ON SUCCESSIVE APPROXIMATIONS FOR NONEXPANSIVE MAPPINGS IN BANACH SPACES†

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Let X be a Banach space and K a convex subset of X. A mapping T of K into K is called a *nonexpansive mapping* if $\|T(x)-T(y)\| \leqq \|x-y\|$ for all $x, y \in K$.

In general, it is not the case for nonexpansive mappings T that the sequences of Picard iterates $\{T^n(x)\}$ converge to fixed points of T, and thus when such fixed points exist other approximation techniques are needed. One such technique is to form the mapping

$$S_\lambda = \lambda I + (1-\lambda)T \qquad (0<\lambda<1),$$

and then show that under certain circumstances the Picard iterates of S_λ converge to a fixed point of T. The first such result was obtained by Krasnoselskii [7], who proved that if K is a closed convex subset of a uniformly convex Banach space and if T is a nonexpansive mapping of K into a compact subset of K, then for any $x \in K$ the sequence of iterates $\{S_\lambda^n(x)\}$, for $\lambda = 1/2$, converges to a fixed point of T. It was noted by Schaefer [8] that this theorem holds for arbitrary $\lambda \in (0, 1)$ and subsequently Edelstein [4] proved the corresponding result in a strictly convex Banach space. Even more recently, Browder and Petryshyn have obtained Krasnoselskii's theorem as a corollary of their results in [3].

Our purpose in this note is to observe that mappings more general than those of type S_λ yield similar convergence theorems.

THEOREM 1. *Let K be a convex subset of a Banach space and T a nonexpansive mapping of K into itself. Define the mapping $S: K \to K$ by*

$$S = \alpha_0 I + \alpha_1 T + \alpha_2 T^2 + \ldots + \alpha_k T^k,$$

where $\alpha_i \geqq 0$, $\alpha_1 > 0$, and $\sum_{i=0}^{k} \alpha_i = 1$. Then $S(x) = x$ if and only if $T(x) = x$.

Proof. Suppose $S(x) = x$. Then

$$x = \sum_{i=1}^{k} \beta_i T^i(x),$$

where $\beta_i = \alpha_i/(1-\alpha_0)$. Thus $x \in \operatorname{conv}\{T(x), T^2(x), \ldots, T^k(x)\}$. Let

$$\delta = \sup\{\|u-v\| : u, v \in \{x, T(x), T^2(x), \ldots, T^k(x)\}\}.$$

Because T is nonexpansive, for some integer $p \geqq 1$,

$$\|x - T^p(x)\| = \delta. \qquad (*)$$

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Assume $\delta > 0$, and let p be the smallest positive integer such that (*) holds. Since $\alpha_1 > 0$,

$$x = \beta_1 T(x) + (1-\beta_1)z,$$

where $z \in \operatorname{conv}\{T^2(x), T^3(x), \ldots, T^k(x)\}$ and $0 < \beta_1 \leqq 1$; thus

$$\begin{aligned}\delta = \|x - T^p(x)\| &= \|\beta_1 T(x) + (1-\beta_1)z - T^p(x)\| \\ &\leqq \beta_1 \|T(x) - T^p(x)\| + (1-\beta_1)\|z - T^p(x)\| \\ &\leqq \beta_1 \delta + (1-\beta_1)\delta = \delta.\end{aligned}$$

This implies $\|T(x) - T^p(x)\| = \delta$, yielding $\|x - T^{p-1}(x)\| \geqq \delta$. This gives a contradiction if $p > 1$. However, if $p = 1$ the preceding argument yields $\|T(x) - T(x)\| \geqq \delta > 0$, which is absurd. Thus, $\delta = 0$ and $x = T(x)$. Since the converse is obvious, the theorem is proved. (We should remark that the stipulation $\alpha_1 > 0$ in Theorem 1 is necessary to rule out the possibility that a fixed point of S is merely a point at which T is periodic.)

Next we prove that in uniformly convex spaces the mapping S is *asymptotically regular*; that is,

$$\lim_{n\to\infty} \|S^{n+1}(x) - S^n(x)\| = 0 \qquad (x \in K).$$

This result is patterned after Theorem 5 in Browder and Petryshyn **[3]**.

THEOREM 2. *Let X be uniformly convex and let T and S be as defined in Theorem* 1. *If T has at least one fixed point then the mapping S is asymptotically regular.*

Proof. Let $x \in K$. Define the sequence $\{x_n\}$ by $x_n = S^n x$, $n = 1, 2, \ldots$. Suppose u is a fixed point of T in K. Then the sequence $\{\|x_n - u\|\}$ is nonincreasing (since S is nonexpansive and $S(u) = u$), and we may suppose $\lim_{n\to\infty} \|x_n - u\| = d \geqq 0$. Assume $d > 0$. (If $d = 0$ there is clearly nothing to prove.) Then (adopting the notation $T^0 = I$) we have

$$\begin{aligned}x_{n+1} - u &= S(x_n) - u \\ &= \sum_{i=0}^{k} \alpha_i T^i(x_n) - u \\ &= \alpha_0(x_n - u) + (1-\alpha_0)z_n,\end{aligned}$$

where

$$z_n = \frac{1}{1-\alpha_0} \sum_{i=1}^{k} \alpha_i (T^i(x_n) - u).$$

Since

$$\|T^i(x_n) - u\| = \|T^i(x_n) - T^i(u)\| \leqq \|x_n - u\|$$

and $\sum_{i=0}^{k} \alpha_i = 1$ it follows that $\limsup_{n\to\infty} \|z_n\| \leqq d$. Also $\lim_{n\to\infty} \|x_n - u\| = d$, $\lim_{n\to\infty} \|x_{n+1} - u\| = d$.

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Because X is uniformly convex it must be the case that

$$\lim_{n\to\infty} \|x_n - u - z_n\| = 0.$$

However, $x_{n+1} - x_n = (1-\alpha_0)(x_n - u - z_n)$ and so $\lim_{n\to\infty} (x_{n+1} - x_n) = 0$, completing the proof.

The above results and Theorem 6 of [3] yield the following corollary.

COROLLARY. *Let X be a uniformly convex Banach space and T a nonexpansive compact mapping of X into X (i.e., T maps bounded subsets of X into precompact subsets of X) which has at least one fixed point. Then if the mapping S is defined as in Theorem* 1, *for each $x_0 \in X$ the sequence $\{S^n(x_0)\}$ converges to a fixed point of T.*

Proof. Since S is asymptotically regular and has the same fixed points as T, the conclusion is a direct consequence of Theorem 6 of Browder–Petryshyn [3] if it is the case that $I-S$ maps bounded closed subsets of X into closed subsets of X. Let H be a bounded closed subset of X and suppose $\lim_{n\to\infty}(h_n - Sh_n) = z$, $h_n \in H$. We need to show that $z \in (I-S)[H]$. Since T is a compact mapping, some subsequence $\{T(h_{n_j})\}$ of $\{T(h_n)\}$ converges; say $T(h_{n_j}) \to v$ as $j\to\infty$. Fix i between 1 and k. Continuity of T implies $T^i(h_{n_j}) \to T^{i-1}(v)$ as $j\to\infty$. Thus by repeatedly choosing subsequences, we may obtain a subsequence $\{\bar{h}_n\}$ of $\{h_n\}$ which has the property:

$$\lim_{n\to\infty} T^i(\bar{h}_n) = w_i \in X \qquad (i = 1, \ldots, k).$$

Now

$$(I-S)(\bar{h}_n) = \bar{h}_n - \sum_{i=0}^{k} \alpha_i T^i(\bar{h}_n)$$

$$= (1-\alpha_0)\bar{h}_n - \sum_{i=1}^{k} \alpha_i T^i(\bar{h}_n).$$

Since $\bar{h}_n - S(\bar{h}_n) \to z$ as $n\to\infty$ it follows that

$$\lim_{n\to\infty} (1-\alpha_0)\bar{h}_n = z + \sum_{i=1}^{k} \alpha_i w_i.$$

This implies that $\{\bar{h}_n\}$ converges, say to $h \in H$ (since H is closed). Hence $h - Sh = z$, which completes the proof.

We conclude by giving an analogue of Theorem 7 of Browder [2].

THEOREM 3. *Let X be a uniformly convex Banach space, K a closed bounded convex subset of X, and T a nonexpansive mapping of K into K. Let*

$$S = \sum_{i=0}^{k} \alpha_i T^i$$

where $\alpha_i \geqq 0$, $\alpha_1 > 0$, and $\sum_{i=0}^{k} \alpha_i = 1$. Suppose T has at most one fixed point y in K. Then for each x_0 in K the sequence $\{S^n(x_0)\}$ converges weakly to y in K.

Proof. Since S is nonexpansive, Theorem 3 of [2] implies that $I-S$ is demiclosed. This means that if $\{u_j\}$ converges weakly to u_0 in K and $(I-S)(u_j)$ converges strongly to w, then $(I-S)(u_0) = w$.

Now let $x_n = S^n(x_0)$, $n = 1, 2, \ldots$, and suppose $\{x_{n_i}\}$ converges weakly to u_0. By Theorem 2, S is asymptotically regular so

$$\lim_{i\to\infty} (I-S)(x_{n_i}) = \lim_{i\to\infty} (S^{n_i}(x_0) - S^{n_i+1}(x_0)) = 0$$

and thus demiclosedness of $I-S$ implies

$$(I-S)(u_0) = 0.$$

Thus u_0 is a fixed point of S. However, by Theorem 1 the fixed points of S and T coincide. Therefore u_0 is the unique fixed point of T and it follows that every weakly convergent subsequence of $\{x_n\}$ converges weakly to u_0. If $\{x_n\}$ does not converge weakly to u_0 then there exists a weak neighborhood W of u_0 and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with the property that $x_{n_k} \notin W$, $k = 1, 2, \ldots$. However, reflexivity of X and boundedness of $\{x_n\}$ imply that some subsequence of $\{x_{n_k}\}$ converges weakly, and by what we have just shown, *this* weakly convergent subsequence must converge to u_0. This implies that terms of the sequence $\{x_{n_k}\}$ must lie in W—a contradiction. Therefore, $\{S^n(x_0)\}$ converges weakly to u_0.

We might remark that the existence of at least one fixed point for T in K follows from a theorem proved independently by Browder [1], Göhde [5], and Kirk [6]. In general, this fixed point is not unique, but it will be unique for strictly contractive mappings (i.e., mappings T for which $\|T(x) - T(y)\| < \|x - y\|$ when $x \neq y$).

ADDED IN PROOF. Using Theorem 1, one may also obtain Theorems 2 and 3 as direct consequences ol their analogues in [2] and [3] by applying the original theorems to the mapping

$$R = \left(\frac{1}{1-\alpha_0}\right) \sum_{i=1}^{k} \alpha_i T^i.$$

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