

AN ASPECT OF THE INVARIANT OF DEGREE 4 OF THE BINARY QUINTIC

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Introduction

1. A binary form of odd degree,

$$f(x, y) \equiv \sum_{r=0}^{2m+1} \binom{2m+1}{r} a_r x^{2m+1-r} y^r \equiv a_x^{2m+1} \equiv b_x^{2m+1},$$

has a quadratic covariant Γ , $(ab)^{2m} a_x b_x$ in Aronhold's notation, and the discriminant Δ of Γ is an invariant of f . For $m=2$ Δ was obtained by Cayley in 1856 [3, p. 274]; it was curiosity as to how Δ could be interpreted geometrically that triggered the writing of this note. An interpretation, in projective space $[2m+1]$, that does not seem to be on record, of Γ and Δ is found below. If $m=1$ one has merely the Hessian and discriminant of a binary cubic whose interpretations in the geometry of the twisted cubic are widely known [5, pp. 241-2].

There is a standard mapping

$$f(x, y) \equiv \sum_{r=0}^n \binom{n}{r} a_r x^{n-r} y^r \rightarrow (a_0, a_1, \dots, a_n)$$

of a binary n -ic onto a point P in a projective space $[n]$ wherein (x_0, x_1, \dots, x_n) are homogeneous coordinates. The focus, so to call it, of this map is the rational normal curve C in Clifford's canonical form

$$x_i = (-t)^i, \quad i = 0, 1, \dots, n \tag{1.1}$$

its points map those f that are perfect n th powers. For $n=4$ the geometry of the rational normal quartic was used by Brusotti [2] to interpret the concomitants of a binary quartic, but only odd integers $n=2m+1$ concern us here. For $m=2$ a note [8] by Todd is relevant but his account does not involve, as ours will, the manifolds generated by the spaces osculating C .

These osculating spaces dominate a paper [1] by Baker who shows that the coefficients in the polarised form $(ab)^{2r} a_x^{n-2r} b_y^{n-2r}$ provide all the quadrics containing all the osculating $[r-1]$ of C . Our concern is restricted here to $n=2m+1$, $r=m$; this

produces a net N of quadrics. But we identify the singular quadrics of N , their envelope being our main objective.

The rational normal quintic and the net of quadrics containing all its tangents

2. Suppose therefore, until after Section 6, that $n = 5$ in (1.1). The chordal [4] spanned by the five points $t = \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ of C is

$$e_5x_0 + e_4x_1 + e_3x_2 + e_2x_3 + e_1x_4 + x_5 = 0$$

where

$$\theta^5 - e_1\theta^4 + e_2\theta^3 - e_3\theta^2 + e_4\theta - e_5 \equiv \prod_{j=0}^4 (\theta - \alpha_j),$$

and this is so whatever confluences may occur among the α_j . In particular $\omega_4(\alpha)$, the osculating [4] at $t = \alpha$, is

$$\alpha^5x_0 + 5\alpha^4x_1 + 10\alpha^3x_2 + 10\alpha^2x_3 + 5\alpha x_4 + x_5 = 0 \tag{2.1}$$

so that the zeros x/y of f are the parameters of the contacts of those ω_4 that contain P . Osculating spaces of lower dimension are identified by simultaneous linear equations; for example, three equations identifying $\omega_2(\alpha)$ are

$$\alpha^3x_k + 3\alpha^2x_{k+1} + 3\alpha x_{k+2} + x_{k+3} = 0, \quad k = 0, 1, 2. \tag{2.2}$$

The basic geometry of C is described in [6], its ranks being given on p. 95. The facts are that

- the tangents ω_1 generate a scroll Ω_2^8 ,
- the osculating planes ω_2 generate a threefold Ω_3^9 ,
- the osculating solids ω_3 generate a primal Ω_4^8 .

It is sometimes convenient to homogenise (1.1) as $x_i = (-1)^i u^i v^{5-i}$; then partial differentiations produce points spanning ω_1, ω_2 , and so on. In particular $\omega_1(t)$ is spanned by

$$a: 0, \quad -1, \quad 2t, \quad -3t^2, \quad 4t^3, \quad -5t^4 \tag{2.3}$$

and

$$b: 5, \quad -4t, \quad 3t^2, \quad -2t^3, \quad t^4, \quad 0.$$

3. Consider now the possibility of Ω_2^8 lying on a quadric $Q: \sum a_{rs}x_r x_s = 0$; the matrix (a_{rs}) is symmetric, $a_{rs} = a_{sr}$, both r and s running from 0 to 5; products $x_r x_s (r \neq s)$ each occur twice, squares only once. If $\omega_1(t)$ is on Q then Q contains both a and b —conditions I_1 and I_2 of incidence—while a, b are also conjugate—condition J . All three conditions are linear in the a_{rs} ; each demands that an octavic polynomial in t is zero,

while if all ω_1 are on Q these polynomials are to be zero *identically*, whatever t . But in any of I_1, I_2, J those a_{rs} with the same $r+s$ multiply the same power of t , this power being $r+s-2$ in I_1 (wherein no a_{0s} occurs), $r+s$ in I_2 and $r+s-1$ in J (wherein a_{00} does not occur); so parallels to the secondary diagonal of (a_{rs}) can be handled independently. A glance at the information provided by the lower and higher values of $r+s$ will show that all a_{rs} are zero for which $r+s$ is any of 0, 1, 2, 3; 7, 8, 9, 10 so that the only non-zero a_{rs} are those in the secondary diagonal ($r+s=5$) itself and the two contiguous parallels ($r+s=4, 6$).

The three conditions are

$$I_1: 0 = a_{11} - 4a_{12}t + (4a_{22} + 6a_{13})t^2 - (8a_{14} + 12a_{23})t^3 + \dots$$

$$J: 0 = -5a_{01} + (4a_{11} + 10a_{02})t - (11a_{12} + 15a_{03})t^2 + (20a_{04} + 14a_{13} + 6a_{22})t^3 + \dots$$

$$I_2: 0 = 25a_{00} - 40a_{01}t + (16a_{11} + 30a_{02})t^2 - (20a_{03} + 24a_{12})t^3 + (10a_{04} + 16a_{13} + 9a_{22})t^4 + \dots$$

giving in succession

$$a_{11} = a_{01} = a_{00} = a_{12} = a_{02} = a_{03} = 0$$

while working down from t^8 as has been here worked upwards from the constant terms one would find

$$a_{44} = a_{54} = a_{55} = a_{43} = a_{53} = a_{52} = 0.$$

When $r+s=4$ the three conditions are

$$6a_{13} + 4a_{22} = 20a_{04} + 14a_{13} + 6a_{22} = 10a_{04} + 16a_{13} + 9a_{22} = 0,$$

three linearly dependent constraints upon the a_{rs} which hold when $a_{04} : -a_{13} : a_{22}$ are in the ratios 1:4:6 of the binomial coefficients and so affording a quadric $x_0x_4 - 4x_1x_3 + 3x_2^2 = 0$ featuring a familiar trinomial. As C is invariant under the involutory permutation $(x_0x_5)(x_1x_4)(x_2x_3)$ imposed by $t \leftrightarrow t^{-1}$, i.e. by harmonic inversion in the two planes

$$x_0 = x_5, x_1 = x_4, x_2 = x_3 \quad \text{and} \quad x_0 = -x_5, x_1 = -x_4, x_2 = -x_3$$

Ω_2^8 also lies on the quadric $x_5x_1 - 4x_4x_2 + 3x_3^2 = 0$ as could also have been found by imposing I_1, I_2, J . As for $r+s=5$ the conditions require

$$4a_{14} + 6a_{23} = 25a_{05} + 17a_{14} + 13a_{23} = 4a_{14} + 6a_{23} = 0$$

of which the first and third both give $a_{14} : a_{23} = -3 : 2$, a consistency explained by the

invariance under $(x_0x_5)(x_1x_4)(x_2x_3)$. J then demands

$$a_{05}:a_{14}:a_{23} = 1:-3:2$$

so that $x_0x_5 - 3x_1x_4 + 2x_2x_3 = 0$ also contains Ω_2^8 . The outcome is that (cf. [1], pp. 137 and 143)

Ω_2^8 is the base surface of a net N of quadrics Q ,

say

$$p(x_0x_4 - 4x_1x_3 + 3x_2^2) - q(x_0x_5 - 3x_1x_4 + 2x_2x_3) + r(x_1x_5 - 4x_2x_4 + 3x_3^2) = 0. \tag{3.1}$$

Readers acquainted with texts on invariants will have remarked that, on replacing x_i by a_i , the three quadrics on which we have based N become the coefficients in Γ ([3], p. 273; [7], p. 206). But the geometry has more to say.

The cones of the net and the interpretation of their envelope

4. The symmetric matrix of a quadric of N is

$$(N) = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & p & -q \\ \cdot & \cdot & \cdot & -4p & 3q & r \\ \cdot & \cdot & 6p & -2q & -4r & \cdot \\ \cdot & -4p & -2q & 6r & \cdot & \cdot \\ p & 3q & -4r & \cdot & \cdot & \cdot \\ -q & r & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

The Laplace expansion on the top three and bottom three rows, in which only two non-zero products occur, or a triple Laplace expansion on the two top, two middle and two bottom rows, in which only four non-zero triple products occur, gives

$$|(N)| = 36(rp - q^2)^3$$

so that the only cones in N have $q^2 = rp$ or, say,

$$p:q:r = \rho^2:-\rho:1.$$

When this substitution is made in (N) only a single linear dependence between the rows emerges, namely

$$R_1 - \rho R_2 + \rho^2 R_3 - \rho^3 R_4 + \rho^4 R_5 - \rho^5 R_6 \equiv 0$$

so that the resulting matrix has rank 5 and the cone has a single point for vertex, indeed

the point on C with $t=\rho$. The cones of N thus compose a family of index 2 and their envelope, the locus of points P such that the two cones of N passing through P coincide, is the quartic primal K with equation

$$(x_0x_5 - 3x_1x_4 + 2x_2x_3)^2 = 4(x_0x_4 - 4x_1x_3 + 3x_2^2)(x_1x_5 - 4x_2x_4 + 3x_3^2). \tag{4.1}$$

Ω_2^8 is a double surface on K .

This is the sought interpretation. The situation may be described as follows.

There is a net N of quadrics containing all the tangents of a rational normal quintic C ; the singular members of N are all point-cones with vertices on C . Through any point P pass two of these cones, their vertices having for parameters on C the two zeros of a quadratic covariant of the binary quintic f mapped by P . If P is such that these two cones are coincident it lies on the quartic primal K and maps an f for which the invariant of degree 4 is zero.

Each cone of N touches K along a quartic threefold containing Ω_2^8 . In the geometry of the non-singular plane quartic the contacts of any two contact conics of the same system are eight points on a conic, and the analogous circumstance holds for K ; any non-singular quadric of N meets K in a pair of the quartic threefolds. These are the contacts of those cones for which ρ satisfies $p - 2q\rho + r\rho^2 = 0$.

Now that the equation of K has been found it is apparent that K contains Ω_3^9 , i.e. every osculating plane ω_2 . For it manifestly contains $\omega_2(0)$, whose equations (cf. 2.2) are $x_3 = x_4 = x_5 = 0$, while it has been defined geometrically in reference to the whole of C with no restriction to any coordinate system.

5. Since a rational plane quintic has six nodes or their equivalent a plane of general position in [5] meets six chords of C : the chords of C generate a sextic threefold M_3^6 . The surface common to M_3^6 and a quadric of N includes Ω_2^8 ; the residue is a quartic scroll. For any chord of C is, as λ, μ vary, traced by the point $x_i = (-1)^i(\lambda\phi^i + \mu\psi^i)$; the result of substituting these x_i in (3.1) is

$$\lambda\mu\{p + q(\phi + \psi) + r\phi\psi\}(\phi - \psi)^4 = 0$$

so that those chords which, in addition to all tangents, lie on (3.1) are joins of the pairs of the involution

$$p + q(\phi + \psi) + r\phi\psi = 0.$$

Such joins are known ([6], p. 97) to generate a rational normal quartic scroll. But the involution can degenerate, the scroll becoming the quartic cone of chords through a point of C ; this occurs whenever $q^2 = rp, p:q:r = \rho^2:-\rho:1$, the "involution" consisting of all chords through $t = \rho$.

As C , on the double surface of K , is at least a double curve any chord of C that meets K at a point not on C lies wholly on K ; the surface, of order 24, common to K and M_3^6

is a scroll. As Ω_2^8 is on M_3^6 and is double on K it counts twice in the intersection, leaving an octavic residue; this turns out to be Ω_2^8 again. For when x_i is replaced by $(-1)^i(\lambda\phi^i + \mu\psi^i)$ in (4.1) the outcome is $\{\lambda\mu(\phi + \psi)(\phi - \psi)^4\}^2 = 4\{\lambda\mu(\phi - \psi)^4\}\{\lambda\mu\phi\psi(\phi - \psi)^4\}$

$$\lambda^2\mu^2(\phi - \psi)^{10} = 0$$

so that no chord of C lies entirely on K unless it is a tangent: K meets M_3^6 in Ω_2^8 reckoned thrice.

The net of quadrics and the osculating planes

6. The developable surface Ω_2^8 has the same tangent plane $\omega_2(t)$ at every point of $\omega_1(t)$. Since the tangent prime at any point of $\omega_1(t)$ to any quadric Q of N contains the tangent plane of Ω_2^8 there $\omega_2(t)$ is in the polar solid of $\omega_1(t)$ with respect to Q . But this solid meets Q in a pair of planes through $\omega_1(t)$ so that $\omega_2(t)$, unless it lies on Q , meets Q only in $\omega_1(t)$ counted twice. It can be proved by correspondence theory that two osculating planes of C lie wholly on Q and these, in the present context, can be identified by elementary algebra. For the three equations (2.2) are equivalent to asserting that, for $s=0, 1, 2, 3$ the four fractions $(\alpha^2x_s + 2\alpha x_{s+1} + x_{s+2})/(-\alpha)^s$ are equal and so, again by (2.2), the points of $\omega_2(\alpha)$ satisfy the three quadratic conditions

$$\sum_{r=0}^3 (-1)^r \binom{3}{r} (\alpha^2 x_{3-r} + 2\alpha x_{4-r} + x_{5-r}) x_{k+r} = 0, \quad k=0, 1, 2.$$

If $k=0$ the terms in α^2 cancel one another, as do those in α when $k=1$ and those without α when $k=2$. The full equation when $k=2$ is

$$\alpha^2(x_3x_2 - 3x_2x_3 + 3x_1x_4 - x_0x_5) + 2\alpha(x_4x_2 - 3x_3^2 + 3x_2x_4 - x_1x_5) = 0$$

i.e.

$$\alpha^2(x_0x_5 - 3x_1x_4 + 2x_2x_3) + 2\alpha(x_1x_5 - 4x_2x_4 + 3x_3^2) = 0$$

with the quadrics of N clearly “declaring their interest”. When the other two equations with $k=0, 1$ are handled similarly it appears that the points of $\omega_2(\alpha)$ satisfy

$$\frac{x_0x_4 - 4x_1x_3 + 3x_2^2}{1} = \frac{x_0x_5 - 3x_1x_4 + 2x_2x_3}{-2\alpha} = \frac{x_1x_5 - 4x_2x_4 + 3x_3^2}{\alpha^2}$$

so that, by (3.1), $\omega_2(\alpha)$ lies wholly on Q when $p + 2q\alpha + r\alpha^2 = 0$. The two osculating planes coincide when Q is a cone.

An alternative identification of these two ω_2 uses the fact that $\omega_2(t)$ is spanned by (Section 2) a, b on $\omega_1(t)$ and any third point c of $\omega_2(t)$ not collinear with them; $\omega_2(t)$ is traced by $\lambda a + \mu b + \nu c$ as λ, μ, ν vary. Since $\omega_2(t)$ meets each quadric Q of N in $\omega_1(t)$ repeated the substitution of the six members of the coordinate vector $\lambda a + \mu b + \nu c$ for the

x_i in (3.1) must either produce zero, when $\omega_2(t)$ would lie on Q , or give $v^2=0$. Now $\omega_2(\alpha)$ meets $x_2=x_3=0$, which is skew to $\omega_1(\alpha)$ save when α is 0 or ∞ , at $(3, -\alpha, 0, 0, \alpha^4, -3\alpha^5)$ and the multiplier of v^2 after the substitution from this vector is

$$p(3\alpha^4) - q(-9\alpha^5 + 3\alpha^5) + r(3\alpha^6)$$

or

$$3\alpha^4(p + 2q\alpha + r\alpha^2).$$

Thus $\omega_2(\alpha)$ lies on that pencil of Q for which $p + 2q\alpha + r\alpha^2 = 0$ while each Q contains two $\omega_2(\alpha)$ which, as above, coincide when Q is a cone.

The generalisation

7. There is a strictly analogous interpretation of the quadratic covariant $(ab)^{2m}a_x b_x$ of a binary form of order $2m + 1$, as well as of the discriminant Δ of this quadratic. As the geometry for $m=2$ has been described at length it will suffice to state the facts for higher values of m without elaboration.

All the ω_{m-1} of a rational normal curve C in $[2m + 1]$ lie on the quadrics Q of a net N [1, p. 142]. These ω_{m-1} generate [9, p. 201] an $\Omega_m^{m(m+2)}$ lying on the $\Omega_{m+1}^{(m+1)^2}$ generated by the ω_m , and $\omega_m(t)$ is the tangent $[m]$ of $\Omega_m^{m(m+2)}$ at every point of $\omega_{m-1}(t)$. Each ω_m is on a pencil of Q and meets those Q on which it does not lie in ω_{m-1} repeated; each Q contains two ω_m which coincide when Q is a cone. The cones of N are point-cones with vertices on C , and their envelope is a quartic primal K . If P lies on K it maps a binary $(2m + 1) \cdot ic$ for which the invariant Δ is zero. K contains $\Omega_{m+1}^{(m+1)^2}$.

All elements of the symmetric matrix for Q are zero save those in the secondary diagonal and its two contiguous parallels. In these parallels the binomial coefficients $\binom{2m}{k}$ appear with alternating signs; the coefficients in the secondary diagonal itself give zero when added to the two contiguous ones in the same row or column. For $m=3$ the matrix is

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & p & -q \\ \cdot & \cdot & \cdot & \cdot & \cdot & -6p & 5q & r \\ \cdot & \cdot & \cdot & \cdot & 15p & -9q & -6r & \cdot \\ \cdot & \cdot & \cdot & -20p & 5q & 15r & \cdot & \cdot \\ \cdot & \cdot & 15p & 5q & -20r & \cdot & \cdot & \cdot \\ \cdot & -6p & -9q & 15r & \cdot & \cdot & \cdot & \cdot \\ p & 5q & -6r & \cdot & \cdot & \cdot & \cdot & \cdot \\ -q & r & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

with determinant a multiple of $(q^2 - rp)^4$ and, should $q^2 = rp$, rank 7. So the quartic

invariant of a_x^7 appears on replacing x_i by a_i in

$$\begin{aligned} & (x_0x_7 - 5x_1x_6 + 9x_2x_5 - 5x_3x_4)^2 \\ & = 4(x_0x_6 - 6x_1x_5 + 15x_2x_4 - 10x_3^2)(x_1x_7 - 6x_2x_6 + 15x_3x_5 - 10x_4^2) \end{aligned}$$

and so reproducing the expression given by Cayley [4, p. 316]. Of course all the coefficients occurring here are patent in

$$(ab)^6 a_x b_x \equiv (a_1b_2 - a_2b_1)^6 (a_1x + a_2y)(b_1x + b_2y)$$

but the geometry surely merits being recognised.

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