which is Mr. Higham's expression with a slightly modified notation.

I am, Sir,
Yours faithfully,
ABRAHAM LEVINE.

National Life Assurance Society,
5 *December* 1895.

UNIFORM SENIORITY.

To the Editor of the Journal of the Institute of Actuaries.

SIR,—In Part II of the *Text-Book,* Mr. King gives an investigation into the most general law of human mortality which will give Simpson's rule for joint-life annuities, and deduces as that law the function of Gompertz. It has occurred to me that a similar investigation into the most general law of mortality that will permit the substitution of *two lives of equal ages* for any two given lives, might perhaps interest some of the readers of the *Journal.*

In order to make this substitution we must be able, for any given values of x and y, to determine w, so that ${}_np_{ww}={}_np_{xy}$ for all values of n. This may be otherwise expressed thus:

$$2\log {}_np_w=\log {}_np_x+\log {}_np_y$$

for all values of n. Differentiating with respect to n, and changing the sign of both sides of the equation, we get

$$2\mu_{w+n}=\mu_{x+n}+\mu_{y+n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

for all values of n. Whence differentiating again, we have

$$2\frac{d\mu_{w+n}}{dn}=\frac{d\mu_{x+n}}{dn}+\frac{d\mu_{y+n}}{dn},$$

or, what is the same thing,

$$2\frac{d\mu_{w+n}}{dw}=\frac{d\mu_{x+n}}{dx}+\frac{d\mu_{y+n}}{dy} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

for all values of n. Putting now $n=o$ in equations (1) and (2), we get

$$2\mu_w=\mu_x+\mu_y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$2\frac{d\mu_w}{dw}=\frac{d\mu_x}{dx}+\frac{d\mu_y}{dy} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Supposing now x and y to vary so that w remains constant, we get, by differentiation with respect to x,

$$\frac{d\mu_x}{dx}+\frac{d\mu_y}{dy}\cdot\frac{dy}{dx}=o \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\frac{d^2\mu_x}{dx^2}+\frac{d^2\mu_y}{dy^2}\cdot\frac{dy}{dx}=o \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

from which, by eliminating $\frac{dy}{dx}$, we get

$$\frac{\frac{d^2\mu_x}{dx^2}}{\frac{d\mu_x}{dx}} = \frac{\frac{d^2\mu_y}{dy^2}}{\frac{d\mu_y}{dy}} = k \text{ (say)} \quad . \quad . \quad . \quad . \quad (7)$$

since the equation holds for any values of x and y. From this by integrating we get

$$\log \frac{d\mu_x}{dx} = kx + l \text{ (say)} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

or

$$\frac{d\mu_x}{dx} = e^{kx+l} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

whence integrating again,

$$\mu_x = m + \frac{1}{k} e^{kx+l} = m + \frac{1}{k} e^l . e^{kx} \quad . \quad . \quad . \quad . \quad (10)$$

Putting now $e^k = c$, $\frac{1}{k} e^l = \text{B}$, and $m = \text{A}$, we get the familiar expression

$$\mu_x = \text{A} + \text{B}c^x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

This is equivalent to

$$\frac{d \log l_x}{dx} = -\text{A} - \text{B}c^x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

whence by integration

$$\log l_x = p - \text{A}_x - \frac{\text{B}}{\log c} c^x \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where p is an arbitrary constant. Putting now $p = \log k$, $\text{A} = -\log s$, and $\frac{\text{B}}{\log c} = -\log g$, we get

$$\log l_x = \log k + x \log s + c^x \log g \quad . \quad . \quad . \quad . \quad (14)$$

or

$$l_x = k s^x g^{c^x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

which is the well-known expression of Makeham's law. Thus we see that Makeham's is the most general law to which the principle of uniform seniority can be applied.

R. HENDERSON.

Ottawa,
3 *August* 1895.

[The proposition proved above was referred to by the late Mr. Woolhouse so long ago as 1870 (*J.I.A.*, xv, 402), in the following words, viz.: "It may further be stated that a rigid analytical proof might be given that Mr. Makeham's formula, which includes that of Gompertz, is the most general form of function possible to which a law of uniform seniority can in any way be applicable."—ED. *J.I.A.*]

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