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General study on limit cycle bifurcation near a double eight figure loop

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In this article, we explore the bifurcation problem of limit cycles near the double eight figure loop (compound cycle with a 2-polycycle connecting two homoclinic loops). A general theory is established to find the lower bound of the maximal number of limit cycles (isolated periodic orbits) near the double eight figure loop. The Liénard system, a well-known nonlinear dynamical model, appears in a natural way in physics, chemistry, engineering, and so on, where periodic phenomena play a relevant role. As an application, we investigate an $(n+1)$th-order generalized Liénard system and prove the system has at least $7[\frac{n}{6}]+2[\frac{r}{2}]-[\frac{r}{4}]$ limit cycles near the double eight figure loop for any $n \geq 5$ and $r = \mathrm{mod(n, 6)}$, and their distribution is also gained.

Keywords: heteroclinic loop; homoclinic loop; Liénard system; limit cycle; Melnikov function

2010 Mathematics Subject Classification: 34C07; 37G15

1. Introduction

Consider the planar near-Hamiltonian system

$$\dot{x} = H_y(x, y) + \varepsilon f(x, y, \delta), \quad \dot{y} = -H_x(x, y) + \varepsilon g(x, y, \delta), \tag{1.1}$$

where the parameter $|\varepsilon| \ll 1$, the vector parameter $\delta \in D \subset \mathbf{R}^n$ with D compact, H, f, g are polynomial functions in x and y. The system (1.1) describes a widely researched dynamical system that not only has numerous applications in vital areas such as celestial mechanics, molecular dynamics, statistical mechanics, and quantum mechanics [22, 30] but also is related to the weakened Hilbert's 16th, as proposed by Anorld [1], which is an important subject of investigation in the qualitative theory of plane differential systems.

Let $\omega = g(x, y, \delta)dx - f(x, y, \delta)dy$ be a 1-form. The weakened Hilbert's 16th problem is to find an upper bound for the number of isolated zeros of the first-order Melnikov function (Abelian integral)

$$M(h, \delta) = \oint_{\Gamma_h} \omega,$$

where $\Gamma_h = \{H(x, y) = h, h \in J\}$ is defined as a family of continuous and close curves, with J being an open interval, and it is related to the lower bound of the maximum number of limit cycles for the system (1.1).

An essential approach to determining the isolated zero number of $M(h, \delta)$ is to investigate its expansion in h near a centre, a homoclinic loop, or a heteroclinic loop (see [9–11, 14, 18, 20]). The asymptotic expansion of the first-order Melnikov function $M(h, \delta)$ near the homoclinic or heteroclinic loop will be briefly described below, including the formulas for the coefficients of the first few terms in the expansion.

Dulac [7] and Roussarie [28] gained the expression of the expansion of $M(h, \delta)$ near and inside the homoclinic loop L with a hyperbolic saddle,

$$M(h, \delta) = \sum_{j \geq 0} \left(c_{2j}(\delta) + c_{2j+1}(\delta)\,(h - h_s) \ln|h - h_s|\right)(h - h_s)^j, \quad 0 < h_s - h \ll 1, \tag{1.2}$$

where $H(S) = h_s$, $c_0(\delta) = \oint_L \omega\,|_{\varepsilon=0}$, and $c_1(\delta) = -\frac{1}{|\lambda|}\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right)\Big|_S$ with the eigenvalues $\pm\lambda$, and the expression of the coefficient c_1 was obtained by Han and Ye [19]. For detailed calculations of higher-order coefficients in the expansion of $M(h, \delta)$, refer to [18, 19, 29, 31].

Suppose system $(1.1)|_{\varepsilon=0}$ has a polycycle L with m hyperbolic saddles S_i for $i = 1, \ldots, m$, encircling the family of periodic orbits Γ_h near it, where $H(S_1) = h_s$. Jiang and Han [21] proved the asymptotic expansion of $M(h, \delta)$ near the heteroclinic loop, and it is in the same form as (1.2). Obviously, the first coefficient $c_0(\delta) = \oint_L \omega\,|_{\varepsilon=0}$ remains unchanged. The coefficient $c_1(\delta) = \sum_{i=1}^{m} -\frac{1}{|\lambda_i|}\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right)\Big|_{S_i}$ with the eigenvalues $\pm\lambda_i$ of S_i was given by Li et al. [24] (see [13, 18, 32] and its references for other results on the calculation of coefficients).

The coefficients $c_i(\delta)$ in the expansion of the first-order Melnikov function $M(h, \delta)$ can be utilized to acquire a lower bound on the maximum number of limit cycles: supposing there exists a constant n such that

$$0 < c_0(\delta) \ll -c_1(\delta) \ll c_2(\delta) \ll \cdots \ll (-1)^{n-1} c_{n-1}(\delta) \ll (-1)^n c_n(\delta),$$

this implies the integral $M(h, \delta)$ can possess n zeros near $h = h_s$. Thereby the system (1.1) has n limit cycles near the homoclinic or heteroclinic loop.

In recent years, numerous investigations have focused on the number of limit cycles inside and outside the figure-of-eight loop (double homoclinic loop) or the two-saddle loop (heteroclinic loop). This research interest not only stems from the qualitative theory of ordinary differential equations [8, 11, 17, 18, 24–26, 29, 31, 44] but also relates to various practical issues, such as the generalized Rayleigh–Liénard oscillator and the Van der Pol–Duffing oscillator [6, 9, 12, 33].

In this article, we investigate the limit cycles near a double eight figure loop, which is a compound cycle with a 2-polycycle connecting two homoclinic loops, see [15, 16, 38, 43]. Different from the articles mentioned above, the highlight of the research is that it can be considered both kinds of bifurcations (homoclinic loop and heteroclinic loop) at the same time. Furthermore, we establish a general theory for finding a lower bound of the maximal number of limit cycles near the double eight figure loop by calculating the algebraic structure (generators) of the first-order Melnikov function. The advantages of this approach are numerous. For instance, it is more convenient to gain the expressions for the coefficients of the higher-order terms in the expansion of $M(h)$, and the results on the number of limit cycles for the system are applicable to any degree n of perturbation. An example is given as an application to illustrate these advantages.

Consider a generalized Liénard system of type (m, n) defined by

$$\dot{x} = y, \quad \dot{y} = -g(x) + \varepsilon f(x)y, \tag{1.3}$$

where $g(x)$ and $f(x)$ are polynomials of degree m and n, respectively. Note that a necessary condition for the Hamiltonian system $(1.3)|_{\varepsilon=0}$ to possess a double eight figure loop is $m \geq 5$. In this article, we examine the scenario where $m = 5$, chosen for computational convenience, and based on the extensive research by many scholars on the problem of limit cycles of the system (1.3) when $m = 5$ with n fixed to a certain value [2–4, 23, 27, 34, 35, 37, 39–42], and in addition it is related to the complex Ginzburg–Landau equation [5]. For example, Xiong and Zhong [34], Zhang et al. [40], and Yang and Zhao [37] discussed the limit cycles or the zeros of first-order Melnikov function (Abelian integral) of the Liénard system

$$\dot{x} = y, \quad \dot{y} = -x(x^2 - 2)(x^2 - \tfrac{1}{2}) + \varepsilon f(x)y, \tag{1.4}$$

with $0 < |\varepsilon| \ll 1$ for $f(x) = \sum_{i=0}^{2} a_i x^{2i}$, $\sum_{i=0}^{4} a_i x^i$, and $\sum_{i=0}^{4} a_i x^{2i}$, respectively. Xu and Li [35] studied the limit cycles of the following system

$$\dot{x} = y, \quad \dot{y} = -x(x^2 - 1)(x^2 - 9) + \varepsilon f(x)y, \tag{1.5}$$

where $0 < |\varepsilon| \ll 1$ and $f(x) = \sum_{i=0}^{5} a_i x^{2i}$. It is obvious that the unperturbed systems $(1.4)|_{\varepsilon=0}$ and $(1.5)|_{\varepsilon=0}$ both have a double eight figure loop.

Motivated by [4, 34, 35, 37, 40, 43], in this article, we consider a more generic system of the form

$$\dot{x} = y, \quad \dot{y} = -x(x^2 - a)(x^2 - b) + \varepsilon f(x)y, \tag{1.6}$$

where $0 < |\varepsilon| \ll 1$, $f(x) = \Sigma_{i=0}^{n} a_i x^i$, $(a_0, a_1, \ldots, a_n) \in D \subset \mathbb{R}^{n+1}$ with D being a compact subset, and $a > b > 0$, which will ensure the unperturbed system $(1.6)|_{\varepsilon=0}$ exists a double eight figure loop. Without loss of generality, we suppose $a > 1$ and $b = 1$. Otherwise, one can utilize a variable transformation

$$x \to \sqrt{b}x, \quad y \to b\sqrt{b}y, \quad t \to \frac{1}{b}t,$$

on the system (1.6) to satisfy this assumption. Compared to existing works, on the one hand, system (1.6) includes the two systems (1.4) and (1.5) described above, and on the other hand the number of limit cycles we obtain for the system holds for any value of n.

The remaining sections of this article are organized as follows. In §2, we build a general bifurcation theory to obtain the lower bound of the maximal number of limit cycles near the double eight figure loop by finding the algebraic structure of the first-order Melnikov function. As an application, we also research the system (1.6) and provide a more optimal result for the lower bound of the maximal number of the limit cycles near the double eight figure loop in §3.

2. A general bifurcation theory to a double eight figure loop

Assume that system $(1.1)|_{\varepsilon=0}$ has a double eight figure loop (compound cycle) $\Gamma \subset G$, which is defined below:

$$\Gamma = \left(\bigcup_{i=1}^{2} L_i\right) \bigcup \Gamma^2, \quad \Gamma^2 = \left(S_1 \cup \widetilde{L}_1\right)\left(S_2 \cup \widetilde{L}_2\right),$$

where the 2-polycycle Γ^2 is composed of two hyperbolic saddles S_1, S_2 with $H(S_1) = h_s$ and two heteroclinic orbits $\widetilde{L}_1, \widetilde{L}_2$ satisfying

$$\omega\left(\widetilde{L}_1\right) = S_1, \alpha\left(\widetilde{L}_1\right) = S_2, \omega\left(\widetilde{L}_2\right) = S_2, \alpha\left(\widetilde{L}_2\right) = S_1,$$

and L_1, L_2 are two homoclinic loops outside Γ^2. There are three centers $O(0,0)$, C_1, C_2 surrounded by Γ^2, L_1, L_2, respectively, where $H(O) = 0, H(C_1) = h_{C_1}, H(C_2) = h_{C_2}$ (see Figure 1).

Near the double eight figure loop Γ, there are four families of periodic orbits: $\Gamma(h)$, which is located outside Γ for $0 < h - h_s < \mu$; and $L_1(h), L_2(h), L_3(h)$, which are located inside Γ for $-\mu < h - h_s < 0$, where $0 < \mu \ll 1$. Then the four first-order Melnikov functions are as follows

$$\begin{aligned} M(h) &= \oint_{\Gamma(h)} g\mathrm{d}x - f\mathrm{d}y, \quad 0 < h - h_s < \mu, \\ M^j(h) &= \oint_{L_j(h)} g\mathrm{d}x - f\mathrm{d}y, \quad -\mu < h - h_s < 0, \quad j = 1, 2, 3. \end{aligned} \tag{2.1}$$

From (1.2), one has

$$\begin{aligned} M(h) &= \varphi(h)(h - h_s)\ln(h - h_s) + N(h), \quad 0 < h - h_s < \mu, \\ M^j(h) &= \varphi_j(h)(h - h_s)\ln|h - h_s| + N_j(h), \quad -\mu < h - h_s < 0, \quad j = 1, 2, 3, \end{aligned} \tag{2.2}$$

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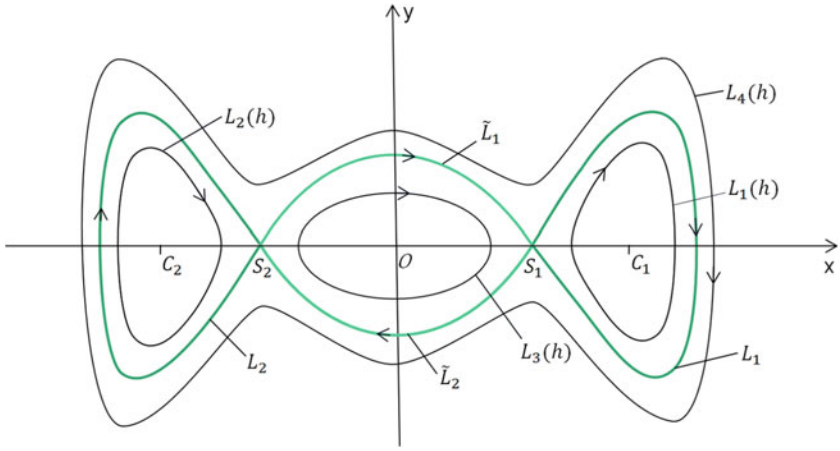

Figure 1. Periodic orbits near Γ.

where $\varphi_j(h), N_j(h)$, and $N(h)$ are analytic functions defined for $0 < |h| < \mu$ and satisfy the following convergent expansions

$$\varphi(h) = \sum_{i\geq 0} c_{2i+1}(h-h_s)^i, \quad \varphi_j(h) = \sum_{i\geq 0} c^j_{2i+1}(h-h_s)^i,$$
$$N_j(h) = \sum_{i\geq 0} c^j_{2i}(h-h_s)^i, \quad N(h) = \sum_{i\geq 0} c_{2i}(h-h_s)^i.$$

From [18] or Theorem 3.2.9 of the book [16], it holds that

$$\varphi_3(h) = \varphi_1(h) + \varphi_2(h). \tag{2.3}$$

The following theorem further gives the relationships between the functions $\varphi_j(h), \varphi(h), N_j(h)$, and $N(h)$.

THEOREM 2.1 *Assume the functions M(h) and $M_j(h)$ $(j = 1, 2, 3)$ defined by (2.1) satisfy (2.2), where H, f, and g are polynomial functions on G containing double eight figure loop Γ. It follows for $0 < |h| < \mu$ that*

$$\frac{1}{2}\varphi(h) = \varphi_1(h) + \varphi_2(h), \tag{2.4}$$

and

$$N(h) = N_1(h) + N_2(h) + N_3(h). \tag{2.5}$$

In fact, by applying Theorem 1.1 in [17] respectively to the right and left part of the y-axis of the phase portrait in Figure 1 one can directly obtain the conclusion of Theorem 2.1. For brevity, we omit the detailed proof here.

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Depending on (2.3), (2.4), and (2.5), one has

$$\begin{aligned} M^j(h) &= \sum_{i\geq 0}(c^j_{2i} + c^j_{2i+1}(h-h_s)\ln|h-h_s|)(h-h_s)^i,\ 0<h_s-h\ll 1,\ j=1,2,\\ M^3(h) &= \sum_{i\geq 0}(c^j_{2i} + (c^1_{2i+1}+c^2_{2i+1})(h-h_s)\ln|h-h_s|)(h-h_s)^i,\ 0<h_s-h\ll 1,\\ M(h) &= \sum_{i\geq 0}\left(c^1_{2i}+c^2_{2i}+c^3_{2i}+2(c^1_{2i+1}+c^2_{2i+1})(h-h_s)\ln|h-h_s|\right)(h-h_s)^i,\\ &\qquad 0<h-h_s\ll 1. \end{aligned} \tag{2.6}$$

Specifically, if system (1.1) is centrally symmetric, that is,

$$H(-x,-y)=H(x,y),\ f(-x,-y,\delta)=-f(x,y,\delta),\ g(-x,-y,\delta)=-g(x,y,\delta),$$

then it follows that $c^1_{2i}=c^2_{2i}$ and $c^1_{2i+1}=c^2_{2i+1}$.

At this time, (1.1) is also a $\mathbb{Z}_2$-equivariant system. Furthermore, we present the following theorem.

THEOREM 2.2 *Suppose that system (1.1) is centrally symmetric.*

(i) If there exists a $\delta_0\in D$, such that

$$c^1_{2i}(\delta_0)=c^3_{2i}(\delta_0)=c^1_{2j+1}(\delta_0)=0,\ i,j=0,1,\ldots,n-1,$$

$$c^1_{2n}(\delta_0), c^3_{2n}(\delta_0)>0,$$

and

$$\operatorname{rank}\frac{\partial\left(c^1_0,c^3_0,c^1_1,c^1_2,c^3_2,c^1_3,\ldots,c^1_{2n-2},c^3_{2n-2},c^1_{2n-1}\right)}{\partial(\delta_1,\ldots,\delta_s)}(\delta_0)=3n,$$

then system (1.1) can exist 7n limit cycles near the double eight figure loop Γ for some (ε,δ) near $(0,\delta_0)$.

(ii) If there exists a $\delta_0\in D$, such that

$$c^1_{2i}(\delta_0)=c^3_{2j}(\delta_0)=c^1_{2j+1}(\delta_0)=0,\ i=0,1,\ldots,n,\ j=0,1,\ldots,n-1,$$

$$c^3_{2n}(\delta_0)c^1_{2n+1}(\delta_0)<0,$$

and

$$\operatorname{rank}\frac{\partial\left(c^1_0,c^3_0,c^1_1,c^1_2,c^3_2,c^1_3,\ldots,c^1_{2n-2},c^3_{2n-2},c^1_{2n-1},c^1_{2n}\right)}{\partial(\delta_1,\ldots,\delta_s)}(\delta_0)=3n+1,$$

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then system (1.1) can exist $7n+2$ *limit cycles near the double eight figure loop* Γ *for some* (ε, δ) *near* $(0, \delta_0)$*.*

(iii) If there exists a $\delta_0 \in D$*, such that*

$$c_{2i}^1(\delta_0) = c_{2i}^3(\delta_0) = c_{2j+1}^1(\delta_0) = 0,\ i = 0, 1, \ldots, n, j = 0, 1, \ldots, n-1,$$

$$c_{2n+1}^1(\delta_0) \neq 0,$$

and

$$\operatorname{rank} \frac{\partial \left(c_0^1, c_0^3, c_1^1, c_2^1, c_2^3, c_3^1, \ldots, c_{2n}^1, c_{2n}^3\right)}{\partial \left(\delta_1, \ldots, \delta_s\right)} (\delta_0) = 3n+2,$$

then system (1.1) can exist $7n+3$ *limit cycles near the double eight figure loop* Γ *for some* (ε, δ) *near* $(0, \delta_0)$*.*

Proof. The proof follows a similar approach to theorem 1.2 in [17]. For clarity, we will focus on proving part (i) as outlined below. By the assumptions, there exists a $\delta_0 \in D$ such that

$$\left|c_0^1\right| \ll \left|c_1^1\right| \ll \left|c_2^1\right| \ll \cdots \ll \left|c_{2n-2}^1\right| \ll \left|c_{2n-1}^1\right| \ll \left|c_{2n}^1\right|,$$

$$\left|c_0^3\right| \ll \left|c_1^1\right| \ll \left|c_2^3\right| \ll \cdots \ll \left|c_{2n-2}^3\right| \ll \left|c_{2n-1}^1\right| \ll \left|c_{2n}^3\right|,$$

and $c_{4n}^1, c_{4n}^3, c_{4n+1}^1, c_{4n+2}^1, c_{4n+2}^3, c_{4n+3}^1$ with symbols $+, +, -, -, -, +$, respectively.

For $0 < h_s - h \ll 1$, the expansion of $M^1 = M^2$ in (2.6) includes terms such as c_0^1, $c_1^1(h-h_s)\ln|h-h_s|$, $c_2^1(h-h_s)$, $c_3^1(h-h_s)^2\ln|h-h_s|$, ..., which have symbols $+, -, +, -$, respectively, and follow this pattern repetitively. These terms satisfy $\left|c_0^1\right| \ll \left|c_1^1\right| \ll \left|c_2^1\right| \ll \cdots \ll \left|c_{2n-1}^1\right| \ll \left|c_{2n}^1\right|$. As a result, both M^1 and M^2 possess at least $2n$ simple zeros for $0 < h_s - h \ll 1$.

Following a similar analysis as described above, we find that M^3 (resp., M) has at least $2n$ (resp., n) simple zeros for $0 < h_s - h \ll 1$ (resp., $0 < h - h_s \ll 1$). Consequently, system (1.1) has at least $7n$ limit cycles near the double figure eight loop. This concludes the proof of the theorem. □

THEOREM 2.3 *Suppose that system (1.1) is non-centrally symmetric.*

(i) If there exists a $\delta_0 \in D$*, such that*

$$c_i^1(\delta_0) = c_i^2(\delta_0) = c_{2j}^3(\delta_0) = 0,\ i = 0, 1, \ldots, 2n-1,\ j = 0, 1, \ldots, n-1,$$

$$c_{2n}^1(\delta_0), c_{2n}^2(\delta_0), c_{2n}^3(\delta_0) > 0,$$

and

$$\operatorname{rank} \frac{\partial\left(c_0^1, c_0^2, c_0^3, c_1^1, c_1^2, \ldots, c_{2n-2}^1, c_{2n-2}^2, c_{2n-2}^3, c_{2n-1}^1, c_{2n-1}^2\right)}{\partial\left(\delta_1, \ldots, \delta_s\right)}\left(\delta_0\right)=5n,$$

then system (1.1) can exist 7n limit cycles near the double eight figure loop Γ *for some* (ε, δ) *near* $(0, \delta_0)$.

(ii) If there exists a $\delta_0 \in D$, *such that*

$$c_i^1(\delta_0)=c_i^2(\delta_0)= \ c_{2j}^3(\delta_0)=0,\ i=0,1, \ldots, 2n,\ j=0,1, \ldots, n-1,$$

$$c_{2n}^3(\delta_0) c_{2n+1}^1(\delta_0)<0, c_{2n}^3(\delta_0) c_{2n+1}^2(\delta_0)<0,$$

and

$$\operatorname{rank} \frac{\partial\left(c_0^1, c_0^2, c_0^3, c_1^1, c_1^2, \ldots, c_{2n-2}^3, c_{2n-1}^1, c_{2n-1}^2, c_{2n}^1, c_{2n}^2\right)}{\partial\left(\delta_1, \ldots, \delta_s\right)}\left(\delta_0\right)=5n+2,$$

then system (1.1) can exist $7n+2$ *limit cycles near the double eight figure loop* Γ *for some* (ε, δ) *near* $(0, \delta_0)$.

(iii) If there exists a $\delta_0 \in D$, *such that*

$$c_i^1(\delta_0)=c_i^2(\delta_0)=c_{2j}^3(\delta_0)=0,\ i=0,1, \ldots, 2n,\ j=0,1, \ldots, n,$$

$$c_{2n+1}^1(\delta_0) c_{2n+1}^2(\delta_0) \neq 0,$$

and

$$\operatorname{rank} \frac{\partial\left(c_0^1, c_0^2, c_0^3, c_1^1, c_1^2, \ldots, c_{2n}^1, c_{2n}^2, c_{2n}^3\right)}{\partial\left(\delta_1, \ldots, \delta_s\right)}\left(\delta_0\right)=5n+3,$$

then system (1.1) can exist $7n+3$ *limit cycles near the double eight figure loop* Γ *for some* (ε, δ) *near* $(0, \delta_0)$.

Proof. From the assumptions, there exists a $\delta_0 \in D$ such that

$$\begin{aligned}
& \left|c_0^1\right| \ll\left|c_1^1\right| \ll\left|c_2^1\right| \ll \cdots \ll\left|c_{2n-2}^1\right| \ll\left|c_{2n-1}^1\right| \ll\left|c_{2n}^1\right|, \\
& \left|c_0^2\right| \ll\left|c_1^2\right| \ll\left|c_2^2\right| \ll \cdots \ll\left|c_{2n-2}^2\right| \ll\left|c_{2n-1}^2\right| \ll\left|c_{2n}^2\right|, \\
& \left|c_0^3\right| \ll\left|c_1^1+c_1^2\right| \ll\left|c_2^3\right| \ll \cdots \ll\left|c_{2n-2}^3\right| \ll\left|c_{2n-1}^1+c_{2n-1}^2\right| \ll\left|c_{2n}^3\right| .
\end{aligned}$$

And the symbols for $c_{4n}^1, c_{4n}^2, c_{4n}^3, c_{4n+1}^1, c_{4n+1}^2, c_{4n+2}^1, c_{4n+2}^2, c_{4n+2}^3, c_{4n+3}^1, c_{4n+3}^2$ are $+,+,+,-,-,-,-,-,+,+$ respectively.

Similar to the proof of Theorem 2.2, we omit the details. This finishes the proof. □

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3. An application

Note that system (1.6) represents a Hamiltonian system with symmetry along the y-axis, defined by the Hamiltonian $H(x, y) = \frac{1}{2}y^2 + \frac{1}{2}ax^2 - \frac{1}{4}(1 + a)x^4 + \frac{1}{6}x^6$ for $a > 1$. This system possesses five equilibrium points: three elementary centres at $O(0, 0)$, $C_1(\sqrt{a}, 0)$, and $C_2(-\sqrt{a}, 0)$; and two hyperbolic saddles at $S_1(1, 0)$ and $S_2(-1, 0)$. The centres C_1 and C_2 are each encircled by homoclinic loops L_1 and L_2, respectively, with $H(C_1) = H(C_2) = h_c = \frac{1}{4}a^2 - \frac{1}{12}a^3$, and $H(S_1) = H(S_2) = h_s = \frac{1}{4}a - \frac{1}{12}$. The origin $O(0, 0)$ is surrounded by a 2-polycycle Γ^2. Furthermore, for $0 < |h - h_s| \ll 1$, the equation $H(x, y) = h$ defines four families of periodic orbits: $L_1(h)$ and $L_2(h)$ for $h_c < h < h_s$, $L_3(h)$ for $0 < h < h_s$, and $L_4(h)$ for $h > h_s$. Figure 1 illustrates the phase portrait of $(1.6)|_{\varepsilon=0}$. We have the following theorem.

THEOREM 3.1 *For all $n \geq 5$, the Liénard system (1.6) can exist $7[\frac{n}{6}] + 2[\frac{r}{2}] - [\frac{r}{4}]$ limit cycles near the double eight figure loop for some (ε, δ) near $(0, \delta_0)$, where $r = \mathrm{mod}(\mathrm{n}, 6)$.*

REMARK 3.2. To the best of our knowledge, this number of limit cycles is maximal that we have been able to find so far near the double eight figure loop. Xu and Li [35] (resp., Xiong and Zhong [34]) proved that the system (1.5) (resp., (1.4)), with $f(x) = \sum_{i=0}^{5} a_i x^{2i}$ (resp., $\sum_{i=0}^{4} a_i x^{2i}$), has 10 (resp., 9) limit cycles near the double eight figure loop, which is consistent with taking $n = 10$ (resp., $n = 8$) in the Theorem 3.1 for the more general system (1.6).

Let $I_k^i(h) = \oint_{L_i(h)} x^k y dx$ and $I_k(h) = \oint_{L(h)} x^k y dx$ be integrals over the curves $L_i(h)$ and $L(h)$, respectively, as defined in §2 and illustrated in Figure 1. Based on the classification of these curves, we derive the following four first-order Melnikov functions.

LEMMA 3.3. *For $n \geq 5$ and $0 < |h - h_s| \ll 1$.*

(i) If $L_i(h)$ near the homoclinic loop L_i for $i = 1, 2$, and $L_i(h)$ near the 2-polycycle Γ^2 for $i = 3$, $M^i(h)$ can be written as

$$M^i(h) = \sum_{k=0}^{4} P_k(h) I_k^i(h), \quad for \quad i = 1, 2, 3. \tag{3.1}$$

(ii) If L(h) near the double eight figure loop Γ, one has

$$M(h) = \sum_{k=0}^{4} P_k(h) I_k(h), \tag{3.2}$$

where $P_k(h)$ are polynomials of h, $\deg P_i(h) \leqslant [\frac{n-k}{6}]$ for $k = 0, 1, 2, 3, 4$. The notation $[s]$ is defined as the integer part of s.

In particular, the terms $I_1(h), I_3(h), I_1^i(h)$, and $I_3^i(h)$ $(i = 1, 2, 3)$ in Eqs. (3.1) and (3.2) will not appear if $f(-x) = f(x)$.

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Proof. The idea of proof is similar to the proposition 2.4 in [44]. From the definition of $\{L_1(h), L_2(h), L_3(h), L(h)\} \triangleq \Gamma_h$, it can be observed that

$$\frac{1}{2}y^2 + \frac{1}{2}ax^2 - \frac{1}{4}(1+a)x^4 + \frac{1}{6}x^6 = h. \tag{3.3}$$

Multiplying the (3.3) by $x^l y^m$ and integrating it over Γ_h, we have

$$\frac{1}{2}I_{l,m+2} + \frac{1}{2}aI_{l+2,m} - \frac{1}{4}(1+a)I_{l+4,m} + \frac{1}{6}I_{l+6,m} = hI_{l,m}, \tag{3.4}$$

where $I_{i,j} = \oint_{\Gamma_h} x^i y^j \, dx$.

It follows from (3.3) that

$$y\frac{\partial y}{\partial x} + ax - (1+a)x^3 + x^5 = 0. \tag{3.5}$$

Multiplying (3.5) by $x^{l-5}y^m$ $(l \geq 5)$ and integrating it over Γ_h by parts, one has

$$-\frac{l-5}{m+2}I_{l-6,m+2} + aI_{l-4,m} - (1+a)I_{l-2,m} + I_{l,m} = 0. \tag{3.6}$$

Hence by (3.4) and (3.6), we obtain

$$I_{l+6,m} = \frac{3(2m+l+5)}{2(3m+l+7)}(1+a)I_{l+4,m} - \frac{3(m+l+3)}{3m+l+7}aI_{l+2,m} + \frac{6(l+1)}{3m+l+7}hI_{l,m}. \tag{3.7}$$

If $l = 5$ in (3.6), it is direct that

$$I_{5,m} = (1+a)I_{3,m} - aI_{1,m}.$$

Therefore, it follows from taking $m = 1$ in (3.7) that the generators of $M(h)$ (resp., $M^i(h)$) are $I_0(h), I_1(h)$, $I_2(h)$, $I_3(h)$, and $I_4(h)$ (resp., $I_0^i(h), I_1^i(h)$, $I_2^i(h)$, $I_3^i(h)$, and $I_4^i(h)$). By induction in n, it is easy to see the dimensions of $P_0(h)$, $P_1(h)$, $P_2(h)$, $P_3(h)$, and $P_4(h)$.

It can be proved in a similar way if $f(-x) = f(x)$. This finishes the proof. □

Proof of Theorem 3.1. From Lemma 3.3, we suppose for $k = 0, 1, 2, 3, 4$ that

$$P_k(h) = m_0^k + m_1^k(h-h_s) + m_2^k(h-h_s)^2 + \cdots + m^k_{[\frac{n-k}{6}]}(h-h_s)^{[\frac{n-k}{6}]}. \tag{3.8}$$

Setting $f(x) = \Sigma_0^n a_i x^i$, it is easy to verify that

$$\left| \frac{\partial\left(m_0^0, m_0^1, m_0^2, m_0^3, m_0^4, \ldots, m^0_{[\frac{n}{6}]}, m^1_{[\frac{n-1}{6}]}, m^2_{[\frac{n-2}{6}]}, m^3_{[\frac{n-3}{6}]}, m^4_{[\frac{n-4}{6}]}\right)}{\partial\left(a_0, a_1, a_2, a_3, a_4, a_6, a_7, a_8, a_9, a_{10}, \ldots, a_n\right)} \right| \neq 0,$$

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where $5 \neq mod(n,6)$. If $f(-x) = f(x)$, one has

$$\left| \frac{\partial \left(m_0^0, m_0^2, m_0^4, \ldots, m^0_{[\frac{n}{6}]}, m^2_{[\frac{n-2}{6}]}, m^4_{[\frac{n-4}{6}]} \right)}{\partial \left(a_0, a_2, a_4, a_6, a_8, a_{10}, \ldots, a_n \right)} \right| \neq 0,$$

where n is even. And hence, the coefficients in (3.8) are independent.

For $0 < |h - h_s| \ll 1$, assume that $I_k(h)$ and $I_k^i(h)$ for $i = 1, 2, 3$ in Lemma 3.3 can be represented as

$$\begin{aligned} I_k^i(h) &= a_{k,0}^i + \widetilde{a}_{k,1}^i (h - h_s) \ln |h - h_s| + \cdots \\ &\quad + \widetilde{a}_{k,n}^i (h - h_s)^n \ln |h - h_s| + a_{k,n}^i (h - h_s)^n + O((h - h_s)^n), \\ I_k(h) &= a_{k,0}^4 + \widetilde{a}_{k,1}^4 (h - h_s) \ln |h - h_s| + \cdots \\ &\quad + \widetilde{a}_{k,n}^4 (h - h_s)^n \ln |h - h_s| + a_{k,n}^4 (h - h_s)^n + O((h - h_s)^n). \end{aligned} \tag{3.9}$$

(i) Centrally symmetric. If $[\frac{n}{6}] = [\frac{n-2}{6}] = [\frac{n-4}{6}] = s$ in (3.8), we set

$$\begin{aligned} \boldsymbol{\delta}^1 &\triangleq (m_0^0, m_0^2, m_0^4, m_1^0, m_1^2, m_1^4, \ldots, m_s^0, m_s^2, m_s^4), \\ \boldsymbol{E}_1 &\triangleq \frac{\partial \left(c_0^1, c_0^3, c_1^1, c_2^1, c_2^3, c_3^1, \ldots, c_{2s}^1, c_{2s}^3, c_{2s+1}^1 \right)}{\partial \boldsymbol{\delta}^1}, \end{aligned}$$

where the coefficients c_i^j appear in (2.6).

Substituting (3.8) and (3.9) into (3.1), it follows from (2.6) that

$$\boldsymbol{E}_1 = \begin{pmatrix} a_{0,0}^1 & a_{2,0}^1 & a_{4,0}^1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_{0,0}^3 & a_{2,0}^3 & a_{4,0}^3 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \widetilde{a}_{0,1}^1 & \widetilde{a}_{2,1}^1 & \widetilde{a}_{4,1}^1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_{0,1}^1 & a_{2,1}^1 & a_{4,1}^1 & a_{0,0}^1 & a_{2,0}^1 & a_{4,0}^1 & \cdots & 0 & 0 & 0 \\ a_{0,1}^3 & a_{2,1}^3 & a_{4,1}^3 & a_{0,0}^3 & a_{2,0}^3 & a_{4,0}^3 & \cdots & 0 & 0 & 0 \\ \widetilde{a}_{0,2}^1 & \widetilde{a}_{2,2}^1 & \widetilde{a}_{4,2}^1 & \widetilde{a}_{0,1}^1 & \widetilde{a}_{2,1}^1 & \widetilde{a}_{4,1}^1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{0,s}^1 & a_{2,s}^1 & a_{4,s}^1 & a_{0,s-1}^1 & a_{2,s-1}^1 & a_{4,s-1}^1 & \cdots & a_{0,0}^1 & a_{2,0}^1 & a_{4,0}^1 \\ a_{0,s}^3 & a_{2,s}^3 & a_{4,s}^3 & a_{0,s-1}^3 & a_{2,s-1}^3 & a_{4,s-1}^3 & \cdots & a_{0,0}^3 & a_{2,0}^3 & a_{4,0}^3 \\ \widetilde{a}_{0,s+1}^1 & \widetilde{a}_{2,s+1}^1 & \widetilde{a}_{4,s+1}^1 & \widetilde{a}_{0,s}^1 & \widetilde{a}_{2,s}^1 & \widetilde{a}_{4,s}^1 & \cdots & \widetilde{a}_{0,1}^1 & \widetilde{a}_{2,1}^1 & \widetilde{a}_{4,1}^1 \end{pmatrix}.$$

Let $\boldsymbol{F}_0 = \begin{pmatrix} a_{0,0}^1 & a_{2,0}^1 & a_{4,0}^1 \\ a_{0,0}^3 & a_{2,0}^3 & a_{4,0}^3 \\ \widetilde{a}_{0,1}^1 & \widetilde{a}_{2,1}^1 & \widetilde{a}_{4,1}^1 \end{pmatrix}$, $A(x) = \sqrt{\frac{1}{2}a - \frac{1}{6} - ax^2 + \frac{1}{2}(1+a)x^4 - \frac{1}{3}x^6}$ and $\gamma_1 = \arcsin\left(\frac{\sqrt{2}}{\sqrt{3a-1}}\right)$, where $\widetilde{a}_{0,1}^1 = \widetilde{a}_{2,1}^1 = \widetilde{a}_{4,1}^1 = -\frac{1}{\sqrt{2(a-1)}}$,

$$\begin{aligned}
a_{0,0}^1 =& 2\int_1^{\frac{1}{2}\sqrt{6a-2}} A(x)\mathrm{d}x \\
=& \frac{3\sqrt{2}}{16}\sqrt{a-1}(a+1) - \frac{\sqrt{3}}{16}(3a-1)(a-3)\left(\gamma_1 - \frac{\pi}{2}\right), \\
a_{2,0}^1 =& 2\int_1^{\frac{1}{2}\sqrt{6a-2}} x^2 A(x)\mathrm{d}x \\
=& \frac{\sqrt{2}}{64}\sqrt{a-1}(9a^2 - 14a + 9) - \frac{\sqrt{3}}{192}(3\,a-5)(3a-1)^2\left(\gamma_1 - \frac{\pi}{2}\right), \\
a_{4,0}^1 =& 2\int_1^{\frac{1}{2}\sqrt{6a-2}} x^4 A(x)\mathrm{d}x \\
=& \frac{3\sqrt{2}}{1024}\sqrt{a-1}\left(45\,a^3 - 73\,a^2 + 23\,a + 13\right) - \frac{\sqrt{3}}{1024}(5a-7)(3a-1)^3\left(\gamma_1 - \frac{\pi}{2}\right), \\
a_{0,0}^3 =& 2\int_{-1}^1 A(x)\mathrm{d}x \\
=& \frac{3\sqrt{2}}{8}\sqrt{a-1}(a+1) - \frac{\sqrt{3}}{8}(3a-1)(a-3)\gamma_1, \\
a_{2,0}^3 =& 2\int_{-1}^1 x^2 A(x)\mathrm{d}x \\
=& \frac{\sqrt{2}}{32}\sqrt{a-1}\left(9\,a^2 - 14\,a + 9\right) - \frac{\sqrt{3}}{96}(3\,a-5)(3\,a-1)^2\gamma_1, \\
a_{4,0}^3 =& 2\int_{-1}^1 x^4 A(x)\mathrm{d}x \\
=& \frac{3\sqrt{2}}{512}\sqrt{a-1}\left(45\,a^3 - 73\,a^2 + 23\,a + 13\right)\sqrt{3} - \frac{\sqrt{3}}{512}(5\,a-7)(3\,a-1)^3\gamma_1.
\end{aligned}$$

From $|\boldsymbol{F}_0| = -\frac{3\sqrt{3}}{128}\pi\,(3\,a-1)(a-1)^3 \neq 0$, it follows that $|\boldsymbol{E}_1| = |\boldsymbol{F}_0|^{s+1} \neq 0$. By Theorem 2.2 (iii), the system (1.6) has at least $7[\frac{n}{6}] + 3$ limit cycles near $h = h_s$.

For $[\frac{n}{6}] = [\frac{n-2}{6}] = [\frac{n-4}{6}] + 1 = s$, there are $3s+2$ free coefficients m_0^0, m_0^2, m_0^4, m_1^0, m_1^2, m_1^4, $\ldots$, m_{s-1}^0, $m_{s-1}^2, m_{s-1}^4, m_s^0, m_s^2$. Let

$$\begin{aligned}
\boldsymbol{\delta}^2 &\triangleq (m_0^0, m_0^2, m_0^4, m_1^0, m_1^2, m_1^4, \ldots, m_{s-1}^0, m_{s-1}^2, m_{s-1}^4, m_s^0), \\
\boldsymbol{E}_2 &\triangleq \frac{\partial\left(c_0^1, c_0^3, c_1^1, c_2^1, c_2^3, c_3^1, \ldots, c_{2s-2}^1, c_{2s-2}^3, c_{2s-1}^1, c_{2s}^1\right)}{\partial\boldsymbol{\delta}^2}.
\end{aligned}$$

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It can be seen that $|\boldsymbol{E}_2| = a_{0,0}^1 \cdot |\boldsymbol{F}_0|^s \neq 0$. Note that $c_{2s}^3 c_{2s+1}^1 = (a_{2,0}^3 m_s^2)(\widetilde{a}_{2,1}^1 m_s^2) < 0$ when $\boldsymbol{\delta}^2 = \mathbf{0}$. Applying Theorem 2.2 (ii), the system has at least $7[\frac{n}{6}] + 2$ limit cycles near $h = h_s$.

If $[\frac{n}{6}] = [\frac{n-2}{6}] + 1 = [\frac{n-4}{6}] + 1 = s$, it implies that the $3s+1$ coefficients $m_0^0, m_0^2, m_0^4, m_1^0, m_1^2, m_1^4, \ldots, m_{s-1}^0, m_{s-1}^2, m_{s-1}^4, m_s^0$ are free. Let

$$\begin{aligned}
\boldsymbol{\delta}^3 &\triangleq (m_0^0, m_0^2, m_0^4, m_1^0, m_1^2, m_1^4, \ldots, m_{s-1}^0, m_{s-1}^2, m_{s-1}^4), \\
\boldsymbol{E}_3 &\triangleq \frac{\partial \left(c_0^1, c_0^3, c_1^1, c_2^1, c_2^3, c_3^1, \ldots, c_{2s-2}^1, c_{2s-2}^3, c_{2s-1}^1\right)}{\partial \boldsymbol{\delta}^3},
\end{aligned}$$

then we obtain $|\boldsymbol{E}_3| = |\boldsymbol{F}_0|^s \neq 0$. In addition, it is not hard to verify that $c_{2s}^1 = a_{0,0}^1 m_s^0$, $c_{2s}^3 = a_{0,0}^3 m_s^0$, $c_{2s}^4 = c_{2s}^1 + c_{2s}^3 = (a_{0,0}^1 + a_{0,0}^3) m_s^0$ when $\boldsymbol{\delta}^3 = \mathbf{0}$. Since the parameter m_s^0 is free and $a_{0,0}^1, a_{0,0}^3 > 0$, we suppose $c_{2s}^1, c_{2s}^3, c_{2s}^4 > 0$. Utilizing Theorem 2.2 (i), the system has at least $7[\frac{n}{6}]$ limit cycles near $h = h_s$.

(ii) Non-centrally symmetric. Set $[\frac{n}{6}] = [\frac{n-1}{6}] = [\frac{n-2}{6}] = [\frac{n-3}{6}] = [\frac{n-4}{6}] = s$ in (3.8) and

$$\begin{aligned}
\widehat{\boldsymbol{\delta}}^1 &\triangleq (m_0^0, m_0^1, m_0^2, m_0^3, m_0^4, \ldots, m_s^0, m_s^1, m_s^2, m_s^3, m_s^4), \\
\widehat{\boldsymbol{E}}_1 &\triangleq \frac{\partial \left(c_0^1, c_0^2, c_0^3, c_1^1, c_1^2, \ldots, c_{2s}^1, c_{2s}^2, c_{2s}^3, c_{2s+1}^1, c_{2s+1}^2\right)}{\partial \widehat{\boldsymbol{\delta}}^1}.
\end{aligned}$$

Combining with (2.6) and substituting (3.8) and (3.9) into (3.1), one determines that $\widehat{\boldsymbol{E}}_1$ can be expressed as

$$\begin{pmatrix}
a_{0,0}^1 & a_{1,0}^1 & a_{2,0}^1 & a_{3,0}^1 & a_{4,0}^1 & \cdots & 0 & 0 & 0 & 0 & 0 \\
a_{0,0}^1 & -a_{1,0}^1 & a_{2,0}^1 & -a_{3,0}^1 & a_{4,0}^1 & \cdots & 0 & 0 & 0 & 0 & 0 \\
a_{0,0}^3 & 0 & a_{2,0}^3 & 0 & a_{4,0}^3 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\widetilde{a}_{0,1}^1 & \widetilde{a}_{1,1}^1 & \widetilde{a}_{2,1}^1 & \widetilde{a}_{3,1}^1 & \widetilde{a}_{4,1}^1 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\widetilde{a}_{0,1}^1 & -\widetilde{a}_{1,1}^1 & \widetilde{a}_{2,1}^1 & -\widetilde{a}_{3,1}^1 & \widetilde{a}_{4,1}^1 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{0,s}^1 & a_{1,s}^1 & a_{2,s}^1 & a_{3,s}^1 & a_{4,s}^1 & \cdots & a_{0,0}^1 & a_{1,0}^1 & a_{2,0}^1 & a_{3,0}^1 & a_{4,0}^1 \\
a_{0,s}^1 & -a_{1,s}^1 & a_{2,s}^1 & -a_{3,s}^1 & a_{4,s}^1 & \cdots & a_{0,0}^1 & -a_{1,0}^1 & a_{2,0}^1 & -a_{3,0}^1 & a_{4,0}^1 \\
a_{0,s}^3 & 0 & a_{2,s}^3 & 0 & a_{4,s}^3 & \cdots & a_{0,0}^3 & 0 & a_{2,0}^3 & 0 & a_{4,0}^3 \\
\widetilde{a}_{0,s+1}^1 & \widetilde{a}_{1,s+1}^1 & \widetilde{a}_{2,s+1}^1 & \widetilde{a}_{3,s+1}^1 & \widetilde{a}_{4,s+1}^1 & \cdots & \widetilde{a}_{0,1}^1 & \widetilde{a}_{1,1}^1 & \widetilde{a}_{2,1}^1 & \widetilde{a}_{3,1}^1 & \widetilde{a}_{4,1}^1 \\
\widetilde{a}_{0,s+1}^1 & -\widetilde{a}_{1,s+1}^1 & \widetilde{a}_{2,s+1}^1 & -\widetilde{a}_{3,s+1}^1 & \widetilde{a}_{4,s+1}^1 & \cdots & \widetilde{a}_{0,1}^1 & -\widetilde{a}_{1,1}^1 & \widetilde{a}_{2,1}^1 & -\widetilde{a}_{3,1}^1 & \widetilde{a}_{4,1}^1
\end{pmatrix}.$$

Let $\widehat{\boldsymbol{F}}_0 = \begin{pmatrix}
a_{0,0}^1 & a_{1,0}^1 & a_{2,0}^1 & a_{3,0}^1 & a_{4,0}^1 \\
a_{0,0}^1 & -a_{1,0}^1 & a_{2,0}^1 & -a_{3,0}^1 & a_{4,0}^1 \\
a_{0,0}^3 & 0 & a_{2,0}^3 & 0 & a_{4,0}^3 \\
\widetilde{a}_{0,1}^1 & \widetilde{a}_{1,1}^1 & \widetilde{a}_{2,1}^1 & \widetilde{a}_{3,1}^1 & \widetilde{a}_{4,1}^1 \\
\widetilde{a}_{0,1}^1 & -\widetilde{a}_{1,1}^1 & \widetilde{a}_{2,1}^1 & -\widetilde{a}_{3,1}^1 & \widetilde{a}_{4,1}^1
\end{pmatrix}$, in which all coefficients of $\widehat{\boldsymbol{F}}_0$, except for the following six, have been determined under the condition of central symmetry,

$$
\begin{aligned}
a_{1,0}^1 =&2\int_1^{\frac{1}{2}\sqrt{6a-2}} xA(x)\mathrm{d}x=\frac{3\sqrt{2}}{10}\sqrt{a-1}(a-1)^2,\\
a_{3,0}^1 =&2\int_1^{\frac{1}{2}\sqrt{6a-2}} x^3A(x)\mathrm{d}x=\frac{3\sqrt{2}}{70}\sqrt{a-1}(6a+1)(a-1)^2,\\
a_{1,0}^3 =&2\int_{-1}^{1} xA(x)\mathrm{d}x=0,\ a_{3,0}^3=2\int_{-1}^{1} x^3A(x)\mathrm{d}x=0,\ \widetilde{a}_{1,1}^1=\widetilde{a}_{3,1}^1=-\frac{1}{\sqrt{2(a-1)}}.
\end{aligned}
$$

It follows that $|\widehat{\boldsymbol{F}}_0|=\frac{27\sqrt{3}}{1120}\pi(3a-1)(a-1)^6\neq 0$, which gives $|\widehat{\boldsymbol{E}}_1|=|\widehat{\boldsymbol{F}}_0|^{s+1}\neq 0$. By Theorem 2.3 (iii), the system (1.6) has at least $7[\frac{n}{6}]+3$ limit cycles near $h=h_s$.

If $[\frac{n}{6}]=[\frac{n-1}{6}]=[\frac{n-2}{6}]=[\frac{n-3}{6}]=[\frac{n-4}{6}]+1=s$, the $5s+4$ coefficients m_0^0, m_0^1, m_0^2, m_0^3, m_0^4, $\ldots$, m_{s-1}^0, $m_{s-1}^1, m_{s-1}^2, m_{s-1}^3, m_{s-1}^4, m_s^0, m_s^1, m_s^2, m_s^3$ are independent. Let

$$
\begin{aligned}
\widehat{\boldsymbol{\delta}}^2 &\triangleq (m_0^0,m_0^1,m_0^2,m_0^3,m_0^4,\ldots,m_{s-1}^0,m_{s-1}^1,m_{s-1}^2,m_{s-1}^3,m_{s-1}^4,m_s^0,m_s^1),\\
\widehat{\boldsymbol{E}}_2 &\triangleq \frac{\partial\left(c_0^1,c_0^2,c_0^3,c_1^1,c_1^2,\ldots,c_{2s-2}^1,c_{2s-2}^2,c_{2s-2}^3,c_{2s-1}^1,c_{2s-1}^2,c_{2s}^1,c_{2s}^2\right)}{\partial\widehat{\boldsymbol{\delta}}^2}.
\end{aligned}
$$

It is not difficult to verify that

$$
|\widehat{\boldsymbol{E}}_2|=\begin{vmatrix} a_{0,0}^1 & a_{1,0}^1\\ a_{0,0}^1 & -a_{1,0}^1\end{vmatrix}\cdot|\widehat{\boldsymbol{F}}_0|^s\neq 0.
$$

For m_s^2 and m_s^3, one has

$$
c_{2s}^3=a_{2,0}^3m_s^2,\quad c_{2s+1}^1=\widetilde{a}_{2,1}^1m_s^2+\widetilde{a}_{3,1}^1m_s^3,\quad c_{2s+1}^2=\widetilde{a}_{2,1}^1m_s^2-\widetilde{a}_{3,1}^1m_s^3,
$$

when $\widehat{\boldsymbol{\delta}}^2=\mathbf{0}$. Setting $|m_s^3|\ll|m_s^2|$, one has $c_{2s}^3c_{2s+1}^1<0, c_{2s}^3c_{2s+1}^2<0$ from $a_{2,0}^3\widetilde{a}_{2,1}^1<0$. Taking Theorem 2.3 (ii) into account, the system (1.6) has at least $7[\frac{n}{6}]+2$ limit cycles near $h=h_s$.

When $[\frac{n}{6}]=[\frac{n-1}{6}]=[\frac{n-2}{6}]=[\frac{n-3}{6}]+1=[\frac{n-4}{6}]+1=s$, we have $5s+3$ free coefficients $m_0^0,m_0^1,m_0^2,m_0^3,m_0^4$, $\ldots$, m_{s-1}^0,m_{s-1}^1, $m_{s-1}^2,m_{s-1}^3,m_{s-1}^4,m_s^0,m_s^1,m_s^2$. Owing to $|\widehat{\boldsymbol{E}}_2|\neq 0$, we will consider m_s^2 below. If $\widehat{\boldsymbol{\delta}}^2=\mathbf{0}$, it holds that

$$
c_{2s}^3=a_{2,0}^3m_s^2,\quad c_{2s+1}^1=\widetilde{a}_{2,1}^1m_s^2,\quad c_{2s+1}^2=\widetilde{a}_{2,1}^1m_s^2,
$$

which imply $c_{2s}^3c_{2s+1}^1<0, c_{2s}^3c_{2s+1}^2<0$. From Theorem 2.3 (ii), the system (1.6) has at least $7[\frac{n}{6}]+2$ limit cycles near $h=h_s$.

For $[\frac{n}{6}]=[\frac{n-1}{6}]=[\frac{n-2}{6}]+1=[\frac{n-3}{6}]+1=[\frac{n-4}{6}]+1=s$, there are $5s+2$ free coefficients m_0^0,m_0^1,m_0^2, m_0^3,m_0^4, $\ldots$, $m_{s-1}^0,m_{s-1}^1,m_{s-1}^2,m_{s-1}^3,m_{s-1}^4,m_s^0,m_s^1$. Let

$$
\begin{aligned}
\widehat{\boldsymbol{\delta}}^3 &\triangleq (m_0^0,m_0^1,m_0^2,m_0^3,m_0^4,\ldots,m_{s-1}^0,m_{s-1}^1,m_{s-1}^2,m_{s-1}^3,m_{s-1}^4),\\
\widehat{\boldsymbol{E}}_3 &\triangleq \frac{\partial\left(c_0^1,c_0^2,c_0^3,c_1^1,c_1^2,\ldots,c_{2s-2}^1,c_{2s-2}^2,c_{2s-2}^3,c_{2s-1}^1,c_{2s-1}^2\right)}{\partial\widehat{\boldsymbol{\delta}}^3},
\end{aligned}
$$

and hence $|\widehat{\boldsymbol{E}}_3| = |\widehat{\boldsymbol{F}}_0|^s \neq 0$. It follows that

$$\begin{aligned} c_{2s}^1 &= a_{0,0}^1 m_s^0 + a_{1,0}^1 m_s^1, \quad c_{2s}^2 = a_{0,0}^1 m_s^0 - a_{1,0}^1 m_s^1, \\ c_{2s}^3 &= a_{0,0}^3 m_s^0, \quad c_{2s}^4 = c_{2s}^1 + c_{2s}^2 + c_{2s}^3 = (2a_{0,0}^1 + a_{0,0}^3) m_s^0, \end{aligned}$$

when $\widehat{\boldsymbol{\delta}}^3 = \mathbf{0}$. By virtue of the fact that the coefficients m_s^0 and m_s^1 are independent, we assume $|m_s^1| \ll |m_s^0|$. Naturally, we arrive at $c_{2s}^1, c_{2s}^2, c_{2s}^3, c_{2s}^4 > 0$ from $a_{0,0}^1, a_{0,0}^3 > 0$. By Theorem 2.3 (i), the system (1.6) has at least $7[\frac{n}{6}]$ limit cycles near $h = h_s$.

If $[\frac{n}{6}] = [\frac{n-1}{6}] + 1 = [\frac{n-2}{6}] + 1 = [\frac{n-3}{6}] + 1 = [\frac{n-4}{6}] + 1 = s$, we gain $5s + 1$ free coefficients $m_0^0, m_0^1, m_0^2, m_0^3, m_0^4, \ldots, m_{s-1}^0, m_{s-1}^1, m_{s-1}^2, m_{s-1}^3, m_{s-1}^4, m_s^0$. By $|\widehat{\boldsymbol{E}}_3| = |\widehat{\boldsymbol{F}}_0|^s \neq 0$, we first consider the coefficient m_s^0. It follows from $\widehat{\boldsymbol{\delta}}^3 = \mathbf{0}$ that

$$\begin{aligned} c_{2s}^1 &= a_{0,0}^1 m_s^0, \quad c_{2s}^2 = a_{0,0}^1 m_s^0, \\ c_{2s}^3 &= a_{0,0}^3 m_s^0, \quad c_{2s}^4 = c_{2s}^1 + c_{2s}^2 + c_{2s}^3 = (2a_{0,0}^1 + a_{0,0}^3) m_s^0. \end{aligned}$$

Combining m_s^0 is independent and $a_{0,0}^1, a_{0,0}^3 > 0$, one gets $c_{2s}^1, c_{2s}^2, c_{2s}^3, c_{2s}^4 > 0$. As a result of Theorem 2.3 (i), the system (1.6) has at least $7[\frac{n}{6}]$ limit cycles near $h = h_s$. □

4. Conclusions

A double eight figure loop is one of the common topological structures in differential system. Moreover, one can utilize it to investigate the simultaneous existence of two (homoclinic loop and heteroclinic loop) bifurcations. This article we establish a general theory to find the lower bound of the maximal number of limit cycles near the double eight figure loop with hyperbolic saddles. In addition, the new approach facilitates the computation of expressions for higher-order coefficients in the expansion of the first-order Melnikov function (Abelian integral) $M(h)$ near the double eight figure loop by finding the algebraic structure (generators) of $M(h)$, and the conclusion gained for the limit cycles can be valid for the perturbation with any degree n.

As an application of our theory and inspired by [4, 34, 35, 37, 40, 43], we study the number of limit cycles in an $(n+1)$th-order generalized Liénard differential system, whose unperturbed system is a Hamiltonian with double eight figure loop passing two hyperbolic saddles. Besides the fact that system (1.6) can contain the two systems (1.4) and (1.5), and the result for the limit cycles holds for any $n \geq 5$.

After receiving notification of the acceptance of the article, we noticed Yang and Han's recent work [36], which also studied the problem of limit cycles near a double eight figure loop.

4. Declaration of competing interest

There is no competing interest.

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