathfrak{g}_1$ is called a Maurer–Cartan element of \mathfrak{g} if it satisfies the following Maurer–Cartan equation:

$$d\theta + \frac{1}{2}[\theta, \theta] = 0. \tag{18}$$

Proposition 5.3 ([27]). Let $(\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k, [\cdot, \cdot], d)$ be a differential graded Lie algebra and let $\mu \in \mathfrak{g}_1$ be a Maurer–Cartan element. Then the map

$$d_\mu : \mathfrak{g} \longrightarrow \mathfrak{g}, \quad d_\mu(x) := d(x) + [\mu, x], \quad \forall x \in \mathfrak{g},$$

is a differential on the graded Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$. For any $v \in \mathfrak{g}_1$, the sum $\mu + v$ is a Maurer–Cartan element of the differential graded Lie algebra $(\mathfrak{g}, [\cdot, \cdot], d)$ if and only if v is a Maurer–Cartan element of the differential graded Lie algebra $(\mathfrak{g}, [\cdot, \cdot], d_\mu)$.

Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ and $(\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}})$ be Lie algebras and $\rho : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{h})$ be an action of \mathfrak{g} on \mathfrak{h} . Consider the graded vector space

$$\mathcal{C}^*(\mathfrak{g}, \mathfrak{h}) := \bigoplus_{k \geq 0} \text{Hom}(\wedge^k \mathfrak{g}, \mathfrak{h}).$$

Define $d : \text{Hom}(\wedge^m \mathfrak{g}, \mathfrak{h}) \rightarrow \text{Hom}(\wedge^{m+1} \mathfrak{g}, \mathfrak{h})$ by

$$\begin{aligned} (df)(x_1, \dots, x_{m+1}) &= \sum_{i=1}^{m+1} (-1)^{m+i} \rho(x_i) f(x_1, \dots, \hat{x}_i, \dots, x_{m+1}) \\ &\quad + \sum_{1 \leq i < j \leq m+1} (-1)^{m+i+j-1} f([x_i, x_j]_{\mathfrak{g}}, x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{m+1}), \end{aligned} \tag{19}$$

for all $f \in \text{Hom}(\wedge^m \mathfrak{g}, \mathfrak{h})$. Define a skew-symmetric bracket operation $[[\cdot, \cdot]] : \text{Hom}(\wedge^m \mathfrak{g}, \mathfrak{h}) \times \text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{h}) \rightarrow \text{Hom}(\wedge^{m+n} \mathfrak{g}, \mathfrak{h})$ by

$$\begin{aligned} &[[f_1, f_2]](x_1, x_2, \dots, x_{m+n}) \\ &= (-1)^{mn+1} \sum_{\sigma \in \mathbb{S}(m, n)} (-1)^\sigma [f_1(x_{\sigma(1)}, \dots, x_{\sigma(m)}), f_2(x_{\sigma(m+1)}, \dots, x_{\sigma(m+n)})]_{\mathfrak{h}} \end{aligned} \tag{20}$$

for all $f_1 \in \text{Hom}(\wedge^m \mathfrak{g}, \mathfrak{h})$ and $f_2 \in \text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{h})$. Here $\mathbb{S}_{(m,n)}$ denotes the set of all (m,n) -shuffles.

Note that for all $u, v \in \mathfrak{h}$, $\llbracket u, v \rrbracket = -\llbracket v, u \rrbracket$.

Proposition 5.4. *With the above notations, $(\mathcal{C}^*(\mathfrak{g}, \mathfrak{h}), \llbracket \cdot, \cdot \rrbracket, d)$ is a differential graded Lie algebra. Its Maurer–Cartan elements are precisely crossed homomorphisms from \mathfrak{g} to \mathfrak{h} with respect to the action ρ .*

Proof. In short, the graded Lie algebra $(\mathcal{C}^*(\mathfrak{g}, \mathfrak{h}), \llbracket \cdot, \cdot \rrbracket)$ is obtained via the derived bracket [23, 46]. In fact, the Nijenhuis–Richardson bracket $[\cdot, \cdot]_{\text{NR}}$ associated to the direct sum vector space $\mathfrak{g} \oplus V$ gives rise to a graded Lie algebra $(\oplus_{k \geq 0} \text{Hom}(\wedge^k(\mathfrak{g} \oplus \mathfrak{h}), \mathfrak{g} \oplus \mathfrak{h}), [\cdot, \cdot]_{\text{NR}})$. Obviously, $\oplus_{k \geq 0} \text{Hom}(\wedge^k \mathfrak{g}, \mathfrak{h})$ is an abelian subalgebra. We denote the Lie brackets $[\cdot, \cdot]_{\mathfrak{g}}$ and $[\cdot, \cdot]_{\mathfrak{h}}$ by $\mu_{\mathfrak{g}}$ and $\mu_{\mathfrak{h}}$, respectively. Since ρ is an action of the Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$, we deduce that $\mu_{\mathfrak{g}} + \rho$ is a semidirect product Lie algebra structure on $\mathfrak{g} \oplus \mathfrak{h}$. Thus, $\mu_{\mathfrak{g}} + \rho$ and $\mu_{\mathfrak{h}}$ are Maurer–Cartan elements of the graded Lie algebra $(\mathcal{C}^*(\mathfrak{g} \oplus \mathfrak{h}, \mathfrak{g} \oplus \mathfrak{h}), [\cdot, \cdot]_{\text{NR}})$. Define a differential $d_{\mu_{\mathfrak{h}}}$ on $(\mathcal{C}^*(\mathfrak{g} \oplus \mathfrak{h}, \mathfrak{g} \oplus \mathfrak{h}), [\cdot, \cdot]_{\text{NR}})$ via

$$d_{\mu_{\mathfrak{h}}} := [\mu_{\mathfrak{h}}, \cdot]_{\text{NR}}.$$

Further, we define the derived bracket on the graded vector space $\oplus_{k \geq 0} \text{Hom}(\wedge^k \mathfrak{g}, \mathfrak{h})$ by

$$\llbracket f_1, f_2 \rrbracket := (-1)^{m-1} [[\mu_{\mathfrak{h}}, f_1]_{\text{NR}}, f_2]_{\text{NR}}, \quad \forall f_1 \in \text{Hom}(\wedge^m \mathfrak{g}, \mathfrak{h}), f_2 \in \text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{h}),$$

which is exactly the bracket given by (20). By $[\mu_{\mathfrak{h}}, \mu_{\mathfrak{h}}]_{\text{NR}} = 0$, we deduce that $(\mathcal{C}^*(\mathfrak{g}, \mathfrak{h}), \llbracket \cdot, \cdot \rrbracket)$ is a graded Lie algebra.

Moreover, by $\text{Im} \rho \subset \text{Der}(\mathfrak{h})$, we have $[\mu_{\mathfrak{g}} + \rho, \mu_{\mathfrak{h}}]_{\text{NR}} = 0$. We define a linear map $d =: [\mu_{\mathfrak{g}} + \rho, \cdot]_{\text{NR}}$ on the graded space $\mathcal{C}^*(\mathfrak{g} \oplus \mathfrak{h}, \mathfrak{g} \oplus \mathfrak{h})$. We obtain that d is closed on the subspace $\oplus_{k \geq 0} \text{Hom}(\wedge^k \mathfrak{g}, \mathfrak{h})$ and is given by (19).

By $[\mu_{\mathfrak{g}} + \rho, \mu_{\mathfrak{g}} + \rho]_{\text{NR}} = 0$, we obtain that $d \circ d = 0$. Moreover, by $[\mu_{\mathfrak{g}} + \rho, \mu_{\mathfrak{h}}]_{\text{NR}} = 0$, we deduce that d is a derivation of $(\mathcal{C}^*(\mathfrak{g}, \mathfrak{h}), \llbracket \cdot, \cdot \rrbracket)$. Therefore, $(\mathcal{C}^*(\mathfrak{g}, \mathfrak{h}), \llbracket \cdot, \cdot \rrbracket, d)$ is a differential graded Lie algebra.

Finally, for a degree 1 element $H \in \text{Hom}(\mathfrak{g}, \mathfrak{h})$, we have

$$(dH + \frac{1}{2} \llbracket H, H \rrbracket)(x, y) = \rho(x)(Hy) - \rho(y)(Hx) - H[x, y]_{\mathfrak{g}} + [Hx, Hy]_{\mathfrak{h}}.$$

Thus, Maurer–Cartan elements are precisely crossed homomorphisms from $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ to $(\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}})$ with respect to the action ρ . The proof is finished. \square

Let $H : \mathfrak{g} \rightarrow \mathfrak{h}$ be a crossed homomorphism with respect to the action ρ . Since H is a Maurer–Cartan element of the differential graded Lie algebra $(\mathcal{C}^*(\mathfrak{g}, \mathfrak{h}), \llbracket \cdot, \cdot \rrbracket, d)$ by Proposition 5.4, it follows from Proposition 5.3 that $d_H := d + \llbracket H, \cdot \rrbracket$ is a graded derivation on the graded Lie algebra $(\mathcal{C}^*(\mathfrak{g}, \mathfrak{h}), \llbracket \cdot, \cdot \rrbracket)$ satisfying $d_H^2 = 0$. Therefore, $(\mathcal{C}^*(\mathfrak{g}, \mathfrak{h}), \llbracket \cdot, \cdot \rrbracket, d_H)$ is a differential graded Lie algebra. This differential graded Lie algebra can control deformations of crossed homomorphisms. We have obtained the following result.

Theorem 5.5. *Let $H : \mathfrak{g} \rightarrow \mathfrak{h}$ be a crossed homomorphism with respect to the action ρ . For a linear map $H' : \mathfrak{g} \rightarrow \mathfrak{h}$, $H + H'$ is still a crossed homomorphism from \mathfrak{g} to \mathfrak{h} with*

respect to the action ρ if and only if H' is a Maurer–Cartan element of the differential graded Lie algebra $(\mathcal{C}^*(\mathfrak{g}, \mathfrak{h}), [[\cdot, \cdot]], d_H)$.

5.2. Cohomologies of crossed homomorphisms

In this subsection, we define cohomologies of a crossed homomorphism, which can be used to study linear deformations in Section 5.3.

Recall that ρ_H defined by (4) is a representation of \mathfrak{g} on \mathfrak{h} . Let $d_{\rho_H} : \text{Hom}(\wedge^k \mathfrak{g}, \mathfrak{h}) \rightarrow \text{Hom}(\wedge^{k+1} \mathfrak{g}, \mathfrak{h})$ be the corresponding Chevalley–Eilenberg coboundary operator. More precisely, for all $f \in \text{Hom}(\wedge^k \mathfrak{g}, \mathfrak{h})$ and $x_1, \dots, x_{k+1} \in \mathfrak{g}$, we have

$$\begin{aligned}
 & d_{\rho_H} f(x_1, \dots, x_{k+1}) \\
 &= \sum_{i=1}^{k+1} (-1)^{i+1} \rho(x_i) f(x_1, \dots, \hat{x}_i, \dots, x_{k+1}) + \sum_{i=1}^{k+1} (-1)^{i+1} [Hx_i, f(x_1, \dots, \hat{x}_i, \dots, x_{k+1})]_{\mathfrak{h}} \\
 &+ \sum_{1 \leq i < j \leq k+1} (-1)^{i+j} f([x_i, x_j]_{\mathfrak{g}}, x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{k+1}).
 \end{aligned}
 \tag{21}$$

It is obvious that $u \in \mathfrak{h}$ is closed if and only if $\rho(x)u + [Hx, u]_{\mathfrak{h}} = 0$ for all $x \in \mathfrak{g}$, and $f \in \text{Hom}(\mathfrak{g}, \mathfrak{h})$ is closed if and only if

$$\rho(x_1)f(x_2) - \rho(x_2)f(x_1) + [Hx_1, f(x_2)]_{\mathfrak{h}} - [Hx_2, f(x_1)]_{\mathfrak{h}} - f([x_1, x_2]_{\mathfrak{g}}) = 0, \quad \forall x_1, x_2 \in \mathfrak{g}.$$

Definition 5.6. Let $H : \mathfrak{g} \rightarrow \mathfrak{h}$ be a crossed homomorphism with respect to the action ρ . Denote by $\mathcal{C}^k(\mathfrak{g}, \mathfrak{h}) = \text{Hom}(\wedge^k \mathfrak{g}, \mathfrak{h})$ and $(\mathcal{C}^*(\mathfrak{g}, \mathfrak{h}), d_{\rho_H})$ the above cochain complex. Denote the set of k -cocycles by $\mathcal{Z}^k(\mathfrak{g}, \mathfrak{h})$ and the set of k -coboundaries by $\mathcal{B}^k(\mathfrak{g}, \mathfrak{h})$. Denote by

$$\mathcal{H}^k(\mathfrak{g}, \mathfrak{h}) = \mathcal{Z}^k(\mathfrak{g}, \mathfrak{h}) / \mathcal{B}^k(\mathfrak{g}, \mathfrak{h}), \quad k \geq 0,
 \tag{22}$$

the k th cohomology group, which will be taken to be the k th cohomology group for the crossed homomorphism H .

Comparing the coboundary operators d_{ρ_H} given above and the operators $d_H = d + [[H, \cdot]]$ defined by the Maurer–Cartan element H , we have the following.

Proposition 5.7. *Let $H : \mathfrak{g} \rightarrow \mathfrak{h}$ be a crossed homomorphism. Then we have*

$$d_{\rho_H} f = (-1)^{k-1} d_H f, \quad \forall f \in \text{Hom}(\wedge^k \mathfrak{g}, \mathfrak{h}).$$

Proof. Indeed, for all $x_1, x_2, \dots, x_{k+1} \in \mathfrak{g}$ and $f \in \text{Hom}(\wedge^k \mathfrak{g}, \mathfrak{h})$, we have

$$\begin{aligned}
 & (-1)^{k-1} (d_H f)(x_1, x_2, \dots, x_{k+1}) \\
 &= (-1)^{k-1} (df + [[H, f]])(x_1, \dots, x_{k+1}) \\
 &= \sum_{i=1}^{i+1} (-1)^{i+1} \rho(x_i) f(x_1, \dots, \hat{x}_i, \dots, x_{k+1})
 \end{aligned}$$

$$\begin{aligned}
 &+ \sum_{1 \leq i < j \leq k+1} (-1)^{i+j} f([x_i, x_j]_{\mathfrak{g}}, x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{k+1}) \\
 &+ (-1)^{k-1} (-1)^{k+1} \sum_{\sigma \in \mathbb{S}_{(1, k)}} (-1)^\sigma [Hx_{\sigma(1)}, f(x_{\sigma(2)}, \dots, x_{\sigma(k+1)})]_{\mathfrak{h}} \\
 = &\sum_{i=1}^{i+1} (-1)^{i+1} \rho(x_i) f(x_1, \dots, \hat{x}_i, \dots, x_{k+1}) \\
 &+ \sum_{1 \leq i < j \leq k+1} (-1)^{i+j} f([x_i, x_j]_{\mathfrak{g}}, x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{k+1}) \\
 &+ \sum_{i=1}^{k+1} (-1)^{i-1} [Hx_i, f(x_1, \dots, \hat{x}_i, \dots, x_{k+1})]_{\mathfrak{h}} \\
 = &(d_{\rho_H} f)(x_1, x_2, \dots, x_{k+1}),
 \end{aligned}$$

which implies that $d_{\rho_H} f = (-1)^{k-1} d_H f$. □

At the end of this section, we show that certain homomorphisms between crossed homomorphisms induce homomorphisms between the corresponding cohomology groups. Let H and \tilde{H} be two crossed homomorphisms from \mathfrak{g} to \mathfrak{h} with respect to the action ρ and $(\phi_{\mathfrak{g}}, \phi_{\mathfrak{h}})$ a homomorphism from \tilde{H} to H in which $\phi_{\mathfrak{g}}$ is invertible. For all $k \geq 0$, define

$$\begin{aligned}
 \Phi : \text{Hom}(\wedge^k \mathfrak{g}, \mathfrak{h}) &\rightarrow \text{Hom}(\wedge^k \mathfrak{g}, \mathfrak{h}) \\
 f &\mapsto \phi_{\mathfrak{h}} \circ f \circ (\phi_{\mathfrak{g}}^{-1})^{\otimes k}.
 \end{aligned}$$

Theorem 5.8. *Let H and \tilde{H} be two crossed homomorphisms from \mathfrak{g} to \mathfrak{h} with respect to the action ρ of \mathfrak{g} on \mathfrak{h} and $(\phi_{\mathfrak{g}}, \phi_{\mathfrak{h}})$ be a homomorphism from \tilde{H} to H in which $\phi_{\mathfrak{g}}$ is invertible. Then the above Φ is a cochain map from the cochain complex $(\mathcal{C}^*(\mathfrak{g}, \mathfrak{h}), d_{\rho_{\tilde{H}}})$ to $(\mathcal{C}^*(\mathfrak{g}, \mathfrak{h}), d_{\rho_H})$. Consequently, Φ induces a homomorphism $\Phi_* : \tilde{\mathcal{H}}^k(\mathfrak{g}, \mathfrak{h}) \rightarrow \mathcal{H}^k(\mathfrak{g}, \mathfrak{h})$ between corresponding cohomology groups.*

Proof. By the fact that $(\phi_{\mathfrak{g}}, \phi_{\mathfrak{h}})$ is a homomorphism from \tilde{H} to H , we have

$$\begin{aligned}
 (\Phi(d_{\rho_{\tilde{H}}} f))(x_1, \dots, x_{k+1}) &= \phi_{\mathfrak{h}}(d_{\rho_{\tilde{H}}} f)(\phi_{\mathfrak{g}}^{-1}(x_1), \dots, \phi_{\mathfrak{g}}^{-1}(x_{k+1})) \\
 = \sum_{i=1}^{i+1} (-1)^{i+1} \phi_{\mathfrak{h}} \rho(\phi_{\mathfrak{g}}^{-1}(x_i)) f(\phi_{\mathfrak{g}}^{-1}(x_1), \dots, \hat{x}_i, \dots, \phi_{\mathfrak{g}}^{-1}(x_{k+1})) \\
 &+ \sum_{1 \leq i < j \leq k+1} (-1)^{i+j} \phi_{\mathfrak{h}} f([\phi_{\mathfrak{g}}^{-1}(x_i), \phi_{\mathfrak{g}}^{-1}(x_j)]_{\mathfrak{g}}, \phi_{\mathfrak{g}}^{-1}(x_1), \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, \phi_{\mathfrak{g}}^{-1}(x_{k+1})) \\
 &+ \sum_{i=1}^{k+1} (-1)^{i+1} \phi_{\mathfrak{h}} [\tilde{H}\phi_{\mathfrak{g}}^{-1}(x_i), f(\phi_{\mathfrak{g}}^{-1}(x_1), \dots, \hat{x}_i, \dots, \phi_{\mathfrak{g}}^{-1}(x_{k+1}))]_{\mathfrak{h}} \\
 = \sum_{i=1}^{i+1} (-1)^{i+1} \rho(x_i) \phi_{\mathfrak{h}} f(\phi_{\mathfrak{g}}^{-1}(x_1), \dots, \hat{x}_i, \dots, \phi_{\mathfrak{g}}^{-1}(x_{k+1}))
 \end{aligned}$$

$$\begin{aligned}
 &+ \sum_{1 \leq i < j \leq k+1} (-1)^{i+j} \phi_{\mathfrak{h}} f(\phi_{\mathfrak{g}}^{-1}[x_i, x_j]_{\mathfrak{g}}, \phi_{\mathfrak{g}}^{-1}(x_1), \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, \phi_{\mathfrak{g}}^{-1}(x_{k+1})) \\
 &+ \sum_{i=1}^{k+1} (-1)^{i+1} [H(x_i), \phi_{\mathfrak{h}} f(\phi_{\mathfrak{g}}^{-1}(x_1), \dots, \hat{x}_i, \dots, \phi_{\mathfrak{g}}^{-1}(x_{k+1}))]_{\mathfrak{h}} \\
 &= d_{\rho_H} \Phi(f)(x_1, \dots, x_{k+1}),
 \end{aligned}$$

which implies that Φ is a cochain map. □

Corollary 5.9. *Let H and \tilde{H} be two isomorphic crossed homomorphisms. Then the cohomology groups $\tilde{\mathcal{H}}^k(\mathfrak{g}, \mathfrak{h})$ and $\mathcal{H}^k(\mathfrak{g}, \mathfrak{h})$ are isomorphic for any $k \in \mathbb{Z}_+$.*

5.3. Linear deformations of crossed homomorphisms

In this subsection, we study linear deformations of crossed homomorphisms using the cohomology theory introduced in Subsection 5.2 and show that isomorphic linear deformations are identified with the same class in the second cohomology group. We give the notion of a Nijenhuis element associated to a crossed homomorphism, which gives rise to a trivial deformation.

Definition 5.10. Let $H : \mathfrak{g} \rightarrow \mathfrak{h}$ be a crossed homomorphism with respect to the action ρ and $\mathfrak{H} : \mathfrak{g} \rightarrow \mathfrak{h}$ be a linear map. If $H_t = H + t\mathfrak{H}$ is still a crossed homomorphism from \mathfrak{g} to \mathfrak{h} with respect to the action ρ for all t , we say that \mathfrak{H} generates a **(one-parameter) linear deformation** of the crossed homomorphism H .

It is direct to check that $H_t = H + t\mathfrak{H}$ is a linear deformation of a crossed homomorphism H if and only if for any $x, y \in \mathfrak{g}$,

$$\rho(x)\mathfrak{H}y - \rho(y)\mathfrak{H}x + [Hx, \mathfrak{H}y]_{\mathfrak{h}} + [\mathfrak{H}x, Hy]_{\mathfrak{h}} - \mathfrak{H}[x, y]_{\mathfrak{g}} = 0, \tag{23}$$

$$[\mathfrak{H}x, \mathfrak{H}y]_{\mathfrak{h}} = 0. \tag{24}$$

Note that Equation (23) means that \mathfrak{H} is a 1-cocycle of the crossed homomorphism H .

Definition 5.11. Let H be a crossed homomorphism from \mathfrak{g} to \mathfrak{h} with respect to the action ρ .

- (i) Two linear deformations $H_t^1 = H + t\mathfrak{H}_1$ and $H_t^2 = H + t\mathfrak{H}_2$ are said to be **equivalent** if there exists an $x \in \mathfrak{g}$ such that $(\text{Id}_{\mathfrak{g}} + t\text{ad}_x, \text{Id}_{\mathfrak{h}} + t\rho(x))$ is a homomorphism from H_t^2 to H_t^1 .
- (ii) A linear deformation $H + t\mathfrak{H}$ of a crossed homomorphism H is said to be **trivial** if there exists an $x \in \mathfrak{g}$ such that $(\text{Id}_{\mathfrak{g}} + t\text{ad}_x, \text{Id}_{\mathfrak{h}} + t\rho(x))$ is a homomorphism from H_t to H .

Let $(\text{Id}_{\mathfrak{g}} + t\text{ad}_x, \text{Id}_{\mathfrak{h}} + t\rho(x))$ be a homomorphism from H_t^2 to H_t^1 . Then $\text{Id}_{\mathfrak{g}} + t\text{ad}_x$ and $\text{Id}_{\mathfrak{h}} + t\rho(x)$ are Lie algebra endomorphisms. Thus, we have

$$\begin{aligned}
 (\text{Id}_{\mathfrak{g}} + t\text{ad}_x)[y, z]_{\mathfrak{g}} &= [(\text{Id}_{\mathfrak{g}} + t\text{ad}_x)(y), (\text{Id}_{\mathfrak{g}} + t\text{ad}_x)(z)]_{\mathfrak{g}}, \forall y, z \in \mathfrak{g}, \\
 (\text{Id}_{\mathfrak{h}} + t\rho(x))[u, v]_{\mathfrak{h}} &= [(\text{Id}_{\mathfrak{h}} + t\rho(x))(u), (\text{Id}_{\mathfrak{h}} + t\rho(x))(v)]_{\mathfrak{h}}, \forall u, v \in \mathfrak{h},
 \end{aligned}$$

which implies that x satisfies

$$[[x, y]_{\mathfrak{g}}, [x, z]_{\mathfrak{g}}]_{\mathfrak{g}} = 0, \quad \forall y, z \in \mathfrak{g}, \tag{25}$$

$$[\rho(x)u, \rho(x)v]_{\mathfrak{h}} = 0, \quad \forall u, v \in \mathfrak{h}. \tag{26}$$

Then by Equation (2), we get

$$(H + t\mathfrak{H}_1)(\text{Id}_{\mathfrak{g}} + t\text{ad}_x)(y) = (\text{Id}_{\mathfrak{h}} + t\rho(x))(H + t\mathfrak{H}_2)(y), \quad \forall y \in \mathfrak{g},$$

which implies

$$(\mathfrak{H}_2 - \mathfrak{H}_1)(y) = -\rho(y)Hx - [Hy, Hx]_{\mathfrak{h}}, \tag{27}$$

$$\mathfrak{H}_1[x, y]_{\mathfrak{g}} = \rho(x)(\mathfrak{H}_2y), \quad \forall y \in \mathfrak{g}. \tag{28}$$

Finally, Equation (3) gives

$$(\text{Id}_{\mathfrak{h}} + t\rho(x))\rho(y)(u) = \rho((\text{Id}_{\mathfrak{g}} + t\text{ad}_x)(y))(\text{Id}_{\mathfrak{h}} + t\rho(x))(u), \quad \forall y \in \mathfrak{g}, u \in \mathfrak{h},$$

which implies that x satisfies

$$\rho([x, y]_{\mathfrak{g}})\rho(x) = 0, \quad \forall y \in \mathfrak{g}. \tag{29}$$

Note that Equation (27) means that $\mathfrak{H}_2 - \mathfrak{H}_1 = d_{\rho_H}(-Hx)$. Thus, we have the following.

Theorem 5.12. *Let H be a crossed homomorphism from \mathfrak{g} to \mathfrak{h} with respect to the action ρ . If two linear deformations $H_t^1 = H + t\mathfrak{H}_1$ and $H_t^2 = H + t\mathfrak{H}_2$ are equivalent, then \mathfrak{H}_1 and \mathfrak{H}_2 are in the same cohomology class of $\mathcal{H}^1(\mathfrak{g}, \mathfrak{h}) = \mathcal{Z}^1(\mathfrak{g}, \mathfrak{h})/\mathcal{B}^1(\mathfrak{g}, \mathfrak{h})$ defined in Definition 5.6.*

Definition 5.13. Let H be a crossed homomorphism from \mathfrak{g} to \mathfrak{h} with respect to the action ρ . An element $x \in \mathfrak{g}$ is called a **Nijenhuis element** associated to H if x satisfies Equations (25), (26), (29) and the equation

$$\rho(x)\left(\rho(y)Hx + [Hy, Hx]_{\mathfrak{h}}\right) = 0, \quad \forall y \in \mathfrak{g}. \tag{30}$$

Denote by $\text{Nij}(H)$ the set of Nijenhuis elements associated to a crossed homomorphism H .

By Equations (25)–(29), it is obvious that a trivial linear deformation gives rise to a Nijenhuis element. The following result is in close analogy to the fact that the differential of a Nijenhuis operator on a Lie algebra generates a trivial linear deformation of the Lie algebra [7], justifying the notion of Nijenhuis elements.

Theorem 5.14. *Let H be a crossed homomorphism from \mathfrak{g} to \mathfrak{h} with respect to the action ρ . Then for any $x \in \text{Nij}(H)$, $H_t := H + t\mathfrak{H}$ with $\mathfrak{H} := d_{\rho_H}(-Hx)$ is a linear deformation of the crossed homomorphism H . Moreover, this deformation is trivial.*

We need the following lemma to prove this theorem.

Lemma 5.15. *Let H be a crossed homomorphism from \mathfrak{g} to \mathfrak{h} with respect to the action ρ . Let $\phi_{\mathfrak{g}} : \mathfrak{g} \rightarrow \mathfrak{g}$ and $\phi_{\mathfrak{h}} : \mathfrak{h} \rightarrow \mathfrak{h}$ be Lie algebra isomorphisms such that Equation (3)*

holds. Then $\phi_{\mathfrak{h}}^{-1} \circ H \circ \phi_{\mathfrak{g}}$ is a crossed homomorphism from \mathfrak{g} to \mathfrak{h} with respect to the action ρ .

Proof. It follows from straightforward computations. □

(The proof of Theorem 5.14.). For any Nijenhuis element $x \in \text{Nij}(H)$, we define

$$\mathfrak{H} = d_H(-Hx). \tag{31}$$

By the definition of Nijenhuis elements of H , for any t , $H_t = H + t\mathfrak{H}$ satisfies

$$\begin{aligned} H \circ (\text{Id}_{\mathfrak{g}} + t\text{ad}_x) &= (\text{Id}_{\mathfrak{h}} + t\rho(x)) \circ H_t, \\ (\text{Id}_{\mathfrak{h}} + t\rho(x)) \circ \rho(y) &= \rho((\text{Id}_{\mathfrak{g}} + t\text{ad}_x)(y)) \circ (\text{Id}_{\mathfrak{h}} + t\rho(x)), \quad \forall y \in \mathfrak{g}. \end{aligned}$$

For t sufficiently small, we see that $\text{Id}_{\mathfrak{g}} + t\text{ad}_x$ and $\text{Id}_{\mathfrak{h}} + t\rho(x)$ are Lie algebra isomorphisms. Thus, we have

$$H_t = (\text{Id}_{\mathfrak{h}} + t\rho(x))^{-1} \circ H \circ (\text{Id}_{\mathfrak{g}} + t\text{ad}_x).$$

By Lemma 5.15, we deduce that H_t is a crossed homomorphism from \mathfrak{g} to \mathfrak{h} , for t sufficiently small. Thus, \mathfrak{H} given by Equation (31) satisfies the conditions (23) and (24). Therefore, H_t is a crossed homomorphism for all t , which means that \mathfrak{H} given by Equation (31) generates a deformation. It is straightforward to see that this deformation is trivial.

It is generally not easy to find Nijenhuis elements associated to a crossed homomorphism H from a Lie algebra \mathfrak{g} to \mathfrak{h} . Next, we give examples on some special Lie algebras where the Nijenhuis elements can be explicitly determined.

Example 5.16. Let \mathfrak{g} be a 2-step nilpotent Lie algebra; that is, $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = 0$ and $H : \mathfrak{g} \rightarrow \mathfrak{g}$ a crossed homomorphism with respect to the adjoint action ad of \mathfrak{g} on \mathfrak{g} . It is easy to see that (25), (26), (29), (30) hold for any $x \in \mathfrak{g}$. Therefore, $\text{Nij}(H) = \mathfrak{g}$ for any crossed homomorphism H with respect to the adjoint action ad of \mathfrak{g} on \mathfrak{g} . For example, we can take \mathfrak{g} to be any Heisenberg algebra.

Example 5.17. Consider the unique 2-dimensional nonabelian Lie algebra on \mathbb{C}^2 . The Lie bracket is given by $[e_1, e_2] = e_1$ for a given basis $\{e_1, e_2\}$. For a matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, define

$$He_1 = a_{11}e_1 + a_{21}e_2, \quad He_2 = a_{12}e_1 + a_{22}e_2.$$

H is a crossed homomorphism from \mathbb{C}^2 to \mathbb{C}^2 with respect to the adjoint action if and only if

$$H[e_1, e_2] = [He_1, e_2] + [e_1, He_2] + [He_1, He_2].$$

By a straightforward computation, we conclude that H is a crossed homomorphism if and only if $a_{21} = 0$, $(1 + a_{11})a_{22} = 0$. So we have the following two cases to consider.

(i) If $a_{22} = 0$, then we deduce that any $H = \begin{pmatrix} a_{11} & a_{12} \\ 0 & 0 \end{pmatrix}$ is a crossed homomorphism. In this case, $x = t_1e_1 + t_2e_2$ is a Nijenhuis element of H if and only if $t_2(t_1a_{11} + t_2a_{12}) = 0$. Then for any $t_1 \in \mathbb{C}, t_1e_1$ is a Nijenhuis element for the crossed homomorphism $H = \begin{pmatrix} a_{11} & a_{12} \\ 0 & 0 \end{pmatrix}$.

(ii) If $1 + a_{11} = 0$, then we deduce that any $H = \begin{pmatrix} -1 & a_{12} \\ 0 & a_{22} \end{pmatrix}$ is a crossed homomorphism. In this case, $x = t_1e_1 + t_2e_2$ is a Nijenhuis element of H if and only if $t_2(t_2a_{12} - t_1a_{22} - t_1) = 0$. In particular, $e_1 + e_2$ is a Nijenhuis element for the crossed homomorphism $H = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$.

Example 5.18. For any crossed homomorphism H from a finite-dimensional semisimple Lie algebra \mathfrak{g} over \mathbb{C} to another Lie algebra \mathfrak{h} with respect to any action ρ , we claim that $\text{Nij}(H) = 0$.

Let $x \in \mathfrak{g}$ be a fixed nonzero vector and assume that $\mathfrak{g}_0 = [x, \mathfrak{g}]$ is abelian; that is, (25) holds. We will show that this is impossible.

Denote $n = \dim \mathfrak{g}, \mathfrak{g}_x = \{y \in \mathfrak{g} : [x, y] = 0\}$. Considering the linear map $\text{ad}(x) : \mathfrak{g} \rightarrow \mathfrak{g}$, we see that $\dim \mathfrak{g}_0 + \dim \mathfrak{g}_x = n$. Let (\cdot, \cdot) be a nondegenerate invariant bilinear form on \mathfrak{g} . It is easy to see that $\mathfrak{g}_x = \mathfrak{g}_0^\perp$. From $0 = (0, \mathfrak{g}) = ([x, \mathfrak{g}], [x, \mathfrak{g}]) = ([x, \mathfrak{g}], [[x, \mathfrak{g}], \mathfrak{g}])$ we have

$$[[x, \mathfrak{g}], \mathfrak{g}] \subset \mathfrak{g}_0^\perp = \mathfrak{g}_x.$$

We deduce that $0 = [x, [[x, \mathfrak{g}], \mathfrak{g}]] = [[x, [x, \mathfrak{g}]], \mathfrak{g}]$. Since \mathfrak{g} is semisimple, we see that $[x, [x, \mathfrak{g}]] = 0$. Thus, x is nilpotent. From Jacobson–Morozov theorem, there are elements $f, h \in \mathfrak{g}$ such that

$$[h, x] = 2x, \quad [h, f] = -2f, \quad [x, f] = h.$$

We see that $[[x, h], [x, f]] = 4x \neq 0$. So \mathfrak{g}_0 is noncommutative, which is a contradiction. Therefore, $\text{Nij}(H) = 0$.

6. Conclusion

We introduce the notions of weak representations of Lie–Rinehart algebras and admissible representations of Leibniz pairs. By using crossed homomorphisms between Lie algebras, we construct two actions of the monoidal category of representations of Lie algebras on the category of weak representations of Lie–Rinehart algebras and the category of admissible representations of Leibniz pairs, respectively. In particular, the corresponding bifunctors, called the actions of monoidal categories, unify and generalise various constructions of modules over certain Cartan-type Lie algebras. New representations of some Lie algebras are also constructed using the actions of monoidal categories. To better understand crossed homomorphisms and the actions of monoidal categories, we also give a systematic study of deformations and cohomologies of crossed homomorphisms.

There are some natural questions worthy of consideration in the future:

- (i) Whether the bifunctors F_H and \mathcal{F}_H preserve certain properties of representations. For example, when $F_H(V, M)$ and $\mathcal{F}_H(V, M)$ are simple if both V and M are simple.
- (ii) For two crossed homomorphisms H and H' , under what conditions are the bifunctors F_H and $F_{H'}$ naturally isomorphic?
- (iii) How to classify simple objects in the categories $\text{WRep}_{\mathbb{K}}(\mathcal{L})$ and $\text{ARep}_{\mathbb{K}}(\mathcal{S})$ under certain conditions.

Acknowledgements The authors thank L. Guo and S. Ng for help discussions on the first version of the article. The authors also thank the anonymous referees for helpful comments and suggestions. This research is supported by NSFC (11871190, 11971315, 11922110, 12001228) and NSERC (311907-2020).

Competing Interests. None.

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