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# NEARRINGS OF CONTINUOUS FUNCTIONS FROM TOPOLOGICAL SPACES INTO TOPOLOGICAL NEARRINGS

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ABSTRACT. Let $\lambda$ be a map from the additive Euclidean $n$-group $R^n$ into the space $R$ of real numbers and define a multiplication $*$ on $R^n$ by $v * w = \big(\lambda(w)\big)v$. Then $(R^n, +, *)$ is a topological nearring if and only if $\lambda$ is continuous and $\lambda(av) = a\lambda(v)$ for every $v \in R^n$ and every $a$ in the range of $\lambda$. For any such map $\lambda$ and any topological space $X$ we denote by $N_\lambda(X, R^n)$ the nearring of all continuous functions from $X$ into $(R^n, +, *)$ where the operations are pointwise. The ideals of $N_\lambda(X, R^n)$ are investigated in some detail for certain $\lambda$ and the results obtained are used to prove that two compact Hausdorff spaces $X$ and $Y$ are homeomorphic if and only if the nearrings $N_\lambda(X, R^n)$ and $N_\lambda(Y, R^n)$ are isomorphic.

1. **Introduction.** For information about nearrings, particularly for any terms not defined here, one should consult [1], [3] or [5]. The symbol $R$ will denote the additive topological group of real numbers and $R^n$ will denote the additive topological Euclidean $n$-group. In [2], we studied a class of multiplications $*$ on $R^n$ such that $(R^n, +, *)$ is a topological nearring. We need to recall a definition in order to be more specific. A map $\lambda$ from $R^n$ to $R$ was defined in [2] to be *semilinear* if it is continuous and $\lambda(av) = a\lambda(v)$ for all $v \in R^n$ and all $a \in \mathrm{Ran}(\lambda)$ where $\mathrm{Ran}(\lambda)$ denotes the range of $\lambda$. It was shown in [2] that if one chooses a map $\lambda$ from $R^n$ to $R$ and defines a binary operation $*$ on $R^n$ by $v*w = \lambda(w)v$, then $(R^n, +, *)$ is a topological nearring if and only if $\lambda$ is a semilinear map. We will refer to $*$ as the multiplication which is *induced* by the semilinear map $\lambda$ and we will also refer to $(R^n, +, *)$ as the topological nearring which is *induced* by $\lambda$. When we wish to emphasize the map $\lambda$ we will use the notation $N_\lambda(R^n)$ in place of $(R^n, +, *)$. All this permits us to associate, in a natural way, many nearrings with each topological space. We denote by $N_\lambda(X, R^n)$ the nearring of all continuous functions from $X$ into $N_\lambda(R^n)$ where the operations are pointwise. That is $(f+g)(x) = f(x)+g(x)$ and $(fg)(x) = f(x)*g(x)$ for all $f, g \in N_\lambda(X, R^n)$ where, of course $f(x) * g(x) = \lambda\big(g(x)\big)f(x)$. Throughout the paper, *multiplication in nearrings of continuous functions will always be denoted by juxtaposition of the functions*. People are well aware of the beautiful theory which has been developed for rings of continuous realvalued functions and the question is, "Might there be an analogue for nearrings of continuous functions?" It turns out that the answer appears to be yes, at least for those topological nearrings induced by certain semilinear maps. One hopes, first of all, for the algebraic structure of the nearring $N_\lambda(X, R^n)$ to

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determine the topological structure of the space $X$ (the converse is always there). That is, one would hope that $N_\lambda(X, R^n)$ and $N_\lambda(Y, R^n)$ are isomorphic if and only if $X$ and $Y$ are homeomorphic. Because then, the topological structure of $X$ would be reflected in the algebraic structure of $N_\lambda(X, R^n)$ and similarly, the algebraic structure of $N_\lambda(X, R^n)$ would be reflected in the topological structure of $X$. Our purpose here is to verify just such a theorem for compact Hausdorff spaces. In Section 2, we go over some preliminary considerations and we exhibit various examples of semilinear maps. In Section 3, we examine the ideals of $N_\lambda(X, R^n)$ in some detail and the results obtained there are used to prove the isomorphism theorems in Section 4.

2. **Preliminaries.** We will denote by $\langle x\rangle$ the constant map which maps everything into the point $x$. The domain of the function will be evident from context. We verified in [2] that for any semilinear map $\lambda$ from $R^n$ to $R$, either (1) $\lambda = \langle 0\rangle$, (2) $\lambda = \langle 1\rangle$, (3) $\lambda(0) = 0$ and $\mathrm{Ran}(\lambda) = R$ or (4) $\lambda(0) = 0$ and $\mathrm{Ran}(\lambda) = R^+$ where $R^+$ denotes the nonnegative real numbers. In this paper, we will be concerned only with nonconstant semilinear maps. For any map from a space $X$ into $R^n$, we let $Z(f) = f^{-1}(0)$ and refer to $Z(f)$ as a *zero set* of $X$. The set $X\backslash Z(f)$ is referred to as a *cozero set* of $X$ and is denoted by $CZ(f)$. For any semilinear map $\lambda$ from $R^n$ to $R$, the set

$$\{w \in R^n : \lambda(v + aw) = \lambda(v) \text{ for all } a \in R \text{ and } v \in R^n\}$$

was referred to in [2] as the *core* of $\lambda$ and was denoted by $C(\lambda)$. It was shown there in Theorem (3.7) that $C(\lambda) \subseteq Z(\lambda)$. In addition, it was shown that $C(\lambda)$ is a linear subspace of $R^n$ and that the ideals of $N_\lambda(R^n)$ coincide with the linear subspaces of $C(\lambda)$. Consequently, the nearring $N_\lambda(R^n)$ is simple if and only if $C(\lambda) = \{0\}$. Throughout this paper, we will use the symbol 0 to denote the zero of each $R^n$ as we expect no confusion to result. Finally, the $i$-th coordinate of a vector $v \in R^n$ will be denoted by $v_i$. That is $v = (v_1, v_2, \ldots, v_n)$.

The remainder of this section will consist of various examples of semilinear maps.

EXAMPLE 2.1. Let $L$ be any nonzero linear map from $R^n$ to $R$ and define a map $\lambda$ from $R^n$ to $R$ by either $\lambda(v) = L(v)$ or by $\lambda(v) = |L(v)|$ for all $v \in R^n$. In the former case $\mathrm{Ran}(\lambda) = R$ and $N_\lambda(R^n)$ is actually a ring while in the latter case, $\mathrm{Ran}(\lambda) = R^+$ and $N_\lambda(R^n)$ is a nearring which is not a ring. Certainly, $\lambda$ is a semilinear map in the first case and one easily shows that it is a semilinear map in the second case as well. It follows from the previous theorem that $C(\lambda) = Z(\lambda) = \mathrm{Ker}\, L$, the kernel of $L$, in both cases.

EXAMPLE 2.2. Recall that a polynomial $P(w_1, w_2, \ldots, w_m)$ of degree $r$ in $m$ indeterminates is homogeneous if

$$P(tw_1, tw_2, \ldots, tw_m) = t^r P(w_1, w_2, \ldots, w_m)$$

for all $w \in R^n$ and $t \in R$. Let $1 \le m \le n$, choose a homogeneous polynomial $P$ of degree $r$ such that $Z(P) = \{0\}$ and define a map $\lambda$ from $R^n$ to $R$ by

$$\lambda(w_1, w_2, \ldots, w_m, w_{m+1}, \ldots, w_n) = |P(w_1, w_2, \ldots, w_m)|^{1/r}.$$

Then $\lambda$ is semilinear and $\mathrm{Ran}(\lambda) = R^+$. Furthermore,

$$Z(\lambda) = \{w \in R^n : w_i = 0 \text{ for } 1 \leq i \leq m\}$$

since $Z(P) = \{0\}$. For any $v \in R^n$, $a \in R$ and $w \in Z(\lambda)$, we have

$$\begin{aligned}\lambda(v + aw) &= \lambda\big((v_1, v_2, \ldots, v_m, v_{m+1}, \ldots, v_n) + a(0, 0, \ldots, 0, w_{m+1}, \ldots, w_n)\big)\\ &= \lambda(v_1, v_2, \ldots, v_m, v_{m+1} + aw_{m+1}, \ldots, v_n + aw_n)\\ &= |P(v_1, v_2, \ldots, v_m)|^{1/r}\\ &= \lambda(v).\end{aligned}$$

Consequently, $C(\lambda) = Z(\lambda)$. To get specific examples, define a map $\lambda$ from $R^3$ to $R$ by $\lambda(w_1, w_2, w_3) = (w_1^2 + w_2^2)^{1/2}$ or by $\lambda(w_1, w_2, w_3) = (w_1^2 + w_1w_2 + w_2^2)^{1/2}$. In the former case, the induced multiplication $*$ is given by $v * w = (w_1^2 + w_2^2)^{1/2}v$ and in the latter case, it is given by $v * w = (w_1^2 + w_1w_2 + w_2^2)^{1/2}v$.

EXAMPLE 2.3. In each of the previous examples $C(\lambda)$ is a nonzero linear subspace of $R^n$ which means $N_\lambda(R^n)$ is not simple. This time, define

$$\lambda(v) = \sqrt{\textstyle\sum_{i=1}^n v_i^2}.$$

Then $\lambda$ is a semilinear map where $C(\lambda) = Z(\lambda) = \{0\}$ and, consequently, $N_\lambda(R^n)$ is simple. The multiplication $*$ is, of course, given by $v * w = \|w\|v$.

EXAMPLE 2.4. For this example, define a semilinear map $\lambda$ from $R^2$ into $R$ by $\lambda(v) = \sqrt{|v_1^2 - v_2^2|}$. Evidently, $Z(\lambda) = \{v \in R^2 : v_1 = \pm v_2\}$. Let us determine $C(\lambda)$. Recall that $w \in C(\lambda)$ if and only if $\lambda(v + aw) = \lambda(v)$ for all $v \in R^2$ and all $a \in R$. So $w \in C(\lambda)$ implies that either

$$(2.4.1) \qquad (v_1 + aw_1)^2 - (v_2 + aw_2)^2 = v_1^2 - v_2^2.$$

or

$$(2.4.2) \qquad (v_1 + aw_1)^2 - (v_2 + aw_2)^2 = v_2^2 - v_1^2.$$

Suppose (2.4.2) holds. Since $C(\lambda) \subseteq Z(\lambda)$, we have $w_1^2 = w_2^2$ and we take $a = 1$ in (2.4.2) and get $v_1w_1 - v_2w_2 = v_2^2 - v_1^2$ for all $v \in R^2$. Then take $v_2 = 0$ and get $v_1w_1 = -v_1^2$ for all $v_1$ which is impossible. Consequently, (2.4.2) cannot hold and thus, (2.4.1) must hold. Take $a = 1$ in (2.4.1) and get $v_1w_1 = v_2w_2$. By alternately taking $v_1 = 1, v_2 = 0$ and $v_1 = 0, v_2 = 1$ we see that $w_1 = w_2 = 0$. Therefore, $C(\lambda) = \{0\}$ and $N_\lambda(R^n)$ is simple. Moreover, $C(\lambda)$ is a proper subset of $Z(\lambda)$.

EXAMPLE 2.5. Theorem (3.7) of [2] tells us that $Z(\lambda)$ is an ideal of $N_\lambda(R^n)$ if and only if $C(\lambda) = Z(\lambda)$. In the previous example, $Z(\lambda)$ isn't even a subgroup of $N_\lambda(R^n)$. In this example, $Z(\lambda)$ is an additive subgroup but not an ideal. Define a map $\lambda$ from $R^2$ to $R$ by $\lambda(v) = \sqrt{v_1^2 + |v_1v_2|}$. One easily verifies that $\lambda$ is semilinear and it is evident that $Z(\lambda) = \{v \in R^2 : v_1 = 0\}$. We determine $C(\lambda)$. Suppose $w \in C(\lambda)$. Then $w_1 = 0$ since

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$C(\lambda) \subseteq Z(\lambda)$ and for any $v \in R^2$ and any $a \in R$, we have $\lambda(v+aw) = \lambda(v)$. Since $w_1 = 0$, this implies that $|v_1 v_2 + a v_1 w_2| = |v_1 v_2|$. Take $a = v_1 = 1$ and $v_2 = 0$ and conclude that $w_2 = 0$. Thus, $C(\lambda) = \{0\}$ and $Z(\lambda)$ is not an ideal of $N_\lambda(R^2)$ since $C(\lambda) \neq Z(\lambda)$

In the next section, we get some results about the ideals of $N_\lambda(X, R^n)$ and we use these results to verify the isomorphism theorems.

3. **The Ideals of $N_\lambda(X, R^n)$.** Throughout this paper, the term *ideal* will always mean *two-sided ideal*. Let us recall that a subset $J$ of a nearring $N$ is an ideal if it satisfies the following three conditions:

(A) $J$ is a normal subgroup of the additive group of $N$,

(B) $JN \subseteq J$,

(C) $x(y + a) - xy \in J$ for all $x, y \in N$ and $a \in J$.

For any subset $A$ of $N_\lambda(X, R^n)$ we let $Z(A) = \cap\{Z(f) : f \in A\}$. An ideal $J$ of $N_\lambda(X, R^n)$ is said to be *fixed* if $Z(J) \neq \emptyset$. An ideal which is not fixed is said to be *free*. As is customary, we will refer to an ideal of a nearring which is a proper subset of that nearring as a proper ideal and a maximal ideal will be any proper ideal which is not a proper subset of any other proper ideal. Throughout the remainder of this paper, *it will be assumed that $\lambda$ is a nonconstant semilinear map from $R^n$ to $R$*. The first result of this section shows that for such maps, the nearrings $N_\lambda(X, R^n)$ all share a familiar property.

THEOREM 3.1. *Every proper ideal of $N_\lambda(X, R^n)$ is contained in a maximal ideal.*

PROOF. It is not true, in general that $NJ \subseteq J$ for an ideal $J$ of a nearring $N$ but it is true for zero symmetric nearrings, that is, nearrings $N$ in which $0x = x0 = 0$ for all $x \in N$. To see this, simply take $y = 0$ in (C). It happens that $N_\lambda(X, R^n)$ is a zero symmetric nearring. Because $N_\lambda(X, R^n)$ is a right nearring, we immediately have $\langle 0\rangle f = \langle 0\rangle$ for all $f \in N_\lambda(X, R^n)$. But since $\lambda$ is nonconstant, Theorem (3.3) of [2] tells us that $\lambda(0) = 0$ so we also have

$$(f\langle 0\rangle)(x) = \big(f(x)\big) * 0 = \big(\lambda(0)\big)\big(f(x)\big) = 0$$

for all $x \in X$. That is, $f\langle 0\rangle = \langle 0\rangle$ for all $f \in N_\lambda(X, R^n)$. Now let $J$ be a proper ideal of $N_\lambda(X, R^n)$, let $\mathcal{J}$ denote the partially ordered family of all ideals of $N_\lambda(X, R^n)$ which contain $J$ and let $\mathcal{C}$ be any chain in $\mathcal{J}$. One verifies, in the usual manner that $\cup\mathcal{C}$ is an ideal so the only remaining task is to show that it is proper. Since $\lambda$ is nonconstant, it follows from Theorem (3.3) of [2] that either $\mathrm{Ran}(\lambda) = R$ or $\mathrm{Ran}(\lambda) = R^+$ the nonnegative real numbers. Choose any $v \in R^n$ such that $\lambda(v) = 1$ and note that for any $f \in N_\lambda(X, R^n)$ and any $x \in X$, we have $(f\langle v\rangle)(x) = f(x) * \langle v\rangle(x) = \big(\lambda(v)\big)f(x) = f(x)$ which implies $f\langle v\rangle = f$. If $\langle v\rangle \in \cup\mathcal{C}$, then $\langle v\rangle \in K$ for some $K \in \mathcal{C}$ which implies $K$ is not proper since $N_\lambda(X, R^n)K \subseteq K$. Consequently, $\langle v\rangle \notin \cup\mathcal{C}$ which means $\cup\mathcal{C}$ is a proper ideal and Zorn's Lemma assures us that $J$ is contained in a maximal ideal.

We next introduce a particular ideal of $N_\lambda(X, R^n)$ and derive some of its properties.

DEFINITION 3.2. We denote by $J_C$ the collection of all functions $f \in N_\lambda(X, R^n)$ with the property that there exists a compact subset $K_f$ of $X$ such that $f(x) = 0$ for each $x \in X \backslash K_f$ and we refer to $J_C$ as the *nucleus* of $N_\lambda(X, R^n)$.

THEOREM 3.3. *$J_C$ is an ideal of $N_\lambda(X, R^n)$ and it is a proper ideal if and only if $X$ is not compact.*

PROOF. It is immediate that $J_C$ is a subgroup of the additive subgroup of $N_\lambda(X, R^n)$ which is, of course, abelian. For any $f \in J_C$, there exists a compact subset $K_f$ of $X$ such that $f(x) = 0$ for $x \in X\backslash K_f$ and for any $g \in N_\lambda(X, R^n)$ and $x \in X\backslash K_f$, we have $fg(x) = f(x) * g(x) = \big(\lambda\big(g(x)\big)\big)\big(f(x)\big) = 0$ which means $fg \in J_C$. For any additional $h \in N_\lambda(X, R^n)$ and any $x \in X\backslash K_f$, we have

$$\begin{aligned}\big(g(h+f) - gh\big)(x) &= \big(g(x)\big) * \big(h(x) + f(x)\big) - \big(g(x)\big) * \big(h(x)\big)\\ &= \big(g(x)\big) * \big(h(x)\big) - \big(g(x)\big) * \big(h(x)\big)\\ &= 0\end{aligned}$$

which means $g(h+f) - gh \in J_C$ and we have verified that $J_C$ is an ideal of $N_\lambda(X, R^n)$. If $X$ is compact, it is immediate that $J_C = N_\lambda(X, R^n)$ and if $X$ is not compact, the only constant function which belongs to $J_C$ is $\langle 0 \rangle$.

LEMMA 3.4. *Suppose $X$ is a completely regular Hausdorff space. Then $x \in Z(J_C)$ if and only if $x$ has no compact neighborhood.*

PROOF. Suppose $x$ is contained in a compact neighborhood. Then $x \in G \subseteq K$ for some open subset $G$ of $X$ and some compact subset $K$ of $X$. Choose any point $v \neq 0$ in $R^n$. Then there exists a continuous map $f$ from $X$ to $R^n$ such that $f(x) = v$ and $f(y) = 0$ for $y \in X\backslash G$. Evidently, $f(y) = 0$ for $y \in X\backslash K$ and we conclude that $f \in J_C$. Since $x \notin Z(f)$, we have $x \notin Z(J_C)$.

Suppose, conversely, that $x \notin Z(J_C)$. Then $x \notin Z(f)$ for some $f \in J_C$. This means $f(x) \neq 0$ and there exists a compact subset $K$ of $X$ such that $f(y) = 0$ for $y \in X\backslash K$. Then there is an open subset $G$ of $X$ containing $x$ such that $f(y) \neq 0$ for all $y \in G$. Since $x \in G \subseteq K$, the proof is complete.

The next result is an immediate consequence of the previous Lemma.

THEOREM 3.5. *Let $X$ be a completely regular Hausdorff space. Then the ideal $J_C$ is fixed if and only if $X$ is not locally compact. Furthermore,*

$$Z(J_C) = \{x \in X : x \textit{ has no compact neighborhood}\}.$$

It follows from the previous Theorem, for example, that if $X$ is the space of rational numbers, then the nucleus of $N_\lambda(X, R^n)$ is the zero ideal.

THEOREM 3.6. *Let $X$ be a completely regular Hausdorff space. Then $J_C$ is a nonzero ideal if and only if $X$ contains a point with a compact neighborhood.*

PROOF. Suppose $J_C$ is a nonzero ideal. Then there exists an $f \in J_C$ and a point $x \in X$ such that $f(x) \neq 0$. Then $x \notin Z(J_C)$ and it follows from Theorem 3.5 that $x$ has a compact neighborhood. Suppose conversely, that $x$ has a compact neighborhood. Then $x \notin Z(J_C)$ by Theorem 3.5 which means $f(x) \neq 0$ for some $f \in J_C$. Evidently, $f \neq \langle 0 \rangle$.

Our next result is an immediate consequence of Theorems 3.3 and 3.5.

THEOREM 3.7. *Let $X$ be a completely regular Hausdorff space. Then $J_C$ is a proper free ideal of $N_\lambda(X, R^n)$ if and only if $X$ is locally compact but not compact.*

THEOREM 3.8. *Let $X$ be a locally compact Hausdorff space and suppose $C(\lambda) = Z(\lambda)$. Then every proper ideal of $N_\lambda(X, R^n)$ is fixed if and only if $X$ is compact and the nearring $N_\lambda(R^n)$ is simple.*

PROOF. Suppose first that every proper ideal of $N_\lambda(X, R^n)$ is fixed. It follows immediately from Theorem 3.7 that $X$ is compact. Assume $N_\lambda(R^n)$ is not simple. According to Theorem 3.7 of [2] $C(\lambda)$ is the unique maximal ideal of $N_\lambda(R^n)$ and is a linear subspace of $R^n$. So to say that $N_\lambda(R^n)$ is not simple is equivalent to saying that $C(\lambda)$ is a nonzero linear subspace of $R^n$. Let $J = \{f \in N_\lambda(X, R^n) : \mathrm{Ran}(f) \subseteq C(\lambda)\}$. $J$ is evidently a subgroup of the additive group of $N_\lambda(X, R^n)$. Suppose $h \in J$ and $f \in N_\lambda(X, R^n)$. Then $hf(x) = h(x) * f(x) = \Big(\lambda\big(f(x)\big)\Big)\big(f(x)\big) \in C(\lambda)$ for all $x \in X$ and (B) is satisfied. For $f$, $g \in N_\lambda(X, R^n)$, $h \in J$ and $x \in X$, we have

$$\begin{aligned}\big(f(g+h) - fg\big)(x) &= f(x) * \big(g(x) + h(x)\big) - f(x) * g(x)\\ &= \Big(\lambda\big(g(x) + h(x)\big) - \lambda\big(g(x)\big)\Big)f(x)\\ &= 0 \in C(\lambda)\end{aligned}$$

since $\lambda\big(g(x) + h(x)\big) = \lambda\big(g(x)\big)$. Thus (C) is satisfied and we conclude that $J$ is a proper ideal. To see that $J$ is free, simply choose any nonzero vector $v \in C(\lambda)$ and note that $\langle v\rangle \in J$ but $Z(\langle v\rangle) = \emptyset$. We have now shown that if every proper ideal of $N_\lambda(X, R^n)$ is fixed, then $X$ is compact and $N_\lambda(R^n)$ is simple.

Suppose, conversely, that $X$ is compact and $N_\lambda(R^n)$ is simple. Let $J$ be a proper ideal of $N_\lambda(X, R^n)$ and suppose $J$ is free. Then for each $x \in X$, there exists an element $f_x \in J$ such that $f_x(x) \neq 0$. Then $\{CZ(f_x) : x \in X\}$ is an open cover of $X$ and since $X$ is compact, there exists a finite subcollection $\{CZ(f_{x_i})\}_{i=i}^m$ which also covers $X$. We previously observed that $N_\lambda(X, R^n)J \subseteq J$ since $N_\lambda(X, R^n)$ is zero symmetric. Let $v = (1, 1, \ldots, 1)$ and note that $g_{x_i} = \langle v\rangle f_{x_i} f_{x_i} \in J$. For $1 \leq i \leq m$, define continuous maps $t_{x_i}$ from $X$ into $R$ by $t_{x_i}(y) = \Big(\lambda\big(f_{x_i}(y)\big)\Big)^2$ and observe that for each $y \in X$ and each $i$, we have

$$\begin{aligned}g_{x_i}(y) &= (\langle v\rangle f_{x_i} f_{x_i})(y)\\ &= v * f_{x_i}(y) * f_{x_i}(y)\\ &= \big(t_{x_i}(y), t_{x_i}(y), \ldots, t_{x_i}(y)\big).\end{aligned}$$

Define $g(y) = \sum_{i=1}^m g_{x_i}(y)$ and it is immediate that $g \in J$. Since $N_\lambda(R^n)$ is simple, $C(\lambda) = \{0\}$ which means $Z(\lambda) = \{0\}$. Consequently, $Z(f_{x_i}) = Z(g_{x_i})$ for each $i$. For each $y \in X$, we have $y \in CZ(f_{x_j})$ for some $j$. It then follows that $t_{x_j}(y) > 0$ and since $t_{x_i}(y) \geq 0$ for $1 \leq i \leq n$, it readily follows that $g(y) \neq 0$ for all $y \in X$. Then $\lambda\big(g(y)\big) \neq 0$ for all $y \in X$ since $Z(\lambda) = \{0\}$ and this enables us to define a continuous function $h$ from $X$ to $R^n$ by

$$h(y) = \left(\frac{1}{\lambda\big(g(y)\big)}, \frac{1}{\lambda\big(g(y)\big)}, \ldots, \frac{1}{\lambda\big(g(y)\big)}\right).$$

For any $y \in X$, we then have $(hg)(y) = h(y) * g(y) = v = \langle v\rangle(y)$ which means $hg = \langle v\rangle$ and therefore, $\langle v\rangle \in J$ since $g \in J$. Let any $k \in N_\lambda(X, R^n)$ be given. Since $\lambda(v) \neq 0$, we can define $t(y) = \big(1/\lambda(v)\big)k(y)$ for all $y \in Y$ and it follows that $t\langle v\rangle \in J$ since $\langle v\rangle \in J$. But $t\langle v\rangle = k$ and we have arrived at the contradiction that $J = N_\lambda(X, R^n)$. We conclude, therefore, that every proper ideal of $N_\lambda(X, R^n)$ is fixed.

DEFINITION 3.9. For $x \in X$, we let $M_x = \{f \in N_\lambda(X, R^n) : f(x) = 0\}$.

THEOREM 3.10. *Suppose $N_\lambda(R^n)$ is simple, $C(\lambda) = Z(\lambda)$, and $X$ is a compact Hausdorff space. Then the maximal ideals of the nearring $N_\lambda(X, R^n)$ are precisely the sets of the form $M_x$.*

PROOF. Choose any $x \in X$ and define a map $\varphi$ from $N_\lambda(X, R^n)$ to $N_\lambda(R^n)$ by $\varphi(f) = f(x)$. One easily verifies that $\varphi$ is a surjective homomorphism with $\mathrm{Ker}(\varphi) = M_x$ which means $M_x$ is an ideal of $N_\lambda(X, R^n)$. But it is more. Since the nearring $N_\lambda(R^n)$ is simple, $M_x$ is a maximal ideal. We now show that all maximal ideals are of this form. Let $M$ be any maximal ideal of $N_\lambda(X, R^n)$. According to Theorem 3.8, $M$ is fixed. Choose any $x \in Z(M)$. Then $M \subseteq M_x$ and since $M$ is maximal, we must have $M = M_x$.

Our next task is to get information about the ideals of $N_\lambda(X, R^n)$ without the assumption that $N_\lambda(R^n)$ is simple. To do this, we need some Lemmas. Recall that $C(\lambda)$ is the unique maximal ideal of $N_\lambda(R^n)$ by Theorem (3.7) of [2]. Then $Q_\lambda = N_\lambda(R^n)/C(\lambda)$ is a topological nearring where the topology on $Q_\lambda$ is the quotient topology and we denote by $N(X, Q_\lambda)$ the nearring of all continuous functions from $X$ into $Q_\lambda$ where the operations on $N(X, Q_\lambda)$ are pointwise.

LEMMA 3.11. *Let $X$ be any paracompact Hausdorff space. Then the map $\eta$ from $N_\lambda(X, R^n)$ to $N(X, Q_\lambda)$ defined by $\big(\eta(f)\big)(x) = \langle\langle f(x)\rangle\rangle$ is an epimorphism from $N_\lambda(X, R^n)$ onto $N(X, Q_\lambda)$ where $\langle\langle f(x)\rangle\rangle$ is the equivalence class to which $f(x)$ belongs.*

PROOF. Let $\pi$ be the projection map from $N_\lambda(R^n)$ onto $Q_\lambda$ which is defined by $\pi(x) = \langle\langle x\rangle\rangle$. Since $\big(\eta(f)\big)(x) = \langle\langle f(x)\rangle\rangle = \pi \circ f(x)$ and $\pi$ is continuous, it follows that $\eta(f) \in N(X, Q_\lambda)$ and one readily verifies that $\eta$ is a homomorphism. It only remains for us to show that $\eta$ is surjective so let $g$ be any function in $N(X, Q_\lambda)$ and let $G$ be any open subset of $R^n$. Let $H = \{v \in R^n : \langle\langle v\rangle\rangle \cap G \neq \emptyset\}$ and let $v \in H$. Then $\langle\langle v\rangle\rangle \cap G \neq \emptyset$ which means $v + w \in G$ for some $w \in C(\lambda)$ and it follows that $G - w$ is a neighborhood of $v$. Furthermore, for any $u \in G - w$, we have $u + w \in G$ which implies $\langle\langle u\rangle\rangle \cap G \neq \emptyset$ and this means $u \in H$. Consequently, $H$ is an open subset of $R^n$. Since $Q_\lambda$ has the quotient topology and $H = \pi^{-1}\{\langle\langle v\rangle\rangle \in Q_\lambda : \langle\langle v\rangle\rangle \cap G \neq \emptyset\}$, it follows that $\{\langle\langle v\rangle\rangle \in Q_\lambda : \langle\langle v\rangle\rangle \cap G \neq \emptyset\}$ is open in $Q_\lambda$. Therefore,

$$g^{-1}\{\langle\langle v\rangle\rangle \in Q_\lambda : \langle\langle v\rangle\rangle \cap G \neq \emptyset\} = \{x \in X : g(x) \cap G \neq \emptyset\}$$

is open in $X$ since $g$ is a continuous map from $X$ to $Q_\lambda$. This means that $g$ is a lower semicontinuous function when regarded as a map from $X$ into $2^{R^n}$, the space of all nonempty closed subsets of $R^n$. Furthermore, it is immediate that $\langle\langle v\rangle\rangle$ is convex for each $v \in R^n$ so that $g$ is, in fact, a lower semicontinuous map from $X$ into the subspace of $2^{R^n}$ consisting of all nonempty closed convex subsets of $R^n$. It now follows follows from Theorem (2.1) of [4] that $g$ admits a selection. That is, there exists a continuous function $f$ from $X$ to $R^n$ such that $f(x) \in g(x)$ for each $x \in X$. It follows from this that $\big(\eta(f)\big)(x) = \langle\!\langle f(x)\rangle\!\rangle = g(x)$ for all $x \in X$. That is, $\eta(f) = g$ and we conclude that $\eta$ is an epimorphism.

LEMMA 3.12. *Suppose $C(\lambda) = Z(\lambda)$ and define a map $\lambda_*$ from $Q_\lambda$ into $R$ by $\lambda_*(\langle\!\langle v\rangle\!\rangle) = \lambda(v)$. Then $\lambda_*$ is a well defined continuous map from $Q_\lambda$ into $R$ which has the following properties:*

(3.12.1) $\lambda_*(a\langle\!\langle v\rangle\!\rangle) = a\lambda_*(\langle\!\langle v\rangle\!\rangle)$ *for all* $\langle\!\langle v\rangle\!\rangle \in Q_\lambda$ *and* $a \in \mathrm{Ran}(\lambda_*)$,

(3.12.2) $\langle\!\langle v\rangle\!\rangle * \langle\!\langle w\rangle\!\rangle = \big(\lambda_*(\langle\!\langle w\rangle\!\rangle)\big)\langle\!\langle v\rangle\!\rangle$ *for all* $\langle\!\langle v\rangle\!\rangle, \langle\!\langle w\rangle\!\rangle \in Q_\lambda$,

(3.12.3) $Z(\lambda_*) = \{\langle\!\langle 0\rangle\!\rangle\}$.

PROOF. Recall first that

(3.12.4) $$C(\lambda) = \{w \in R^n : \lambda(v + aw) = \lambda(v) \text{ for all } a \in R \text{ and } v \in R^n\}$$

and suppose $\langle\!\langle u\rangle\!\rangle = \langle\!\langle v\rangle\!\rangle$. Then $u = v+w$ for some $w \in C(\lambda)$ and it follows immediately from (3.12.4) that $\lambda(u) = \lambda(v)$. Consequently, $\lambda_*$ is well defined and it is continuous since $Q_\lambda$ has the quotient topology. Note that $\mathrm{Ran}(\lambda_*) = \mathrm{Ran}(\lambda)$. For any $a \in \mathrm{Ran}(\lambda_*)$ and $\langle\!\langle v\rangle\!\rangle \in Q_\lambda$, we have

$$\lambda_*(a\langle\!\langle v\rangle\!\rangle) = \lambda_*(\langle\!\langle av\rangle\!\rangle) = \lambda(av) = a\lambda(v) = a\lambda_*(\langle\!\langle v\rangle\!\rangle)$$

and (3.12.1) has been verified. For $\langle\!\langle v\rangle\!\rangle, \langle\!\langle w\rangle\!\rangle \in Q_\lambda$, we have

$$\langle\!\langle v\rangle\!\rangle * \langle\!\langle w\rangle\!\rangle = \langle\!\langle v * w\rangle\!\rangle = \langle\!\langle \big(\lambda(w)\big)v\rangle\!\rangle = \lambda(w)\langle\!\langle v\rangle\!\rangle = \big(\lambda_*(\langle\!\langle w\rangle\!\rangle)\big)\langle\!\langle v\rangle\!\rangle$$

which means (3.12.2) is valid. Finally, suppose $\langle\!\langle v\rangle\!\rangle \in Z(\lambda_*)$. Then $\lambda(v) = \lambda_*(\langle\!\langle v\rangle\!\rangle) = 0$ which means $v \in Z(\lambda)$. But $C(\lambda) = Z(\lambda)$ so that $v \in C(\lambda)$ which means $\langle\!\langle v\rangle\!\rangle = \langle\!\langle 0\rangle\!\rangle$ and we have verified (3.12.3).

LEMMA 3.13. *Let $m = \dim C(\lambda)$ and let $\alpha$ be any linear isomorphism from $R^{n-m}$ onto $Q_\lambda$. Define a map $\mu$ from $R^{n-m}$ into $R$ by $\mu = \lambda_* \circ \alpha$. Then $\mu$ is a semilinear map with the property that $Z(\mu) = \{0\}$ and $\alpha$ is a topological isomorphism from $N_\mu(R^{n-m})$ onto $Q_\lambda$.*

PROOF. It is immediate from Lemma (3.12) that $\mu$ is a semilinear map and that $Z(\mu) = \{0\}$. Moreover, it is also immediate that $\alpha$ is a homeomorphism and an additive group isomorphism. We need only verify that it is a multiplicative homomorphism. With this in mind, let $v, w \in R^{n-m}$. It follows from (3.12.2) that

$$\alpha(v * w) = \alpha\Big(\big(\mu(w)\big)v\Big) = \big(\mu(w)\big)\alpha(v) = \Big(\lambda_*\big(\alpha(w)\big)\Big)\alpha(v) = \big(\alpha(v)\big) * \big(\alpha(w)\big)$$

and the proof is complete.

We recall from [1], Definition (3.13), p. 23 that the *right annihilator*, $\mathrm{Ann}(x)$, of an element $x$ of a nearring $N$ is defined by $\mathrm{Ann}(x) = \{n \in N : xn = 0\}$ and the *right annihilator*, $\mathrm{Ann}(X)$, of a nonempty subset $X$ of $N$ is defined by $\mathrm{Ann}(X) = \cap\{\mathrm{Ann}(x) : x \in X\}$.

LEMMA 3.14. *Let $X$ be a paracompact Hausdorff space. Then* $\mathrm{Ann}\big(N_\lambda(R^n)\big) = Z(\lambda)$ *and*

$$\mathrm{Ann}\big(N_\lambda(X, R^n)\big) = \{f \in N_\lambda(X, R^n) : \mathrm{Ran}(f) \subseteq Z(\lambda)\}.$$

*Moreover, the following statements are equivalent:*

(3.14.1) $\mathrm{Ann}\big(N_\lambda(X, R^n)\big)$ *is an ideal of* $N_\lambda(X, R^n)$,

(3.14.2) $\mathrm{Ann}\big(N_\lambda(R^n)\big)$ *is an ideal of* $N_\lambda(R^n)$,

(3.14.3) $C(\lambda) = Z(\lambda)$.

PROOF. It is immediate that $\mathrm{Ann}\big(N_\lambda(R^n)\big) = Z(\lambda)$. Denote $\{f \in N_\lambda(X, R^n) : \mathrm{Ran}(f) \subseteq Z(\lambda)\}$ by $J$ and suppose $f \notin J$. Then $f(x) \notin Z(\lambda)$ for some $x \in X$. Choose any nonzero $v \in R^n$ and we have $(\langle v\rangle f)(x) = v * f(x) = \Big(\lambda\big(f(x)\big)\Big)v \neq 0$. Thus, $\langle v\rangle f \neq \langle 0\rangle$ which means $f \notin \mathrm{Ann}(\langle v\rangle)$ and hence $f \notin \mathrm{Ann}\big(N_\lambda(X, R^n)\big)$. On the other hand, if $f \in J$, then $\mathrm{Ran}(f) \subseteq Z(\lambda)$ and we have $(gf)(x) = g(x) * f(x) = \Big(\lambda\big(f(x)\big)\Big) * g(x) = 0$ which means $gf = \langle 0\rangle$ for all $g \in N_\lambda(X, R^n)$. Consequently, $f \in \mathrm{Ann}\big(N_\lambda(X, R^n)\big)$ and we have verified that $\mathrm{Ann}\big(N_\lambda(X, R^n)\big) = \{f \in N_\lambda(X, R^n) : \mathrm{Ran}(f) \subseteq Z(\lambda)\}$.

Suppose (3.14.1) holds, choose any point $x \in X$ and define a map $\varphi_x$ from $N_\lambda(X, R^n)$ into $N_\lambda(R^n)$ by $\varphi_x(f) = f(x)$. It is a simple matter to verify that $\varphi_x$ is an epimorphism from $N_\lambda(X, R^n)$ onto $N_\lambda(R^n)$ and since $\varphi_x\Big(\mathrm{Ann}\big(N_\lambda(X, R^n)\big)\Big) = \mathrm{Ann}\big(N_\lambda(R^n)\big)$ we conclude that $\mathrm{Ann}\big(N_\lambda(R^n)\big)$ is an ideal of $N_\lambda(R^n)$. Thus, (3.14.1) implies (3.14.2). Theorem (3.7) of [2] tells us that the proper ideals of $N_\lambda(R^n)$ coincide with the linear subspaces of $C(\lambda)$ so that (3.14.2) implies that $Z(\lambda) \subseteq C(\lambda)$. Since we always have $C(\lambda) \subseteq Z(\lambda)$ we see that (3.14.2) implies (3.14.3). To see that (3.14.3) implies (3.14.1), simply observe that $\mathrm{Ker}\,\eta = \{f \in N_\lambda(X, R^n) : \mathrm{Ran}(f) \subseteq C(\lambda)\}$ where $\eta$ is the homomorphism defined in Lemma 3.11. Consequently, when $C(\lambda) = Z(\lambda)$, we have $\mathrm{Ker}\,\eta = \mathrm{Ann}\big(N_\lambda(X, R^n)\big)$ which means $\mathrm{Ann}\big(N_\lambda(X, R^n)\big)$ is an ideal of $N_\lambda(X, R^n)$.

J.R. Clay shows in Proposition (3.15), p. 23 of [1] that the right annihilator of a nearring is a normal subgroup of its additive group. But his nearrings are left nearrings while ours are right nearrings. For instance, in Example 2.4,

$$Z(\lambda) = \{v \in R^2 : v_1 = v_2\} \cup \{v \in R^2 : v_1 = -v_2\}$$

whereas $C(\lambda) = \{0\}$. Consequently, $Z(\lambda)$ is not a subgroup of $R^2$ and $\mathrm{Ann}\big(N_\lambda(X, R^n)\big)$ is not a subgroup of $N_\lambda(X, R^n)$. To see the latter, simply observe that $\langle(1, 1)\rangle, \langle(1, -1)\rangle \in \mathrm{Ann}\big(N_\lambda(X, R^n)\big)$ while $\langle(1, 1)\rangle + \langle(1, -1)\rangle = \langle(2, 0)\rangle \notin \mathrm{Ann}\big(N_\lambda(X, R^n)\big)$. Of course, $Z(\lambda)$ can be an additive subgroup of $N_\lambda(R^n)$ even when $C(\lambda) \neq Z(\lambda)$. This is the case in Example 2.5 where $C(\lambda) = \{0\}$ while $Z(\lambda) = \{v \in R^2 : v_1 = 0\}$. In this case $\mathrm{Ann}\big(N_\lambda(X, R^n)\big)$ is also an additive subgroup of $N_\lambda(X, R^n)$. Actually, it is not difficult to verify that $\mathrm{Ann}\big(N_\lambda(X, R^n)\big)$ is an additive subgroup of $N_\lambda(X, R^n)$ if and only if $Z(\lambda)$

is a subgroup of $N_\lambda(R^n)$. It is immediate that $\mathrm{Ann}\big(N_\lambda(X, R^n)\big)$ is an additive subgroup of $N_\lambda(X, R^n)$ whenever $Z(\lambda)$ is an additive subgroup of $N_\lambda(R^n)$. To see that $Z(\lambda)$ is a subgroup whenever $\mathrm{Ann}\big(N_\lambda(X, R^n)\big)$ is, choose any $x \in X$ and define $\varphi_x(f) = f(x)$. Then $\varphi_x$ is an epimorphism from $N_\lambda(X, R^n)$ onto $N_\lambda(R^n)$ for which $\varphi_x\Big(\mathrm{Ann}\big(N_\lambda(X, R^n)\big)\Big) = Z(\lambda)$.

THEOREM 3.15. *Let $X$ be a paracompact Hausdorff space, let $C(\lambda) = Z(\lambda)$ and let $m = \dim C(\lambda)$. Then there exists a topological isomorphism $\beta$ from $N(X, Q_\lambda)$ onto $N_\mu(X, R^{n-m})$. Let $\eta$ be the epimorphism from $N_\lambda(X, R^n)$ onto $N(X, Q_\lambda)$ defined by $\big(\eta(f)\big)(x) = \langle\langle f(x)\rangle\rangle$. Then $\beta \circ \eta$ is an epimorphism from $N_\lambda(X, R^n)$ onto $N_\mu(X, R^{n-m})$ with the property that* $\mathrm{Ker}(\beta \circ \eta) = \mathrm{Ann}\big(N_\lambda(X, R^n)\big)$.

PROOF. Lemma 3.13 assures that there exists a topological isomorphism $\alpha$ from $N_\mu(R^{n-m})$ onto $Q_\lambda$. Define a map $\beta$ from $N(X, Q_\lambda)$ into $N_\mu(X, R^{n-m})$ by $\beta(f) = \alpha^{-1} \circ f$. It is immediate that $\beta$ is a bijection and that it is an additive isomorphism. Let $f, g \in N(X, Q_\lambda)$, let $x \in X$ and recall from Lemma 3.13 that $\mu = \lambda_* \circ \alpha$. We then appeal to (3.12.2) once again and get

$$\begin{aligned}
\big(\beta(fg)\big)(x) &= \alpha^{-1}\big((fg)(x)\big) = \alpha^{-1}\big(f(x) * g(x)\big) = \alpha^{-1}\Big(\lambda_*\big(g(x)\big)f(x)\Big) \\
&= \Big(\lambda_*\big(g(x)\big)\Big)\alpha^{-1}\big(f(x)\big) = \lambda_* \circ \alpha\big(\alpha^{-1} \circ g(x)\big)\big(\alpha^{-1} \circ f(x)\big) \\
&= \mu\Big(\big(\beta(g)\big)(x)\Big)\Big(\big(\beta(f)\big)(x)\Big) = \big(\beta(f)\big)(x) * \big(\beta(g)\big)(x) \\
&= \big(\beta(f)\beta(g)\big)(x)
\end{aligned}$$

which implies that $\beta(fg) = \beta(f)\beta(g)$ and we have verified that $\beta$ is an isomorphism from $N(X, Q_\lambda)$ onto $N_\mu(X, R^{n-m})$. Then $\beta \circ \eta$ is an epimorphism from $N_\lambda(X, R^n)$ onto $N_\mu(X, R^{n-m})$. Moreover, $\mathrm{Ker}(\beta \circ \eta) = \mathrm{Ann}\big(N_\lambda(X, R^n)\big)$ since $\mathrm{Ker}(\eta) = \mathrm{Ann}\big(N_\lambda(X, R^n)\big)$ and $\beta$ is an isomorphism.

For any $f \in N_\lambda(X, R^n)$, let $P(f) = f^{-1}\big(C(\lambda)\big)$ and for any subset $A \subseteq X$, let $P(A) = \cap\{P(f) : f \in A\}$.

DEFINITION 3.16. An ideal $J$ of the nearring $N_\lambda(X, R^n)$ is said to be *stable* if $\mathrm{Ann}\big(N_\lambda(X, R^n)\big) \subseteq J$ and *unstable* otherwise. It is said to be $C(\lambda)$-*fixed* if $P(J) \neq \emptyset$ and $C(\lambda)$-*free* otherwise.

THEOREM 3.17. *Let $X$ be a locally compact paracompact Hausdorff space and suppose $C(\lambda) = Z(\lambda)$. Then every proper stable ideal of $N_\lambda(X, R^n)$ is $C(\lambda)$-fixed if and only if $X$ is compact.*

PROOF. Theorem 3.15 tells us that $\beta \circ \eta$ is an epimorphism from $N_\lambda(X, R^n)$ onto $N_\mu(X, R^{n-m})$. Suppose first that $X$ is compact and let $J$ be a proper stable ideal of $N_\lambda(X, R^n)$. Then $\beta \circ \eta(J)$ is a proper ideal of $N_\mu(X, R^{n-m})$. It follows from (3.12.3) and the definition of the semilinear map $\mu$ in Lemma 3.13 that $Z(\mu) = \{0\}$ which means $C(\mu) = Z(\mu) = \{0\}$ since $C(\mu) \subseteq Z(\mu)$. Consequently, $N_\mu(X, R^{n-m})$ is simple and it follows from Theorem 3.8 that the ideal $\beta \circ \eta(J)$ is fixed. This means that there exists an $x \in X$ such that $\big(\beta \circ \eta(f)\big)(x) = 0$ for all $f \in J$. But $\big(\beta \circ \eta(f)\big)(x) = \beta\big(\langle\langle f(x)\rangle\rangle\big)$ which means $\langle\langle f(x)\rangle\rangle = \langle\langle 0\rangle\rangle$ since $\beta$ is an isomorphism. This means $f(x) \in C(\lambda)$ for all $f \in J$ and we conclude that $J$ is $C(\lambda)$-fixed.

Now suppose that every proper stable ideal of $N_\lambda(X, R^n)$ is $C(\lambda)$-fixed and this time, let $J$ be any proper ideal of $N_\mu(X, R^{n-m})$. Then $(\beta \circ \eta)^{-1}(J)$ is a proper stable ideal of $N_\lambda(X, R^n)$ which is $C(\lambda)$-fixed and consequently, there exists a point $x \in X$ such that $f(x) \in C(\lambda)$ for each $f \in (\beta \circ \eta)^{-1}(J)$. Let $g$ be any element of $J$ and let $\beta \circ \eta(f) = g$. Then, since $f(x) \in C(\lambda)$, we have

$$g(x) = \big((\beta \circ \eta)(f)\big)(x) = \beta\big(\langle\langle f(x)\rangle\rangle\big) = \beta(\langle\langle 0\rangle\rangle) = 0$$

and this means that $J$ is fixed. Again, we use the fact that $C(\mu) = Z(\mu) = \{0\}$ (which implies that $N_\mu(X, R^{n-m})$ is simple) to conclude from Theorem 3.8 that $X$ is compact.

DEFINITION 3.18. Let $M_x^* = \{f \in N_\lambda(X, R^n) : f(x) \in C(\lambda)\}$.

THEOREM 3.19. *Suppose $C(\lambda) = Z(\lambda)$ and $X$ is a compact Hausdorff space. Then the stable maximal ideals of $N_\lambda(X, R^n)$ are precisely the sets of the form $M_x^*$.*

PROOF. Let $\beta\circ\eta$ be as in Theorem 3.15 and for $x \in X$ define a map $\varphi_x$ from $N_\lambda(X, R^n)$ to $N_\mu(R^{n-m})$ by $\varphi_x(f) = \big(\beta\circ\eta(f)\big)(x)$. Since $\beta\circ\eta$ is an epimorphism from $N_\lambda(X, R^n)$ onto $N_\mu(X, R^{n-m})$, it readily follows that $\varphi_x$ is an epimorphism from $N_\lambda(X, R^n)$ onto $N_\mu(R^{n-m})$ and since $N_\mu(R^{n-m})$ is simple, it follows that $\mathrm{Ker}(\varphi_x)$ is a maximal ideal of $N_\lambda(X, R^n)$. One readily verifies that $\mathrm{Ker}(\varphi_x) = M_x^*$ which means $M_x^*$ is a stable maximal ideal for each $x \in X$. Now let $M$ be any stable maximal ideal. By Theorem 3.17, $M$ is $C(\lambda)$-fixed. Choose any $x \in P(M)$. Then $M \subseteq M_x^*$ which means $M = M_x^*$ since $M$ is maximal.

4. **The Isomorphism Theorems.** In this section, we prove the isomorphism theorems. This section is relatively short since we fashioned all the tools we need for the proof in the previous section. Just as in Lemma 3.11, we let $Q_\lambda = N_\lambda(R^n)/C(\lambda)$ with the quotient topology and, similarly, we let $Q_\rho = N_\rho(R^m)/C(\rho)$ with the quotient topology.

THEOREM 4.1. *Let $\lambda$ and $\rho$ be nonconstant semilinear maps from $R^n$ and $R^m$, respectively, into $R$ such that $C(\lambda) = Z(\lambda)$ and $C(\rho) = Z(\rho)$ and let $X$ and $Y$ be compact Hausdorff spaces. If the nearrings $N_\lambda(X, R^n)$ and $N_\rho(Y, R^m)$ are isomorphic, then*

(4.1.1) *the spaces $X$ and $Y$ are homeomorphic,*

*and*

(4.1.2) *the nearrings $Q_\lambda$ and $Q_\rho$ are topologically isomorphic.*

PROOF. Let $\varphi$ be any isomorphism from $N_\lambda(X, R^n)$ onto $N_\rho(Y, R^m)$. Then $\varphi[M]$ is a stable maximal ideal of $N_\rho(Y, R^m)$ for each stable maximal ideal $M$ of $N_\lambda(X, R^n)$. Consequently, according to Theorem 3.19, there exists, for each $x \in X$, a point $y \in Y$

such that $\varphi[M_x^*] = M_y^*$. Define a bijection $h$ from $X$ onto $Y$ by $h(x) = y$ and note that $\varphi[M_x^*] = M_{h(x)}^*$. For any $f \in N_\lambda(X, R^n)$, we have

$$\begin{aligned} x \in P(f) &\Longleftrightarrow f(x) \in C(\lambda) \Longleftrightarrow f \in M_x^* \\ &\Longleftrightarrow \varphi(f) \in \varphi[M_x^*] \Longleftrightarrow \varphi(f) \in M_{h(x)}^* \\ &\Longleftrightarrow h(x) \in P\big(\varphi(f)\big) \end{aligned}$$

which readily implies that $h[P(f)] = P\big(\varphi(f)\big)$. In a similar manner, $h^{-1}[P(g)] = P\big(\varphi^{-1}(g)\big)$ for each $g \in N_\rho(Y, R^m)$. Since $\{P(f) : f \in N_\lambda(X, R^n)\}$ and $\{P(g) : g \in N_\rho(Y, R^m)\}$ form bases for the closed subsets of $X$ and $Y$ respectively, it follows that $h$ is a homeomorphism and (4.1.1) has been verified.

Now choose any $x \in X$ and let $\varphi[M_x^*] = M_y^*$. Theorem 3.19 tells us that $Q_\lambda$ is topologically isomorphic to $N_\mu(R^{n-r})$ where $r = \dim C(\lambda)$ and we observed in the proof of Theorem 3.19 that there exists an epimorphism $\alpha$ from $N_\lambda(X, R^n)$ onto $N_\mu(R^{n-r})$ whose kernel is $M_x^*$. Consequently, $N_\lambda(X, R^n)/C(\lambda)$ is topologically isomorphic to $N_\mu(R^{n-r})$. In a similar manner, there exists an epimorphism $\beta$ from $N_\rho(Y, R^m)$ whose kernel is $M_y^*$ so that $N_\rho(Y, R^m)/C(\rho)$ is topologically isomorphic to $N_\sigma(R^{m-s})$ where $s = \dim C(\rho)$ and $\sigma$ is the semilinear map induced by $\rho$ just as $\mu$ is induced by $\lambda$ in Lemma 3.13. Moreover, $Q_\rho$ is topologically isomorphic to $N_\sigma(R^{m-s})$. We now wish to define an isomorphism $\psi$ from $N_\mu(R^{n-r})$ onto $N_\sigma(R^{m-s})$ which is suggested by the following diagram.

$$\text{(4.1.3)} \qquad \begin{array}{ccc} N_\lambda(X, R^n) & \xrightarrow{\varphi} & N_\rho(Y, R^m) \\ \big\downarrow{\scriptstyle\alpha} & & \big\downarrow{\scriptstyle\beta} \\ N_\mu(R^{n-r}) & \xrightarrow{\psi} & N_\sigma(R^{m-s}) \end{array}$$

Accordingly, for any $v \in N_\mu(R^{n-r})$, we choose any $f \in N_\lambda(X, R^n)$ such that $\alpha(f) = v$ and we define $\psi(v) = \beta \circ \varphi(f)$. Our first task is to show that $\psi$ is well defined. Suppose $\alpha(f) = \alpha(g)$. Then $f - g \in M_x^*$ which implies $\varphi(f) - \varphi(g) \in M_y^*$. Consequently, $\beta \circ \varphi(f) = \beta \circ \varphi(g)$. It is easily verified that $\psi$ is a homomorphism. It is immediate that Diagram (4.1.3) commutes and since $\varphi$, $\alpha$ and $\beta$ are all surjective, $\psi$ must be surjective as well. Finally, suppose $\psi(v) = 0$ and let $\alpha(f) = v$. Then $\beta\big(\varphi(f)\big) = 0$ which means $\varphi(f) \in M_y^*$. This implies that $f \in M_x^*$ and it follows that $v = \alpha(f) = 0$. Therefore $\psi$ is an isomorphism from $N_\mu(R^{n-r})$ onto $N_\sigma(R^{m-s})$ and since any isomorphism is also a homeomorphism, we conclude that $N_\mu(R^{n-r})$ and $N_\sigma(R^{m-s})$ are topologically isomorphic. Since $Q_\lambda$ is topologically isomorphic to $N_\mu(R^{n-r})$ and $Q_\rho$ is topologically isomorphic to $N_\sigma(R^{m-s})$, (4.1.2) has been verified and the proof is complete.

Our next result follows easily from the previous one.

THEOREM 4.2. *Let $\lambda$ be any nonconstant semilinear map from $R^n$ to $R$ such that $C(\lambda) = Z(\lambda)$ and let $X$ and $Y$ be compact Hausdorff spaces. Then the nearrings $N_\lambda(X, R^n)$ and $N_\lambda(Y, R^n)$ are isomorphic if and only if $X$ and $Y$ are homeomorphic.*

PROOF. It follows from the previous theorem that $X$ and $Y$ are homeomorphic if $N_\lambda(X, R^n)$ and $N_\lambda(Y, R^n)$ are isomorphic. Suppose, conversely, that $h$ is a homeomorphism from $Y$ onto $X$. The map $\varphi$ defined by $\varphi(f) = f \circ h$ is easily shown to be an isomorphism from $N_\lambda(X, R^n)$ onto $N_\lambda(Y, R^n)$.

THEOREM 4.3. *Let $\lambda$ and $\rho$ be nonconstant semilinear maps from $R^n$ and $R^m$, respectively, into $R$ such that $C(\lambda) = Z(\lambda)$ and $C(\rho) = Z(\rho)$. Let $X$ and $Y$ be compact Hausdorff spaces and suppose both $N_\lambda(R^n)$ and $N_\rho(R^m)$ are simple (which, here, is equivalent to requiring that $Z(\lambda) = \{0\}$ and $Z(\rho) = \{0\}$). Then the following statements are equivalent:*

(4.3.1) *The nearrings $N_\lambda(X, R^n)$ and $N_\rho(Y, R^m)$ are isomorphic,*

(4.3.2) *The spaces $X$ and $Y$ are homeomorphic and the nearrings $N_\lambda(R^n)$ and $N_\rho(R^m)$ are topologically isomorphic,*

(4.3.3) *The spaces $X$ and $Y$ are homeomorphic, $n = m$ and there exists a linear isomorphism $\varphi$ from $R^n$ onto $R^m$ such that $\lambda = \rho \circ \varphi$.*

PROOF. Since $N_\lambda(R^n)$ and $N_\rho(R^m)$ are both simple, it follows that $N_\lambda(R^n)$ coincides with $Q_\lambda$ and $N_\rho(R^m)$ coincides with $Q_\rho$. Therefore, it is an immediate consequence of Theorem (4.1) that (4.3.1) implies (4.3.2). Now suppose (4.3.2) holds. It is immediate that $n = m$ and it follows from Theorem (3.14) of [2] that there exists a linear isomorphism $\varphi$ from $R^n$ onto $R^m$ such that $\lambda = \rho \circ \varphi$. Finally, suppose (4.3.3) holds. Let $h$ be any homeomorphism from $Y$ onto $X$ and let $\varphi$ be any linear isomorphism from $R^n$ onto $R^m$. Define a map $\psi$ from $N_\lambda(X, R^n)$ to $N_\rho(Y, R^m)$ by $\big(\psi(f)\big)(y) = \varphi\big(f \circ h(y)\big)$. It follows easily that $\psi$ is an additive isomorphism from $N_\lambda(X, R^n)$ onto $N_\rho(Y, R^m)$. To show that $\psi(fg) = \psi(f)\psi(g)$, let any $y \in Y$ be given and observe that

$$\begin{aligned}
\big(\psi(fg)\big)(y) &= \varphi\big((fg) \circ h(y)\big) = \varphi\Big(f\big(h(y)\big) * g\big(h(y)\big)\Big) \\
&= \varphi\Big(\lambda\Big(g\big(h(y)\big)\Big)f\big(h(y)\big)\Big) = \lambda\Big(g\big(h(y)\big)\Big)\varphi\Big(f\big(h(y)\big)\Big) \\
&= \rho\Big(\varphi\Big(g\big(h(y)\big)\Big)\Big)\varphi\Big(f\big(h(y)\big)\Big) = \varphi\Big(f\big(h(y)\big)\Big) * \varphi\Big(g\big(h(y)\big)\Big) \\
&= \big(\psi(f)(y)\big) * \big(\psi(g)(y)\big) = \big(\psi(f)\psi(g)\big)(y).
\end{aligned}$$

Consequently, $\psi(fg) = \psi(f)\psi(g)$ and we conclude that $\psi$ is an isomorphism from the nearring $N_\lambda(X, R^n)$ onto the nearring $N_\rho(Y, R^m)$. Thus (4.3.3) implies (4.3.1) and the proof is complete.

SOME CONCLUDING REMARKS. Theorem 4.1 tells us that if the nearrings $N_\lambda(X, R^n)$ and $N_\rho(Y, R^m)$ are isomorphic, then $Q_\lambda$ and $Q_\rho$ are topologically isomorphic. However, we cannot conclude from this that $N_\lambda(R^n)$ and $N_\rho(R^m)$ are topologically isomorphic. For example, define a map $\lambda$ from $R^2$ into $R$ by $\lambda(v) = |v_1|$ and define a map $\rho$ from $R^3$ into $R$ by $\rho(w) = |w_1|$. Then

$$C(\lambda) = Z(\lambda) = \{v \in R^2 : v_1 = 0\} \text{ and } C(\rho) = Z(\rho) = \{w \in R^3 : w_1 = 0\}.$$

One can verify that $Q_\lambda = N_\lambda(R^2)/C(\lambda)$ is isomorphic to $Q_\rho = N_\rho(R^3)/C(\rho)$ are isomorphic but $N_\lambda(R^2)$ and $N_\rho(R^3)$ are certainly not isomorphic. Nevertheless, we conjecture that Theorem 4.3 holds without the requirement that the nearrings $N_\lambda(R^n)$ and $N_\rho(R^m)$ be simple.

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