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ON WARING'S PROBLEM IN SUMS OF THREE CUBES FOR SMALLER POWERS

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Abstract

We give an upper bound for the minimum s with the property that every sufficiently large integer can be represented as the sum of s positive kth powers of integers, each of which is represented as the sum of three positive cubes for the cases $2 \leq k \leq 4$.

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1. Introduction

Additive problems involving small powers of positive integers have led to a vast development of new ideas and techniques in the application of the Hardy–Littlewood method, which often cannot be extended to the setting of general kth powers. Finding the least number s such that for every sufficiently large integer n,

$$n = x_1^k + \cdots + x_s^k, \tag{1-1}$$

where $x_i \in \mathbb{N}$, might be among the most studied examples. We denote such a number s by $G(k)$. Let $\mathscr{C}$ be the set of integers represented as a sum of three positive integral cubes. In this work we shall be concerned with the function $G_3(k)$, which we define as the minimum s such that (1-1) is soluble with $x_i \in \mathscr{C}$ for the cases $2 \leq k \leq 4$.

Providing the precise value of $G(k)$ is still an open question for most k, the cases $k = 2, 4$ being precisely the only ones solved. Lagrange showed in 1770 that $G(2) = 4$, and Davenport [2] proved in 1939 the identity $G(4) = 16$. Although it is believed that $G(3) = 4$, the best current upper bound is $G(3) \leq 7$ due to Linnik [8].

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Not very much is known about $\mathscr{C}$. In fact, it is not even known whether it has positive density, the best current lower bound on the cardinality of the set, due to Wooley [22], being

$$\mathcal{N}(X) = |\mathscr{C} \cap [1, X]| \gg X^{\beta},$$

where $\beta = 0.917\,094\,77$. We note that under some unproved assumptions on the zeros of some Hasse–Weil L-functions, Hooley [5, 6] and Heath-Brown [4] showed using different procedures that

$$\sum_{n \le X} r_3(n)^2 \ll X^{1+\varepsilon},$$

where $r_3(n)$ is the number of representations of n as sums of three positive integral cubes. This implies by applying a standard Cauchy–Schwarz argument that $\mathcal{N}(X) \gg X^{1-\varepsilon}$. This lack of understanding of the cardinality of the set also prevents us from understanding its distribution over arithmetic progressions, which often comes into play in the major arc analysis. Therefore, even if the exponents $k = 2, 4$ are well understood for the original problem, it turns out to be much harder when we restrict the variables to lie on $\mathscr{C}$. In this paper we establish the following bounds for $G_3(k)$.

THEOREM 1.1. *One has $G_3(2) \le 8, G_3(3) \le 17$ and $G_3(4) \le 57$.*

We are far from knowing whether these estimates are good or bad, since the only lower bounds that we have for the above quantities are $4 \le G(3) \le G_3(3)$ and $16 = G(4) \le G_3(4)$. For the case $k = 2$, though, we can actually do better. We take, for convenience, an integer $j \ge 0$, and observe that the only solution to

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 2^{6+12j} \tag{1-2}$$

with $x_i \in \mathbb{N}$ is $x_1 = x_2 = x_3 = x_4 = 2^{2+6j}$. This can be seen by taking Equation (1-2) modulo 8, realising that one must have $2 \mid x_i$ for every i and iterating the process. However, one has $2^{2+6j} \equiv 4 \pmod 9$, and no number congruent to 4 (mod 9) can be written as the sum of three cubes. Therefore, there are infinitely many numbers for which (1-2) does not have any solution with $x_i \in \mathscr{C}$. The preceding remark then implies the bound $5 \le G_3(2)$.

Our proof of Theorem 1.1 is based on the application of the Hardy–Littlewood method. In the setting of this paper, the constraint that prevents us from taking fewer variables is the treatment of the minor arcs discussed in Section 2. In order to analyse them, we use an argument of Vaughan [15, Lemma 3.4] to bound certain families of exponential sums over the minor arcs, together with nonoptimal estimates of sums of the shape

$$\sum_{x \le X} a_x^2 \quad \text{where } a_x = \operatorname{card}\{\mathbf{x} \in \mathbb{N}^3 : x = x_1^3 + x_2^3 + x_3^3, \;\; x_2, x_3 \in \mathcal{A}(P, P^{\eta})\}$$

with $\eta > 0$ being a small enough parameter and

$$\mathcal{A}(Y,R) = \{n \in [1,Y] \cap \mathbb{N} : p \mid n \text{ and } p \text{ prime} \Rightarrow p \leq R\}.$$

Here, the reader may find it useful to observe that it is a consequence of Montgomery and Vaughan [9, Theorem 7.2] that

$$\mathrm{card}(\mathcal{A}(P,P^{\eta})) = c_{\eta}P + O(P/\log P)$$

for some constant $c_{\eta} > 0$ which only depends on η. In order to briefly discuss the outcome that follows after applying the argument of Vaughan we introduce the exponential sum

$$W(\alpha) = \sum_{M/2\leq p\leq M}\ \sum_{H/2\leq h\leq H} b_h e(\alpha p^{3k}h^k), \tag{1-3}$$

where $M, H > 0$ and b_h are weights which the reader should think of being the previously defined a_h, and p runs over prime numbers. In order to make the argument work, the parameters M and H must be subjected to the constraint $\max(M^{5-1/k}, M^{2^{k-1}}) \leq H$. The saving over the natural bound HM for $W(\alpha)$ obtained with the method is roughly speaking of size $M^{1/2}H^{-1/24}$, which makes the estimate obtained worse than trivial for $k \geq 5$.

A naive approach to bounding $G_3(k)$ would be to replace each sum of three cubes by the specialisation $3x^3$, and this suggests a bound of the shape $G_3(k) \leq G(3k)$. With this idea in mind, the bounds $G(6) \leq 24$ (due to Vaughan and Wooley [17]), $G(9) \leq 47$ and $G(12) \leq 72$ (due to Wooley [23]) reveal that our methods improve the trivial approach and confirm that we are actually using the three integral cubes nontrivially in our argument. For the cases $k = 2, 3$, we combine the pointwise bound obtained for $W(\alpha)$ over the minor arcs with some restriction estimates involving the coefficients a_m. When $k = 4$, we instead use a bound for a mean value of smooth Weyl sums of exponent 12. The estimate for $W(\alpha)$ obtained here is then robust enough to enable us to gain 15 variables from the trivial approach over the minor arcs and allows us to prune back to a narrower set of major arcs.

The purpose of the present work is to derive upper bounds for the minimum number of variables that guarantee the existence of solutions to Equation (1-1) for smaller values of k. For the alternative problem of establishing the validity of an asymptotic formula for the number of such representations for all $k \geq 2$, the interested reader is referred to [11]. As experts may expect, the minor arc arguments in the analysis of that paper, as opposed to those employed in the present work, rely on estimates stemming from Vinogradov's mean value theorem [24]. Moreover, the major arc discussion follows a standard approach, and the author incorporates the three cubes in the analysis of the singular series. By contrast, the major arc manoeuvres herein entail fixing the two smooth cubes in order to provide robust approximations of the corresponding exponential sums on a wider set of major arcs, and the pruning operations deployed in the discussion involve both these approximations and minor arc type estimates.

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Moreover, the absence of two of the cubes in the corresponding singular series makes the local solubility analysis more tedious and somewhat different from that of [11].

In [10], the author employs the minor arc bound for the case $k = 2$ obtained in the present work to derive an almost-all result for the analogue of Lagrange's four square theorem when the variables are restricted to the set of sums of three positive cubes. Moreover, he makes use of the approximations of the exponential sums obtained in the pruning process herein, and applies several major-arc-type lemmas deduced in this work. Nevertheless, the author incorporates the three cubes in the analysis of the singular series involving four squares, which in turn entails a rather delicate discussion of a different nature regarding the behaviour of such series. This strategy requires the use of thinner major arcs and pruning operations that have little resemblance to the manoeuvres deployed in this work.

In a recent preprint [12] discussing an analogous problem in which the variables are instead restricted to sums of lth powers, the author focuses on obtaining some uniformity with respect to l in the number of variables needed to attain a representation. The novelty of that paper is a pointwise minor arc estimate for a certain exponential sum which is obtained via an application of the large sieve inequality. The major arc analysis is standard and does not present any difficulty. The reader may find it useful to note that the methods employed in the aforementioned paper and the ones recorded herein barely have any similarities, and the nature of the results discussed are totally different.

This paper is organized as follows. In Section 2 we use Vaughan's methods to estimate $W(\alpha)$ and provide bounds for the contribution of the minor arcs which are good enough for our purposes when $k = 2, 3$. We approximate the generating functions of the problem on a narrower set of major arcs in Section 3. In Sections 4, 5 and 6 we only consider the exponents $k = 2, 3$, whereas in Section 7 we prove a theorem for $k = 4$. Sections 4 and 5 are devoted to the study of the singular series and the singular integral, respectively. We then prune back to the narrower set of arcs to exhibit a lower bound for the major arc contribution in Section 6.

Unless otherwise specified, any lower case letter $\mathbf{x}$ written in bold denotes a triple of integers (x_1, x_2, x_3). We write $a \leq \mathbf{V} \leq b$ when $a \leq v_i \leq b$ for $1 \leq i \leq n$. As usual in analytic number theory, for each $x \in \mathbb{R}$ we denote $\exp(2\pi ix)$ by $e(x)$, and for each natural number q, $e(x/q)$ is written as $e_q(x)$. We use $\ll$ and $\gg$ to denote Vinogradov's notation, and write $A \asymp B$ whenever $A \ll B \ll A$. When ε appears in any bound, it will mean that the bound holds for every $\varepsilon > 0$, though the implicit constant may then depend on ε. We adopt the convention that whenever we write δ in our computations we mean that there exists a positive constant δ for which the bound holds. We write $p^r \| n$ to denote that $p^r | n$ but $p^{r+1} \nmid n$.

2. Minor arc estimates

As mentioned in the Introduction, we provide an estimate for the exponential sum $W(\alpha)$ by using methods of Vaughan. We use a Hardy–Littlewood dissection and combine both the bound for $W(\alpha)$ and a restriction estimate of a certain mean value to bound the minor arc contribution for the cases $k = 2, 3$. Our estimate for $W(\alpha)$ is also used in Sections 6 and 7 to prune the major arcs back to a narrower set of arcs. Before going into the proof of the main lemma, it is convenient to write

$$S_k(q, a) = \sum_{r=1}^{q} e_q(ar^k). \tag{2-1}$$

We also introduce the multiplicative function $\tau_k(q)$ by defining $\tau_2(q) = q^{-1/2}$, and

$$\tau_k(p^{uk+v}) = p^{-u-1} \quad \text{when } u \geq 0 \text{ and } 2 \leq v \leq k$$

and

$$\tau_k(p^{uk+1}) = kp^{-u-1/2} \quad \text{when } u \geq 0$$

for $k \geq 3$. Observe that with this definition one has the bound

$$\tau_k(q) \ll q^{-1/k}, \tag{2-2}$$

and the proof of Theorem 4.2 of [16] yields

$$q^{-1}S_k(q, a) \ll \tau_k(q). \tag{2-3}$$

LEMMA 2.1. *Let $2 \leq k \leq 4$. Take $H, M > 0$ such that $\max(M^{5-1/k}, M^{2^{k-1}}) \leq H$. Let $\alpha \in [0, 1)$. Suppose that $\alpha = a/q + \beta$, where $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$, such that $q \leq Y$, and $|\beta| \leq q^{-1}Y^{-1}$, where Y is a parameter in the range $M^k \leq Y \leq H^k M^{2k}$. Then the exponential sum $W(\alpha)$, defined in (1-3), satisfies*

$$W(\alpha) \ll H^{\varepsilon}\Big(HM + \frac{\tau_k(q)HM^2}{1 + M^{3k}H^k|\alpha - a/q|}\Big)^{1/2}\Big(\sum_{H/2\leq h\leq H} |b_h|^2\Big)^{1/2}. \tag{2-4}$$

PROOF. For the sake of simplicity we do not write the limits of summation for p and h throughout the rest of this section. We apply Cauchy–Schwarz to obtain

$$\begin{aligned} W(\alpha) &\ll \Big(\sum_h |b_h|^2\Big)^{1/2}\Big(\sum_h \sum_{p_1,p_2} e(\alpha(p_1^{3k} - p_2^{3k})h^k)\Big)^{1/2} \\ &\ll \Big(\sum_h |b_h|^2\Big)^{1/2}(HM + E(\alpha))^{1/2}, \end{aligned} \tag{2-5}$$

where the term HM comes from the diagonal contribution and

$$E(\alpha) = \sum_h \sum_{p_2<p_1} e(\alpha(p_1^{3k} - p_2^{3k})h^k).$$

In order to estimate $E(\alpha)$ we follow closely the argument of Vaughan [15, Lemma 3.4]. For a given pair of primes (p_1, p_2) we choose $b, r \in \mathbb{N}$ with $(b, r) = 1$, such that $r \leq 2kH^{k-1}$ and $|\alpha(p_1^{3k} - p_2^{3k}) - b/r| \leq (2k)^{-1}r^{-1}H^{1-k}$. Then, if $r > H$, an application of

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Weyl's inequality [16, Lemma 2.4] yields the bound

$$\sum_h e(\alpha(p_1^{3k} - p_2^{3k})h^k) \ll H^{1-2^{1-k}+\varepsilon} \ll H^{1+\varepsilon}M^{-1},$$

where we use the restriction on M at the beginning of the lemma. If, on the other hand, $r \leq H$ we combine Lemmas 6.1 and 6.2 of [16] with (2-3) to obtain

$$\sum_h e(\alpha(p_1^{3k} - p_2^{3k})h^k) \ll \frac{\tau_k(r)H}{1 + H^k|\alpha(p_1^{3k} - p_2^{3k}) - b/r|} + r^{1/2+\varepsilon}.$$

Consequently,

$$E(\alpha) \ll E_0 + H^{1+\varepsilon}M + \sum_{(p_1,p_2)} H^{1/2+\varepsilon} \ll E_0 + H^{1+\varepsilon}M,$$

where

$$E_0 = \sum_{(p_1,p_2)\in\mathcal{A}} \frac{\tau_k(r)H}{1 + H^k|\alpha(p_1^{3k} - p_2^{3k}) - b/r|}$$

and $\mathcal{A}$ is the set of pairs (p_1, p_2) with $p_2 < p_1$, for which $r < (6k)^{-1}M^k$, such that $|\alpha(p_1^{3k} - p_2^{3k}) - b/r| < 2^{-1}r^{-1/k}MH^{-k}$. Note that the contribution of the pairs for which one of the previous two restrictions does not hold is $O(HM)$. For each pair (p_1, p_2), define $n = p_2, l = p_1 - p_2$ and $D = ((n + l)^{3k} - n^{3k})/l$. Then, one finds that

$$E_0 \ll \sum_{(n,l)} \frac{\tau_k(r)H}{1 + H^k|\alpha lD - b/r|}, \tag{2-6}$$

where (n, l) runs over pairs with $1 \leq l \leq M$ and $M/2 \leq n \leq M$ with the property that $(n + l, n) = 1$ and satisfying the aforementioned bounds on r and $|\alpha lD - b/r|$.

We next choose for convenience $c, s \in \mathbb{N}$ satisfying $(c, s) = 1$, with the properties that $s \leq H^kM^{-k}$ and $|\alpha l - c/s| \leq s^{-1}M^kH^{-k}$. By the constraint imposed on M and H at the beginning of the lemma we obtain

$$\left|\frac{c}{s} - \frac{b}{rD}\right|sDr < DrM^kH^{-k} + \frac{1}{2}sr^{1-1/k}MH^{-k} < \frac{3k}{6k}M^{5k-1}H^{-k} + \frac{1}{2}sM^kH^{-k} \leq 1.$$

Therefore, $crD = bs$ and hence the coprimality condition on r and b yields $r|s$. Let $s_0 = s/r$. Then $s_0 \mid D$, whence

$$E_0 \ll \sum_{s_0|s} \tau_k\Big(\frac{s}{s_0}\Big) \sum_{(n,l)} \frac{H}{1 + H^kD|\alpha l - c/s|},$$

where the sum on (n, l) runs over the same range described after (2-6) with the conditions $(n + l, n) = 1$ and $((n + l)^{3k} - n^{3k})/l \equiv 0 \pmod{s_0}$. Once we fix l, using the above constraints one has that the number of such n is bounded above by

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$O((M/s_0+1)s_0^{\varepsilon})$. Consequently, we obtain that $E(\alpha) \ll H^{1+\varepsilon}M + H^{\varepsilon}ME_1$, where

$$E_1 = \sum_{l\in\mathcal{L}} \frac{\tau_k(s)H}{1+H^kM^{3k-1}|\alpha l - c/s|},$$

and $\mathcal{L}$ is the set of integers $l \le M$ for which $s < M^k/2$ and $|\alpha l - c/s| < M^{2-3k}H^{-k}$. Now we choose d, t with $(d,t) = 1$ satisfying $t \le M^{k+1}$ and $|\alpha - d/t| \le t^{-1}M^{-k-1}$. One finds that

$$\left|\frac{c}{ls} - \frac{d}{t}\right| slt < stM^{2-3k}H^{-k} + slM^{-k-1} < \frac{1}{2}M^{3-k}H^{-k} + \frac{1}{2} \le 1.$$

Therefore, one has $ct = dsl$ and hence $s|t$. Let $t_0 = t/s$. Then it follows that $t_0 \mid l$, and on defining $l_0 = l/t_0$ we obtain

$$E_1 \ll \sum_{t_0|t} \tau_k\Big(\frac{t}{t_0}\Big) \sum_{l_0 \le M/t_0} \frac{H}{1+H^kM^{3k-1}l_0t_0|\alpha - d/t|} \ll \frac{\tau_k(t)HM^{1+\varepsilon}}{1+H^kM^{3k}|\alpha - d/t|}.$$

If either $t \ge M^k/2$ or $|\alpha - d/t| \ge 2^{-1}t^{-1/k}H^{-k}M^{1-3k}$, we get $E_1 \ll HM^{\varepsilon}$ and we are done. For the remaining cases, one finds that

$$\left|\frac{a}{q} - \frac{d}{t}\right| qt < \frac{1}{2}qH^{-k}M^{1-3k}t^{1-1/k} + tY^{-1} < \frac{1}{2}YH^{-k}M^{-2k} + \frac{1}{2}M^kY^{-1} \le 1,$$

which implies that $a = d$ and $q = t$, and yields the bound

$$E(\alpha) \ll H^{1+\varepsilon}M + \frac{\tau_k(q)H^{1+\varepsilon}M^2}{1+H^kM^{3k}|\alpha - a/q|}.$$

The combination of this estimate and (2-5) proves the lemma. □

Before describing the application of this lemma in the minor arc treatment, it is convenient to introduce some notation. Let n be a natural number and take $P = n^{1/(3k)}$. Define the parameters

$$\gamma(k) = \frac{3}{3+\max(5-1/k, 2^{k-1})}, \quad M = P^{\gamma(k)}, \quad H = \max(M^{5-1/k}, M^{2^{k-1}}). \tag{2-7}$$

We observe that

$$P^3 = M^3H. \tag{2-8}$$

Note that these choices for M and H maximize the saving obtained for $W(\alpha)$ over the trivial bound in the previous lemma. Take

$$H_1 = (\tfrac{1}{2})^{1/3}H^{1/3}, \quad H_2 = (\tfrac{2}{3})^{1/3}H^{1/3}, \quad H_3 = (\tfrac{1}{6})^{1/3}H^{1/3}.$$

For every triple $\mathbf{x} \in \mathbb{R}^3$, consider the function $T(\mathbf{x}) = x_1^3 + x_2^3 + x_3^3$. Define the sets

$$\mathcal{H} = \left\{(y, \mathbf{y}) \in \mathbb{N}^3 : \frac{P}{2} \le y \le P,\ \mathbf{y} \in \mathcal{A}(P, P^\eta)^2\right\},$$
$$\mathcal{W} = \{(y, \mathbf{y}) \in \mathbb{N}^3 : H_1 \le y \le H_2,\ \mathbf{y} \in \mathcal{A}(H_3, P^\eta)^2\},$$

and the corresponding weights

$$a_x = |\{\mathbf{x} \in \mathcal{H} : x = T(\mathbf{x})\}|, \quad b_h = |\{\mathbf{x} \in \mathcal{W} : h = T(\mathbf{x})\}|,$$

where b_h is the choice that we make for the weights of $W(\alpha)$ in (1-3). We use a_x to define the weighted exponential sum

$$h(\alpha) = \sum_{x \le 3P^3} a_x e(\alpha x^k).$$

Before describing the roles of $h(\alpha)$ and $W(\alpha)$ in the argument, we first produce upper bounds on the L^2-norms of the weights to estimate the minor arc contribution. Let $X > 0$, write

$$f(\alpha; X) = \sum_{x \le X} e(\alpha x^3), \quad f(\alpha; X; X^\eta) = \sum_{x \in \mathcal{A}(X, X^\eta)} e(\alpha x^3),$$

and define the mean value

$$U(X) = \int_0^1 |f(\alpha; X)|^2 |f(\alpha; X; X^\eta)|^4 \, d\alpha.$$

It is a consequence of [22, Theorem 1.2] that $U(X) \ll X^{3+1/4-\tau}$, where $\tau = 0.00128432$. Consequently, on considering the underlying Diophantine equations due to orthogonality, it follows that

$$\sum_{x \le 3P^3} a_x^2 \le U(P) \ll P^{3+1/4-\tau}, \qquad \sum_{H/2 \le h \le H} b_h^2 \le U(H^{1/3}) \ll H^{13/12-\tau/3}. \tag{2-9}$$

The reader may note that we did not write the entire decimal expression of τ, so the bound for $U(X)$ holds for a slightly bigger τ. Therefore, whenever we encounter bounds involving the mean value $U(X)$, we can omit the parameter ε in the exponents.

Take

$$s(k) = 2^k \quad \text{when } k = 2, 3,$$
$$t(2) = 4 \quad \text{and} \quad t(3) = 9. \tag{2-10}$$

For ease of notation we just write s and t instead of $s(k)$ and $t(k)$ throughout the paper. Let $R(n)$ be the number of solutions of the equation

$$n = \sum_{i=1}^{t} T(p_i \mathbf{x}_i)^k + \sum_{i=t+1}^{s+t} T(\mathbf{x}_i)^k,$$

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where $\mathbf{x}_i \in \mathcal{W}$ for p_i prime satisfying $M/2 \le p_i \le M$ when $1 \le i \le t$ and $\mathbf{x}_i \in \mathcal{H}$ for $t+1 \le i \le s+t$. Note that by orthogonality,

$$R(n) = \int_0^1 h(\alpha)^s W(\alpha)^t e(-\alpha n)\, d\alpha.$$

Our goal throughout Sections 2 to 6 is to obtain a lower bound for $R(n)$ for all sufficiently large n. For such purpose, we make use of a Hardy–Littlewood dissection in our analysis. When $1 \le X \le M^k$, we define the major arcs $\mathfrak{M}(X)$ to be the union of

$$\mathfrak{M}(a,q) = \left\{\alpha \in [0,1) : |\alpha - a/q| \le \frac{X}{qn}\right\} \tag{2-11}$$

with $0 \le a \le q \le X$ and $(a,q) = 1$. For simplicity we write

$$\mathfrak{M} = \mathfrak{M}(M^k), \quad \mathfrak{N} = \mathfrak{M}((6k)^{-1}H^{1/3}).$$

We define the minor arcs as $\mathfrak{m} = [0,1) \setminus \mathfrak{M}$ and $\mathfrak{n} = [0,1) \setminus \mathfrak{N}$. This dissection remains valid for the case $k = 4$ and is used in Section 7. We then take $\alpha \in \mathfrak{m}$ and observe that by Dirichlet's approximation there exist nonnegative integers a, q with $(a,q) = 1$ and $1 \le q \le nM^{-k}$ such that

$$|\alpha - a/q| \le \frac{M^k}{qn}.$$

Consequently, one has $q > M^k$ and hence (2-9) and Lemma 2.1 yield the bound

$$W(\alpha) \ll H^{1/2+\varepsilon} M^{1/2} \Big(\sum_{h \le H} b_h^2\Big)^{1/2} \ll H^{1+1/24-\tau/6} M^{1/2}. \tag{2-12}$$

As observed right after (1-3), the reader may find it useful to note that employing the definitions of (2-7) the above estimate is worse than the trivial one HM whenever $k \ge 5$. This explains why we restrict our analysis to the cases $2 \le k \le 4$. In the following proposition, we combine this pointwise bound with some restriction estimates to bound the minor arc contribution.

PROPOSITION 2.2. *When $k = 2, 3$,*

$$\int_{\mathfrak{m}} |h(\alpha)|^s |W(\alpha)|^t\, d\alpha \ll (HM)^t P^{3s-3k-\delta}. \tag{2-13}$$

PROOF. Combining [1, Equation (1.6)] when $k = 2$ and [7, Theorem 4.1] when $k = 3$ with Equation (2-9), we find that

$$\int_0^1 |h(\alpha)|^s\, d\alpha \ll P^{3s/2-3k+\varepsilon} \Big(\sum_{x \le 3P^3} a_x^2\Big)^{s/2} \ll P^{3s-3k+s/8-\delta}.$$

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Therefore, an application of the pointwise bound on the minor arcs in (2-12) yields the estimate

$$\int_{\mathfrak{m}} |h(\alpha)|^s |W(\alpha)|^t \, d\alpha \ll H^{t+t/24} M^{t/2} P^{3s-3k+s/8-\delta}.$$

We define the parameter $\xi(k)$ as $\xi(2) = 0$ and $\xi(3) = 7/92$, and deduce that the proposition then follows after noting by (2-7) and (2-8) that $H^{t/24} M^{t/2} P^{s/8} = M^t P^{-\xi(k)}$. □

Here, knowing the existence of $\delta > 0$ for which (2-13) holds suffices. The reader may observe though that the precise saving over the expected main term that we obtain here is $H^{t\tau/6} P^{\xi(k)+s\tau/2-\varepsilon}$.

3. Approximation of exponential sums over the major arcs

We adapt the argument of Vaughan [16, Theorem 4.1] to estimate the differences between the exponential sums $h(\alpha)$, $W(\alpha)$ and their approximations over the major arcs. Let $\mathbf{y} \in [0, P]^2$ and set

$$C_{\mathbf{y}} = y_1^3 + y_2^3. \tag{3-1}$$

Let $\beta \in \mathbb{R}$ and let p be a prime number. Consider the integrals

$$v_{\mathbf{y}}(\beta) = \int_{P/2}^{P} e(\beta(x^3 + C_{\mathbf{y}})^k) \, dx \quad \text{and} \quad v_{\mathbf{y},p}(\beta) = \int_{H_1}^{H_2} e(\beta p^{3k}(x^3 + C_{\mathbf{y}})^k) \, dx. \tag{3-2}$$

Note that by a change of variables one finds that

$$v_{\mathbf{y}}(\beta) = \int_{M_{\mathbf{y}}}^{N_{\mathbf{y}}} B_{\mathbf{y}}(\gamma) e(\beta\gamma) \, d\gamma, \quad v_{\mathbf{y},p}(\beta) = \int_{M_{\mathbf{y},p}}^{N_{\mathbf{y},p}} B_{\mathbf{y},p}(\gamma) e(\beta\gamma) \, d\gamma, \tag{3-3}$$

where the limits of integration taken are $M_{\mathbf{y}} = (P^3/8 + C_{\mathbf{y}})^k$, $N_{\mathbf{y}} = (P^3 + C_{\mathbf{y}})^k$, $M_{\mathbf{y},p} = (Hp^3/2 + C_{p\mathbf{y}})^k$ and $N_{\mathbf{y},p} = (2Hp^3/3 + C_{p\mathbf{y}})^k$, and the functions inside the integral are defined as

$$B_{\mathbf{y}}(\gamma) = \frac{1}{3k} \gamma^{1/k-1} (\gamma^{1/k} - C_{\mathbf{y}})^{-2/3}, \quad B_{\mathbf{y},p}(\gamma) = \frac{1}{3kp} \gamma^{1/k-1} (\gamma^{1/k} - C_{p\mathbf{y}})^{-2/3}. \tag{3-4}$$

We introduce the auxiliary multiplicative function $w_k(q)$ defined for prime powers by taking

$$w_k(p^{3ku+v}) = \begin{cases} p^{-u-v/3k} & \text{when } u \geq 1 \text{ and } 1 \leq v \leq 3k, \\ p^{-1} & \text{when } u = 0 \text{ and } 2 \leq v \leq 3k, \\ p^{-1/2} & \text{when } u = 0 \text{ and } v = 1. \end{cases} \tag{3-5}$$

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In order to discuss the approximation of $f(\alpha)$ on the major arcs, it is convenient to consider, for $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$, the sums

$$S_{\mathbf{y}}(q, a) = \sum_{r=1}^{q} e_q(a(r^3 + C_{\mathbf{y}})^k) \quad \text{and} \quad V(\alpha, q, a) = q^{-1} \sum_{\mathbf{y}} S_{\mathbf{y}}(q, a) v_{\mathbf{y}}(\beta), \tag{3-6}$$

where $\mathbf{y}$ runs over the set $\mathcal{A}(P, P^{\eta})^2$ of pairs of smooth numbers.

LEMMA 3.1. *Suppose that $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a, q) = 1$. Let $\alpha \in [0, 1)$ and $\beta = \alpha - a/q$. Then we have the estimate*

$$h(\alpha) - V(\alpha, q, a) \ll P^2 q^{1+\varepsilon} w_k(q)(1 + n|\beta|)^{1/2}.$$

Moreover, if $|\beta| \leq (2 \cdot 3^k kq)^{-1} P n^{-1}$ then

$$h(\alpha) - V(\alpha, q, a) \ll P^2 q^{1+\varepsilon} w_k(q). \tag{3-7}$$

PROOF. Let $b \in \mathbb{Z}$ and $\mathbf{y} \in \mathcal{A}(P, P^{\eta})^2$. We define

$$S_{\mathbf{y}}(q, a, b) = \sum_{r=1}^{q} e_q(a(r^3 + C_{\mathbf{y}})^k + br) \quad \text{and} \quad I_{\mathbf{y}}(b) = \int_{P/2}^{P} e(F(\gamma; b))\, d\gamma, \tag{3-8}$$

where

$$F(\gamma; b) = \beta(\gamma^3 + C_{\mathbf{y}})^k - b\gamma/q.$$

Both the complete exponential sum and the integral have roles in the analysis of the main and the error terms. Observe that $h(\alpha)$ can be written as

$$h(\alpha) = \sum_{\mathbf{y} \in \mathcal{A}(P, P^{\eta})^2} h_{\mathbf{y}}(\alpha) \quad \text{with} \quad h_{\mathbf{y}}(\alpha) = \sum_{P/2 \leq x \leq P} e(\alpha(x^3 + C_{\mathbf{y}})^k).$$

Then, by sorting the summation into arithmetic progressions modulo q and applying orthogonality, it follows that

$$h_{\mathbf{y}}(\alpha) = q^{-1} \sum_{-q/2 < b \leq q/2} S_{\mathbf{y}}(q, a, b) \sum_{P/2 \leq x \leq P} e(F(x; b));$$

whence, using [16, Lemma 4.2], we obtain

$$\begin{aligned} h_{\mathbf{y}}(\alpha) - q^{-1} S_{\mathbf{y}}(q, a) v_{\mathbf{y}}(\beta) &= q^{-1} \sum_{\substack{-B < b \leq B \\ b \neq 0}} S_{\mathbf{y}}(q, a, b) I_{\mathbf{y}}(b) \\ &\quad + O\Big(q^{-1} \log(H + 2) \sum_{-q/2 < b \leq q/2} |S_{\mathbf{y}}(q, a, b)|\Big), \end{aligned} \tag{3-9}$$

where $B = (H + 1/2)q$ and $H = \lceil 3^k k P^{-1} n|\beta| + 1/2 \rceil$. Note that by the quasi- multiplicative property, in order to bound $S_{\mathbf{y}}(q, a, b)$ it suffices to consider the case when q is a prime power. For such purposes, we take $q = p^{3ku+v}$. We observe first that by [16, Theorem 7.1] one has that $S_{\mathbf{y}}(q, a, b) \ll q^{1-1/3k+\varepsilon}$. Moreover, when $v \geq 2$ and

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$u = 0$ we can deduce from the proof of the same theorem (see in particular the argument following [16, Equation (7.16)]) that $S_{\mathbf{y}}(p^v, a, b) \ll p^{v-1}$. For the case $q = p$, the work of Weil [19] yields the estimate $S_{\mathbf{y}}(p, a, b) \ll p^{1/2}$ (see [13, Corollary 2F] for an elementary proof of this bound). Therefore, combining these bounds with the definition in (3-5), one finds that

$$S_{\mathbf{y}}(q, a, b) \ll q^{1+\varepsilon} w_k(q). \tag{3-10}$$

Consequently, by (3-9) we have

$$h_{\mathbf{y}}(\alpha) - q^{-1} S_{\mathbf{y}}(q, a) v_{\mathbf{y}}(\beta) \ll q^{\varepsilon} w_k(q) \sum_{\substack{-B<b\le B\\ b\neq 0}} |I_{\mathbf{y}}(b)| + q^{1+\varepsilon} w_k(q) \log(H+2). \tag{3-11}$$

To treat the sum on the right-hand side, we use the methods of the proof of [16, Theorem 4.1]. In his analysis Vaughan classifies the range of integration of $I(b)$ according to the size of $|G'(\gamma)|$, where

$$G(\gamma) = \beta\gamma^k - b\gamma/q \quad \text{and} \quad I(b) = \int_0^X e(G(\gamma))\, d\gamma.$$

We follow Vaughan's analysis closely, dividing the range of integration of $I_{\mathbf{y}}(b)$ according to the size of $|F'(\gamma; b)|$, to obtain

$$\sum_{\substack{-B<b\le B\\ b\neq 0}} |I_{\mathbf{y}}(b)| \ll q^{1+\varepsilon}(1 + n|\beta|)^{1/2}.$$

Since $\log(H + 2) \ll (1 + n|\beta|)^{1/2}$,

$$h_{\mathbf{y}}(\alpha) - q^{-1} S_{\mathbf{y}}(q, a) v_{\mathbf{y}}(\beta) \ll q^{1+\varepsilon} w_k(q)(1 + n|\beta|)^{1/2}.$$

Summing over $\mathbf{y} \in \mathcal{A}(P, P^{\eta})^2$ yields the first statement of the lemma. Note that when $|\beta| \le (2 \cdot 3^k kq)^{-1} P n^{-1}$ and $b \neq 0$ one has $|F'(x; b)| \ge |b|/(2q)$ and $H = 1$. Observing that $F'(x; b)$ is monotonic, partial integration yields

$$\sum_{\substack{-B<b\le B\\ b\neq 0}} |I_{\mathbf{y}}(b)| \ll \sum_{\substack{-B<b\le B\\ b\neq 0}} \frac{q}{|b|} \ll q^{1+\varepsilon}.$$

Combining this estimate with (3-11) and summing over $\mathbf{y} \in \mathcal{A}(P, P^{\eta})^2$, we get (3-7). □

By applying similar methods we can obtain the same type of approximation for the exponential sum $W(\alpha)$. For $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$, and recalling (3-2) and (3-6), we introduce the auxiliary function

$$W(\alpha, q, a) = q^{-1} \sum_{\mathbf{y}, p} S_{p\mathbf{y}}(q, a) v_{\mathbf{y},p}(\beta) \tag{3-12}$$

where $\mathbf{y} \in \mathcal{A}(H_3, P^{\eta})^2$ and $M/2 \le p \le M$.

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LEMMA 3.2. *Suppose that* $(a, q) = 1$ *and* $(p, q) = 1$ *for all primes with* $M/2 \leq p \leq M$. *Let* $\alpha \in [0, 1)$ *and* $\beta = \alpha - a/q$. *Then we have the estimate*

$$W(\alpha) - W(\alpha, q, a) \ll MH^{2/3}q^{1+\varepsilon}w_k(q)(1 + n|\beta|)^{1/2}(\log P)^{-1}.$$

Moreover, if $|\beta| \leq (6kq)^{-1}H^{1/3}n^{-1}$ *then*

$$W(\alpha) - W(\alpha, q, a) \ll MH^{2/3}q^{1+\varepsilon}w_k(q)(\log P)^{-1}.$$

PROOF. In the same way as before, we can express the exponential sum $W(\alpha)$ as

$$W(\alpha) = \sum_{\mathbf{y},p} W_{\mathbf{y},p}(\alpha) \quad \text{where } W_{\mathbf{y},p}(\alpha) = \sum_{H_1 \leq x \leq H_2} e(\alpha p^{3k}(x^3 + C_{\mathbf{y}})^k),$$

and the parameter $C_{\mathbf{y}}$ was defined in (3-1). Sorting the summation into arithmetic progressions modulo q and applying orthogonality, one has that

$$W_{\mathbf{y},p}(\alpha) = q^{-1} \sum_{-q/2 < b \leq q/2} S_{\mathbf{y}}(q, ap^{3k}, b) \sum_{H_1 \leq x \leq H_2} e\Big(\beta p^{3k}(x^3 + C_{\mathbf{y}})^k - \frac{bx}{q}\Big).$$

Because $(q, p) = 1$, a change of variables yields $S_{\mathbf{y}}(q, ap^{3k}) = S_{p\mathbf{y}}(q, a)$. Therefore, the application of the argument of [16, Theorem 4.1] in the same way as above leads to

$$W_{\mathbf{y},p}(\alpha) - q^{-1}S_{p\mathbf{y}}(q, a)v_{\mathbf{y},p}(\beta) \ll q^{1+\varepsilon}w_k(q)(1 + n|\beta|)^{1/2},$$

and if $|\beta| \leq (6kq)^{-1}H^{1/3}n^{-1}$ then

$$W_{\mathbf{y},p}(\alpha) - q^{-1}S_{p\mathbf{y}}(q, a)v_{\mathbf{y},p}(\beta) \ll q^{1+\varepsilon}w_k(q).$$

Summing over the range of $(\mathbf{y}, p)$ described in (3-12) delivers the desired result. □

4. Treatment of the singular series

Unless otherwise specified, in this section and the subsequent two we assume that $k = 2, 3$. We introduce some exponential sums and present upper bounds which we obtain by using the arguments in Vaughan [16, Theorem 7.1]. We also discuss the congruence problem and introduce some divisibility constraints on $C_{\mathbf{y}_i}$ and $C_{p_i\mathbf{y}_i}$ to ensure local solubility. For further purposes, we remind the reader of the definition in (2-10). For the rest of the paper, unless otherwise specified, $\mathbf{Y} = (\mathbf{y}_1, \ldots, \mathbf{y}_{s+t}) \in \mathbb{N}^{2s+2t}$ and $\mathbf{p} = (p_1, \ldots, p_t)$ denote tuples with $\mathbf{y}_i \in \mathcal{A}(P, P^\eta)^2$ for $t + 1 \leq i \leq s + t$ and $\mathbf{y}_i \in \mathcal{A}(H_3, P^\eta)^2$ for $1 \leq i \leq t$, where p_i are primes satisfying $M/2 \leq p_i \leq M$. We take for convenience $q \in \mathbb{N}$ and define

$$S_{\mathbf{Y},\mathbf{p}}(q) = q^{-s-t} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e(-an/q) \prod_{i=1}^{t} S_{p_i\mathbf{y}_i}(q, a) \prod_{i=t+1}^{s+t} S_{\mathbf{y}_i}(q, a).$$

The following technical lemma provides a straightforward upper bound for the previous exponential sum and is used throughout the major arc treatment.

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LEMMA 4.1. *Assume that $2 \le k \le 4$. Let $m \ge 2$. Take $\alpha \le (m-1)/3k$ when $m \ge 3$ and $\alpha = 0$ for $m = 2$. Let $Q \ge 1$. Then, when w_k is defined as in (3-5), one has*

$$\sum_{q \le Q} q^\alpha w_k(q)^m \ll Q^\varepsilon.$$

Moreover, for the case $k = 4$ we also have

$$\sum_{q \le Q} q\tau_4(q)^4 w_4(q) \ll Q^\varepsilon, \tag{4-1}$$

where $\tau_4(q)$ is defined just before Lemma 2.1.

PROOF. By the multiplicative property of $w_k(q)$ it follows that

$$\sum_{q \le Q} q^\alpha w_k(q)^m \ll \prod_{p \le Q} \Big(1 + \sum_{h=1}^{\infty} p^{h\alpha} w_k(p^h)^m\Big) \ll \prod_{p \le Q} (1 + Cp^{-1}) \ll Q^\varepsilon.$$

For the second estimate we use the bound $\tau_4(p^h)^4 \ll p^{-h}$ when $h \ge 2$ to obtain

$$\sum_{h=1}^{\infty} p^h \tau_4(p^h)^4 w_4(p^h) \ll p^{-3/2} + \sum_{h \ge 2} w_4(p^h) \ll p^{-1}.$$

Equation (4-1) then follows by combining the above bound with multiplicativity. □

LEMMA 4.2. *Let $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ be such that $(a, q) = 1$. The functions $S_{\mathbf{y}}(q, a)$ and $S_{\mathbf{Y},\mathbf{p}}(q)$ defined above satisfy*

$$S_{\mathbf{y}}(q, a) \ll q^{1+\varepsilon} w_k(q), \quad S_{\mathbf{Y},\mathbf{p}}(q) \ll q^{1+\varepsilon} w_k(q)^{s+t}. \tag{4-2}$$

As a consequence, for every $Q \ge 1$ and every $\alpha \le ((s + t - 1)/3k) - 1$ it follows that

$$\sum_{q \le Q} q^\alpha |S_{\mathbf{Y},\mathbf{p}}(q)| \ll Q^\varepsilon \quad \text{and} \quad \sum_{q > Q} |S_{\mathbf{Y},\mathbf{p}}(q)| \ll Q^{\varepsilon-\alpha}. \tag{4-3}$$

PROOF. Recalling (3-8), note that $S_{\mathbf{y}}(q, a) = S_{\mathbf{y}}(q, a, 0)$. Therefore, (3-10) yields $S_{\mathbf{y}}(q, a) \ll q^{1+\varepsilon} w_k(q)$; and hence (4-2) holds. These estimates and Lemma 4.1 imply the first inequality in (4-3). Finally, observe that as a consequence we have

$$\sum_{Q \le q \le 2Q} |S_{\mathbf{Y},\mathbf{p}}(q)| \ll Q^{\varepsilon-\alpha},$$

from which the second inequality of (4-3) follows by summing over dyadic intervals. □

We apply the bounds obtained in the previous lemma to a collection of singular series and other related series. For this purpose, it is convenient to define, for tuples $(\mathbf{Y}, \mathbf{p})$ and each prime p, the sums

$$\mathfrak{S}_{\mathbf{Y},\mathbf{p}}(n) = \sum_{q=1}^{\infty} S_{\mathbf{Y},\mathbf{p}}(q), \quad \sigma(p) = \sum_{l=0}^{\infty} S_{\mathbf{Y},\mathbf{p}}(p^l).$$

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LEMMA 4.3. *The singular series* $\mathfrak{S}_{\mathbf{Y},\mathbf{p}}(n)$ *converges absolutely, the identity*

$$\mathfrak{S}_{\mathbf{Y},\mathbf{p}}(n) = \prod_p \sigma(p) \tag{4-4}$$

holds and $0 \leq \mathfrak{S}_{\mathbf{Y},\mathbf{p}}(n) \ll 1$. *We recall (2-10) and (3-1) to the reader. Then one has* $\mathfrak{S}_{\mathbf{Y},\mathbf{p}}(n) \gg 1$ *provided that the following hold.*

(1) *When* $k = 2$, *one has* $C_{p_i\mathbf{y}_i} \equiv 28 \pmod{108}$ *for* $1 \leq i \leq t$ *and* $C_{\mathbf{y}_i} \equiv 28 \pmod{108}$ *for* $t + 1 \leq i \leq s + t$.

(2) *When* $k = 3$, *one has* $C_{p_i\mathbf{y}_i} \equiv 0 \pmod{162}$ *for* $1 \leq i \leq t$ *and* $C_{\mathbf{y}_i} \equiv 0 \pmod{162}$ *for* $t + 1 \leq i \leq s + t$.

As mentioned above, the constraints on $C_{\mathbf{y}_i}$ and $C_{p_i\mathbf{y}_i}$ ensure the local solubility of the problem. Note that the set of tuples with these divisibility conditions has positive density over the set of tuples without these restrictions, since it follows from the proof of Lemma 5.4 of [15] that smooth numbers are well distributed on arithmetic progressions. Therefore, we are still able to get the expected lower bound for the major arc contribution. Observe though that the choices for the constraints are not unique, but for the purpose of this exposition it will suffice to study just one of the possible restrictions.

PROOF. Note that the application of Lemma 4.2 yields the estimate

$$\sigma(p) - 1 \ll p^{-2}. \tag{4-5}$$

This bound and the multiplicative property of $S_{\mathbf{Y},\mathbf{p}}(q)$ imply (4-4), the convergence of the series $\mathfrak{S}_{\mathbf{Y},\mathbf{p}}(n)$ and the desired upper bound for it. To give a more arithmetic description of $\sigma(p)$ it is convenient to introduce

$$\mathcal{M}_n(p^h) = \Big\{\mathbf{X} \in [1, p^h]^{s+t} : \ n \equiv \sum_{i=1}^{t} (x_i^3 + C_{p_i\mathbf{y}_i})^k + \sum_{i=t+1}^{s+t} (x_i^3 + C_{\mathbf{y}_i})^k \pmod{p^h}\Big\}$$

and $M_n(p^h) = |\mathcal{M}_n(p^h)|$. Observe that by a standard argument making use of orthogonality we obtain the relation

$$\sum_{l=0}^{h} S_{\mathbf{Y},\mathbf{p}}(p^l) = p^{(1-s-t)h} M_n(p^h).$$

In view of (4-5), in order to prove the lower bound for $\mathfrak{S}_{\mathbf{Y},\mathbf{p}}(n)$ it suffices to show that $p^{(1-s-t)h} M_n(p^h) \geq C_p$ for some positive constant C_p depending on p. For each p prime, take $\tau \geq 0$ for which $p^\tau \| 3k$. Define $\gamma = \gamma(p) = 2\tau + 1$ and

$$\mathcal{M}_n^*(p^\gamma) = \{\mathbf{X} \in \mathcal{M}_n(p^\gamma) : \ p \nmid x_1, \ \ p \nmid (x_1^3 + C_{p_1\mathbf{y}_1})\}.$$

We take $h \geq \gamma$ for convenience. Our priority for the rest of the proof is to show that $|\mathcal{M}_n^*(p^\gamma)| > 0$, since then an application of Hensel's Lemma yields the bound $M_n(p^h) \geq p^{(s+t-1)(h-\gamma)}$.

For further discussion, it is convenient to consider for a fixed number $C \in \mathbb{N}$ the sets

$$\mathcal{T}_C(p^\gamma) = \{x^3 + C \pmod{p^\gamma}\}, \quad \mathcal{T}_C^*(p^\gamma) = \{x^3 + C \pmod{p^\gamma} : \ p \nmid x, \ \ p \nmid (x^3 + C)\}.$$

Let $p \equiv 1 \pmod 3$. Under this condition one has $p \geq 7$, whence it then follows that $\gamma = 1$ with $|\mathcal{T}_C(p)| = (p+2)/3$ and $|\mathcal{T}_C^*(p)| \geq 1$. If we denote the set of kth powers of the above set by

$$\mathcal{T}_C^k(p^\gamma) = \{y^k \pmod{p^\gamma} : \ y \in \mathcal{T}_C(p^\gamma)\},$$

then $|\mathcal{T}_C^k(p)| \geq \lceil (p+2)/3k \rceil$. One can check that $|\mathcal{T}_C^k(7)| \geq 2$ for every $C \in \mathbb{N}$, and, whenever $p > 7$,

$$(s+t-1)\Big(\Big\lceil \frac{p+2}{3k} \Big\rceil - 1\Big) \geq p;$$

whence Cauchy–Davenport [16, Lemma 2.14] delivers $|\mathcal{M}_n^*(p)| > 0$. When $p \equiv 2 \pmod 3$ and $p > 2$ then $\gamma = 1$, and we further get $|\mathcal{T}_C(p)| = p$ and $|\mathcal{T}_C^*(p)| \geq 1$; whence another application of [16, Lemma 2.14] yields $|\mathcal{M}_n^*(p)| > 0$. For the case $p = 2$ and $k = 2$ the divisibility constraints reduce the problem to the resolution of

$$y_1^6 + \cdots + y_8^6 \equiv n \pmod 8$$

with $y_i \in \mathbb{N}$ and $2 \nmid y_1$, which is straightforward. The case $k = 3$ is also trivial since then one would have $\gamma(2) = 1$. Likewise, if one has $p = 3$ one finds that whenever $C \equiv 1 \pmod{27}$ then $\mathcal{T}_C^2(27) = \{0, 1, 4, 13, 22\}$ and $|\mathcal{T}_C^*(27)| = 3$; so $|\mathcal{M}_n^*(27)| > 0$ when $k = 2$ follows by combining the constraints for $C_{p_i\mathbf{y}_i}$ and $C_{\mathbf{y}_i}$ described above and [16, Lemma 2.14]. Finally, when $k = 3$ we make use of the conditions $C_{\mathbf{y}_i} \equiv 0 \pmod{81}$ and $C_{p_i\mathbf{y}_i} \equiv 0 \pmod{81}$ to reduce the problem to finding a solution for

$$y_1^9 + \cdots + y_{17}^9 \equiv n \pmod{243}$$

with $y_i \in \mathbb{N}$ and $3 \nmid y_1$. The solubility of this congruence is a consequence of [16, Lemma 2.15]. □

5. Singular integral

In this section we analyse the size of the singular integral following the classical approach making use of Fourier's integral theorem. For each pair of tuples $(\mathbf{Y}, \mathbf{p})$ consider

$$J_{\mathbf{Y},\mathbf{p}}(n) = \int_{-\infty}^{\infty} V_{\mathbf{Y},\mathbf{p}}(\beta) e(-n\beta) d\beta \quad \text{where } V_{\mathbf{Y},\mathbf{p}}(\beta) = \prod_{i=1}^{t} v_{\mathbf{y}_i,p_i}(\beta) \prod_{i=t+1}^{s+t} v_{\mathbf{y}_i}(\beta),$$

and $v_{\mathbf{y_i},p_i}(\beta)$ and $v_{\mathbf{y}_i}(\beta)$ are defined in (3-2).

Lemma 5.1. *One has that* $0 \leq J_{\mathbf{Y},\mathbf{p}}(n) \ll P^s H^{t/3} n^{-1}$. *Moreover, whenever* $(\mathbf{Y}, \mathbf{p})$ *satisfies* $M/2 \leq p_i \leq 51M/100$ *for* $1 \leq i \leq t$ *and* $\mathbf{y}_i \leq P/2$ *for* $t+1 \leq i \leq s+t$, *then*

$$J_{\mathbf{Y},\mathbf{p}}(n) \gg P^s H^{t/3} n^{-1}. \tag{5-1}$$

In the following discussion we rewrite $J_{\mathbf{Y},\mathbf{p}}(n)$ as an integral whose size is easier to estimate. The conditions on the tuples described above ensure that we get a suitable range of integration for such an integral. Note that the set of tuples on that range has positive density over the set of tuples without the restrictions, and hence we are still able to get the expected lower bound for the major arc contribution.

PROOF. By using the expressions for $v_{\mathbf{y}}(\beta)$ and $v_{\mathbf{y},p}(\beta)$ in (3-3), we find that

$$J_{\mathbf{Y},\mathbf{p}}(n) = \lim_{\lambda\to\infty} \int_{-\lambda}^{\lambda} \int_{\mathbf{x}\in\mathcal{S}} B_{\mathbf{Y},\mathbf{p}}(\mathbf{x}) e\Big(\beta\Big(\sum_{i=1}^{s+t} x_i - n\Big)\Big)\, d\mathbf{x}\, d\beta,$$

where the function $B_{\mathbf{Y},\mathbf{p}}(\mathbf{x})$ is taken to be

$$B_{\mathbf{Y},\mathbf{p}}(\mathbf{x}) = \prod_{i=1}^{t} B_{\mathbf{y}_i,p_i}(x_i) \prod_{i=t+1}^{s+t} B_{\mathbf{y}_i}(x_i)$$

and we integrate over the set $\mathcal{S} = \prod[M_{\mathbf{y}_i,\mathbf{p}_i}, N_{\mathbf{y}_i,\mathbf{p}_i}] \times \prod[M_{\mathbf{y}_i}, N_{\mathbf{y}_i}]$. Then, by integrating with respect to β and making the change of variables $v = \sum_{i=1}^{s+t} x_i$, we obtain

$$J_{\mathbf{Y},\mathbf{p}}(n) = \lim_{\lambda\to\infty} \int_{S_1}^{S_2} \phi(v) \frac{\sin(2\pi\lambda(v-n))}{\pi(v-n)}\, dv,$$

where $\phi(v)$ is defined as

$$\phi(v) = \int_{\mathbf{x}\in\mathcal{S}'(v)} B_{\mathbf{y}_{s+t}}\Big(v - \sum_{i=1}^{s+t-1} x_i\Big) \prod_{i=1}^{t} B_{\mathbf{y}_i,p_i}(x_i) \prod_{i=t+1}^{s+t-1} B_{\mathbf{y}_i}(x_i)\, d\mathbf{x},$$

the subset $\mathcal{S}'(v) \subset \mathbb{R}^{s+t-1}$ denotes the tuples satisfying

$$x_i \in [M_{\mathbf{y}_i,p_i}, N_{\mathbf{y}_i,p_i}] \quad \text{for } 1 \le i \le t, \qquad x_i \in [M_{\mathbf{y}_i}, N_{\mathbf{y}_i}] \quad \text{for } t+1 \le i \le s+t-1,$$

and

$$M_{\mathbf{y}_{s+t}} \le v - \sum_{i=1}^{s+t-1} x_i \le N_{\mathbf{y}_{s+t}} \tag{5-2}$$

and the above limits of integration are

$$S_1 = \sum_{i=1}^{t} M_{\mathbf{y}_i,\mathbf{p}_i} + \sum_{i=t+1}^{s+t} M_{\mathbf{y}_i}, \qquad S_2 = \sum_{i=1}^{t} N_{\mathbf{y}_i,\mathbf{p}_i} + \sum_{i=t+1}^{s+t} N_{\mathbf{y}_i}.$$

Since $\phi(v)$ is a function of bounded variation, it follows from Fourier's integral theorem (to which we refer the reader to the argument in Davenport [3, pages 21–22] or in [20, Section 9.43]) that $J_{\mathbf{Y},\mathbf{p}}(n) = \phi(n)$, which implies positivity. Note that combining the identity (2-8), the limits of integration defined after (3-3) and Equation (3-4), we find that whenever $\mathbf{x} \in \mathcal{S}'(n)$ then $B_{\mathbf{y}_i,p_i}(x_i) \asymp H^{1/3} n^{-1}$ for $1 \le i \le t$ and $B_{\mathbf{y}_i}(x_i) \asymp P n^{-1}$ for

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$t+1 \le i \le s+t-1$, and one further has

$$B_{\mathbf{y}_{s+t}}\Big(n - \sum_{i=1}^{s+t-1} x_i\Big) \asymp Pn^{-1}.$$

Therefore, combining the previous ideas we obtain the upper bound for $J_{\mathbf{Y},\mathbf{p}}(n)$ stated at the beginning of the lemma. Moreover, if $(\mathbf{Y},\mathbf{p})$ lies in the range described right after that bound, then there exist intervals $I_i \subset [M_{\mathbf{y}_i,p_i}, N_{\mathbf{y}_i,p_i}]$ for $1 \le i \le t$ and $I_i \subset [M_{\mathbf{y}_i}, N_{\mathbf{y}_i}]$ for $t+1 \le i \le s+t-1$ satisfying $|I_i| \asymp n$, with the property that whenever $x_i \in I_i$, (5-2) holds for $v = n$. Consequently, the preceding discussion yields (5-1). □

For the sake of brevity we define the auxiliary functions $h^*(\alpha)$ and $W^*(\alpha)$ by putting

$$h^*(\alpha) = V(\alpha, q, a) \quad \text{and} \quad W^*(\alpha) = W(\alpha, q, a)$$

when $\alpha \in \mathfrak{M}(a,q) \subset \mathfrak{M}$ and $h^*(\alpha) = W^*(\alpha) = 0$ for $\alpha \in \mathfrak{m}$. Here the reader may want to recall (3-6) and (3-12). In the rest of this section we present some bounds for these functions.

LEMMA 5.2. *Let $\beta \in \mathbb{R}$. For every prime p and $\mathbf{y} \in \mathbb{N}^2$ one has*

$$v_{\mathbf{y}}(\beta) \ll \frac{P}{1+n|\beta|} \quad \text{and} \quad v_{\mathbf{y},p}(\beta) \ll \frac{H^{1/3}}{1+n|\beta|}.$$

Moreover, whenever $\alpha \in \mathfrak{M}(a,q) \subset \mathfrak{M}$ we have

$$h^*(\alpha) \ll \frac{q^{\varepsilon} w_k(q) P^3}{1+n|\alpha - a/q|} \quad \text{and} \quad W^*(\alpha) \ll \frac{q^{\varepsilon} w_k(q) MH}{(1+n|\alpha - a/q|)(\log P)}.$$

PROOF. When $|\beta| \le n^{-1}$, the bound for $v_{\mathbf{y}}(\beta)$ follows observing that by (3-3) and the limits of integration given after (3-3) we have

$$v_{\mathbf{y}}(\beta) \ll \int_{M_{\mathbf{y}}}^{N_{\mathbf{y}}} y^{1/k-1}(y^{1/k} - C_{\mathbf{y}})^{-2/3}\, dy \ll P.$$

The function $C_{\mathbf{y}}$ in the above line is defined in Equation (3-1). For the case $|\beta| > n^{-1}$, using the fact that $B_{\mathbf{y}}(y)$ is decreasing and integrating by parts, we have that

$$v_{\mathbf{y}}(\beta) \ll |\beta|^{-1} B_{\mathbf{y}}(M_{\mathbf{y}}) \ll |\beta|^{-1} n^{1/3k-1},$$

which proves the statement. The bound for $v_{\mathbf{y},p}(\beta)$ is proved in a similar way; the proof follows after applying (2-8). Combining these estimates and Lemma 4.2 we get the bounds for $h^*(\alpha)$ and $W^*(\alpha)$. □

6. Major arc contribution

In this section we show that the contribution of the set of narrow arcs $\mathfrak{N}$ is asymptotic to the expected main term. We then prove that the contribution of the remaining arcs is smaller by combining major and minor arc techniques and making use of Lemma 2.1.

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PROPOSITION 6.1. *There exists $\delta > 0$ such that*

$$\int_{\mathfrak{M}} h(\alpha)^s W(\alpha)^t e(-\alpha n)\, d\alpha = \sum_{\mathbf{Y},\mathbf{p}} \mathfrak{S}_{\mathbf{Y},\mathbf{p}}(n) J_{\mathbf{Y},\mathbf{p}}(n) + O(H^t M^t P^{3s-3k-\delta}),$$

where $(\mathbf{Y}, \mathbf{p})$ lies in the range of summation described at the beginning of Section 4.

PROOF. We note first that the triangle inequality yields

$$h(\alpha)^s - h^*(\alpha)^s \ll |h(\alpha) - h^*(\alpha)|(|h^*(\alpha)|^{s-1} + |h(\alpha) - h^*(\alpha)|^{s-1}).$$

Observe that by (2-7) and the definition in (2-11), whenever $\alpha \in \mathfrak{N}(a, q)$ one has that $(1 + n|\beta|)^{-1} \geq qH^{-1/3} \geq qP^{-1}$ and $|\beta| \leq (6kq)^{-1} H^{1/3} n^{-1} \leq (2 \cdot 3^k kq)^{-1} P n^{-1}$ for n sufficiently large. Consequently, Lemma 3.1 applied to $|h(\alpha) - h^*(\alpha)|$ and Lemma 5.2 applied to $|h^*(\alpha)|$ in the above inequality deliver

$$h(\alpha)^s - h^*(\alpha)^s \ll q^{1+\varepsilon} w_k(q)^s P^{3s-1} (1 + n|\beta|)^{-s+1}; \tag{6-1}$$

and by the same reason, whenever $\alpha \in \mathfrak{N}(a, q)$ with $(p, q) = 1$ for all primes p having the property that $M/2 \leq p \leq M$, Lemma 3.2 gives

$$W(\alpha)^t - W^*(\alpha)^t \ll M^t H^{t-1/3} q^{1+\varepsilon} w_k(q)^t (1 + n|\beta|)^{-t+1}. \tag{6-2}$$

We also need a bound on the following quantity to exploit some orthogonality relation when averaging over q. Denote by $N(q, P)$ the number of solutions of the congruence

$$T(p_1\mathbf{x}_1)^k + T(p_2\mathbf{x}_2)^k \equiv T(p_3\mathbf{x}_3)^k + T(p_4\mathbf{x}_4)^k \pmod q,$$

where $\mathbf{x}_i \in [1, H^{1/3}]^3$ and $M/2 \leq p_i \leq M$ with $q \in \mathbb{N}$. By expressing q as the product of prime powers, using the structure of the ring of integers modulo a prime power and noting that the number of primes dividing q is $O((\log q)/\log\log q)$, we obtain

$$N(q, P) \ll q^{\varepsilon} (MH)^4 (\log P)^{-4} (q^{-1} + P^{-1}), \tag{6-3}$$

where we also use the identity (2-8); hence, by orthogonality it follows that

$$\sum_{a=1}^{q} |W(\beta + a/q)|^4 \leq qN(q, P) \ll q^{1+\varepsilon} (MH)^4 (\log P)^{-4} (q^{-1} + P^{-1}). \tag{6-4}$$

Combining (6-1) and (6-4) one has that

$$\int_{\mathfrak{N}} |h(\alpha)^s - h^*(\alpha)^s||W(\alpha)|^t\,d\alpha \ll (HM)^t P^{3s-3k-1} \sum_{q\le H^{1/3}} q^{1+\varepsilon} w_k(q)^s \ll (HM)^t P^{3s-3k-\delta},$$

where we use (2-7) and Lemma 4.1. Before introducing the auxiliary function $W^*(\alpha)$ to replace $W(\alpha)$ we must ensure that the contribution of the arcs having the property that $M/4 < q \le (6k)^{-1}H^{1/3}$ is small enough. By doing so we avoid having to approximate $W(\alpha)$ for the cases when $p \mid q$ for primes p appearing in the definition (1-3) of $W(\alpha)$. Combining Lemma 5.2 with (6-4) one finds that

$$\sum_{M/4<q\le(6k)^{-1}H^{1/3}} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_0^1 |h^*(\beta+a/q)|^s |W(\beta+a/q)|^t\,d\beta \ll (HM)^t P^{3s-3k+\varepsilon} \sum_{M/4<q\le(6k)^{-1}H^{1/3}} w_k(q)^s \ll (HM)^t P^{3s-3k-\delta},$$

where in the last step we apply the definition in (3-5). For the range $q \le M/4$ we always have $(p, q) = 1$ for all primes p with $M/2 \le p \le M$; so we can use (6-2) and Lemma 5.2 to obtain

$$\sum_{q\le M/4} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{\mathfrak{N}(a,q)} |h^*(\alpha)|^s |W(\alpha)^t - W^*(\alpha)^t|\,d\alpha \ll P^{3s-3k} M^t H^{t-1/3} \sum_{q\le M/4} q^{2+\varepsilon} w_k(q)^{s+t} \ll (HM)^t P^{3s-3k-\delta},$$

where in the last line we use (2-7) and apply Lemma 4.1. By Lemmas 4.1 and 5.2 one has that

$$\sum_{q\le M/4} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{|\alpha-a/q|>(6kq)^{-1}H^{1/3}n^{-1}} |h^*(\alpha)|^s |W^*(\alpha)|^t\,d\alpha \ll H^{2t/3-s/3+1/3} M^t P^{3s-3k} \sum_{q\le M/4} q^{s+t+\varepsilon} w_k(q)^{s+t} \ll (HM)^t P^{3s-3k-\delta}.$$

Therefore, using the previous bounds, making a change of variables and combining Lemmas 4.2 and 5.2, we have

$$\int_{\mathfrak{N}} h(\alpha)^s W(\alpha)^t e(-\alpha n)\,d\alpha = \sum_{\mathbf{Y},\mathbf{p}} \mathfrak{S}_{\mathbf{Y},\mathbf{p}}(n) J_{\mathbf{Y},\mathbf{p}}(n) + O((HM)^t P^{3s-3k-\delta}). \tag{6-5}$$

The rest of this section is devoted to ensuring that the contribution of the remaining major arcs is smaller than the main term in the previous equation. Let $R(q, P)$ be the number of solutions of the congruence

$$T(\mathbf{x}_1)^k + T(\mathbf{x}_2)^k \equiv T(\mathbf{x}_3)^k + T(\mathbf{x}_4)^k \pmod q,$$

where $\mathbf{x}_i \in [1, P]^3$. Applying the same argument we use in (6-3) for bounding $N(q, P)$, we find that $R(q, P) \ll q^{\varepsilon} P^{12}(q^{-1} + P^{-1})$, and hence by orthogonality it follows that

$$\sum_{a=1}^{q} |h(\beta + a/q)|^4 \leq qR(q, P) \ll q^{1+\varepsilon} P^{12}(q^{-1} + P^{-1}). \tag{6-6}$$

Moreover, observe that for the case $k = 2$ by a similar argument we get

$$\sum_{a=1}^{q} |h(\beta + a/q)|^2 \ll q^{1+\varepsilon} P^{6}(q^{-1} + P^{-1}). \tag{6-7}$$

We consider for convenience the mean value

$$I_M = \int_{\mathfrak{M}\setminus\mathfrak{N}} |h(\alpha)|^s |W(\alpha)|^t \, d\alpha.$$

Our strategy for the treatment of this integral is to bound $W(\alpha)$ pointwise via Lemma 2.1 and use some major arc estimates. For such purposes, we first define $\Upsilon(\alpha)$ for $\alpha \in [0, 1)$ by taking

$$\Upsilon(\alpha) = \tau_k(q)(1 + n|\alpha - a/q|)^{-1}$$

when $\alpha \in \mathfrak{M}(a, q) \subset \mathfrak{M}$, and $\Upsilon(\alpha) = 0$ otherwise. When $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy the inequality $0 \leq a \leq q \leq M^k$ and $(a, q) = 1$, consider the set of arcs

$$\mathfrak{M}'(a, q) = \left\{\alpha \in [0, 1) : |\alpha - a/q| \leq \frac{M}{q^{1/k} n}\right\} \tag{6-8}$$

and take $\mathfrak{M}'$ to be the union of such arcs. Note that one then has $\mathfrak{M}' \subset \mathfrak{M}$. Observe that for $\alpha \in \mathfrak{M} \setminus \mathfrak{M}'$, the bound on the right-hand side of (2-4) corresponding to the diagonal contribution dominates over the one corresponding to the nondiagonal contribution. Therefore, we can apply the same argument that we use in Proposition 2.2 to estimate the integral over this set. When $\alpha \in \mathfrak{M}'$, the dominating bound is the one corresponding to the nondiagonal terms. Let I'_M be the contribution of $\mathfrak{M}' \setminus \mathfrak{N}$ to the integral I_M. By making use of Lemma 2.1 and (2-9) we obtain that

$$I'_M \ll H^{t+t/24-\delta} M^t \int_{\mathfrak{M}'\setminus\mathfrak{N}} |h(\alpha)|^s \Upsilon(\alpha)^{t/2} \, d\alpha \ll H^{t+t/24-\delta} M^t (I_1 + I_2),$$

where

$$I_i = \int_{\mathfrak{M}'\setminus\mathfrak{N}} |h(\alpha)|^{s-2} G_i(\alpha) \Upsilon(\alpha)^{t/2} \, d\alpha, \quad i = 1, 2,$$

with $G_1(\alpha) = |h^*(\alpha)|^2$ and $G_2(\alpha) = |h(\alpha) - h^*(\alpha)|^2$. In view of the definitions in (2-11) and (6-8) for $\mathfrak{N}$ and $\mathfrak{M}'$, respectively, we make a distinction between the ranges

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$q \le (6k)^{-1}H^{1/3}$ and $(6k)^{-1}H^{1/3} < q \le M^k$. We also combine Lemmas 4.1 and 5.2 with Equations (6-6) and (6-7) and the bound given in (2-2) to obtain

$$\begin{aligned} I_1 &\ll P^{3s-3k}H^{-t/6-1/3} \sum_{q\le(6k)^{-1}H^{1/3}} w_k(q)^2 q^{t/2+1-t/2k+\varepsilon} \\ &\quad + P^{3s-3k} \sum_{(6k)^{-1}H^{1/3}<q\le M^k} w_k(q)^2 q^{1-t/2k+\varepsilon}(q^{-1}+P^{-1}) \ll P^{3s-3k+\varepsilon}H^{-t/6k}. \end{aligned}$$

Likewise, combining Equations (6-6) and (6-7) with Lemmas 3.1 and 4.1 one finds that

$$\begin{aligned} I_2 &\ll P^{3s-3k-2+\varepsilon}H^{-t/6+2/3} \sum_{q\le(6k)^{-1}H^{1/3}} q^{t/2-t/2k} w_k(q)^2 \\ &\quad + P^{3s-3k-2+\varepsilon} \sum_{(6k)^{-1}H^{1/3}<q\le M^k} w_k(q)^2 q^{3-t/2k}(q^{-1}+P^{-1}) \ll P^{3s-3k+\varepsilon}H^{-t/6k}, \end{aligned}$$

where we make use of (2-7). Therefore, we obtain that $I'_M = O((HM)^t P^{3s-3k-\delta})$, whence the proposition follows by combining (6-5) with the previous estimates. □

PROOF OF THEOREM 1.1 WHEN $k = 2, 3$. Note first that Lemma 5.1 ensures positivity for $J_{\mathbf{Y},\mathbf{p}}(n)$ and guarantees that for $(\mathbf{Y},\mathbf{p})$ in the range described there $J_{\mathbf{Y},\mathbf{p}}(n) \gg P^s H^{t/3} n^{-1}$. Similarly, Lemma 4.3 ensures the positivity of $\mathfrak{S}_{\mathbf{Y},\mathbf{p}}(n)$ and implies that for $(\mathbf{Y},\mathbf{p})$ satisfying the local conditions described after (4-4) one has $\mathfrak{S}_{\mathbf{Y},\mathbf{p}}(n) \gg 1$. As observed at the beginning of the lemmas, the intersection of the sets of pairs $(\mathbf{Y},\mathbf{p})$ satisfying these conditions has positive density. Therefore, we find that

$$\sum_{\mathbf{Y},\mathbf{p}} \mathfrak{S}_{\mathbf{Y},\mathbf{p}}(n) J_{\mathbf{Y},\mathbf{p}}(n) \gg (HM)^t P^{3s-3k}(\log P)^{-t}.$$

Propositions 2.2 and 6.1 then yield the bound $R(n) \gg (HM)^t P^{3s-3k}(\log P)^{-t}$, which proves the theorem for $k = 2, 3$. □

7. The case $k = 4$.

In this section we discuss the proof of the theorem for fourth powers. For this purpose, it is convenient to introduce the exponential sum

$$f(\alpha) = \sum_{x\in\mathcal{A}(P,P^\eta)} e(\alpha x^{12}).$$

Let $R_4(n)$ be the number of solutions of the equation

$$n = \sum_{i=1}^{11} T(p_i\mathbf{x}_i)^4 + 81(y_1^{12} + \cdots + y_{46}^{12}),$$

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where $\mathbf{x}_i \in \mathcal{W}$ with $M/2 \leq p_i \leq M$ for $1 \leq i \leq 11$ and $y_i \in \mathcal{A}(P, P^\eta)$ for $1 \leq i \leq 46$. Observe that the sums of three cubes on the right-hand side have been replaced by the specialisation $3y^3$. Note as well that orthogonality yields the identity

$$R_4(n) = \int_0^1 W(\alpha)^{11} f(81\alpha)^{46} e(-\alpha n)\, d\alpha.$$

Our goal throughout this section is to obtain a lower bound for $R_4(n)$ for all sufficiently large n. Recalling (2-7) and (2-12) and using the table of permissible exponents for $k = 12$ from [18], we find that

$$\begin{aligned} \int_{\mathfrak{m}} |W(\alpha)|^{11} |f(81\alpha)|^{46}\, d\alpha &\ll H^{11+11/24-\delta} M^{11/2} \int_0^1 |f(\alpha)|^{46}\, d\alpha \\ &\ll (HM)^{11} P^{34+\Delta_{23}-1/2-\delta}, \end{aligned} \tag{7-1}$$

where $\Delta_{23} = 0.498\,838\,3$; it follows that the minor arc contribution is then $O((HM)^{11} P^{34-\delta})$.

We define a set of narrow major arcs $\mathfrak{P}$ by taking the union of

$$\mathfrak{P}(a, q) = \left\{ \alpha \in [0, 1) : |\alpha - a/q| \leq \frac{R}{n} \right\}$$

with $0 \leq a \leq q \leq R$ and $(a, q) = 1$, where $R = (\log P)^{1/5}$, and consider $\mathfrak{p} = [0, 1) \setminus \mathfrak{P}$. In the next few lines we combine various major and minor arc techniques to prune back to the set of narrow arcs $\mathfrak{P}$. As observed after (6-8), whenever $\alpha \in \mathfrak{M} \setminus \mathfrak{M}'$, the bound on the right-hand side of (2-4) corresponding to the diagonal contribution dominates over the one corresponding to the nondiagonal contribution. Therefore, we can apply the same argument that we applied in (7-1) to obtain that the integral over that set is $O((HM)^{11} P^{34-\delta})$.

We next note that Theorem 1.8 from Vaughan [15] yields

$$\sup_{\mathfrak{n}} |f(81\alpha)| \ll P^{1-\rho+\varepsilon}, \tag{7-2}$$

where $\rho = 0.004\,259$. (Although one could obtain a slightly bigger ρ by applying methods from [23], such improvement in the exponent would have no impact on the argument.) Observe that using the same procedure as in (6-4) and (6-6) we deduce that

$$\sum_{a=1}^{q} |f(81(\beta + a/q))|^{12} \ll q^{1+\varepsilon} P^{12} (q^{-1} + P^{-1}). \tag{7-3}$$

Note as well that whenever $\alpha \in \mathfrak{M}' \setminus \mathfrak{N}$ then $(1 + n|\beta|)^{3/2} \geq H^{1/3} q^{-1}$, and hence Lemmas 3.2 and 5.2 yield

$$W(\alpha) \ll M H^{2/3} q^{1+\varepsilon} w_4(q) (1 + n|\beta|)^{1/2}.$$

By the preceding discussion, together with Lemma 2.1 and Equations (7-2) and (7-3), we obtain

$$\int_{\mathfrak{M}'\setminus\mathfrak{N}} |W(\alpha)|^{11}|f(81\alpha)|^{46}\,d\alpha \ll (HM)^{11}P^{34(1-\rho)}\sum_{q\leq M^4} q^2\tau_4(q)^4 w_4(q)(q^{-1}+P^{-1}).$$

Here, we apply the estimates (2-4) and (2-9) to eight copies of $W(\alpha)$ and the bound for $W(\alpha)$ deduced above to just one of them. Likewise, we make use of the pointwise estimate (7-2) to bound 34 copies of $f(81\alpha)$, and we use the other 12 to exploit the congruence condition via (7-3). We find that the above sum when $q \leq P$ is $O((HM)^{11}P^{34-\delta})$ via Lemma 4.1. Similarly, we use Lemma 4.1 and the bound $qP^{-1} \leq P^{1/11}$, which follows after an application of (2-7), for the range $P \leq q \leq M^4$ to obtain that this contribution is also $O((HM)^{11}P^{34-\delta})$. By the observation made before (6-1), which is still valid for $k = 4$, and Lemma 3.2 we find that whenever $\alpha \in \mathfrak{N}$ then

$$W(\alpha) \ll \frac{q^{\varepsilon}w_4(q)HM}{(1+n|\beta|)(\log P)}.$$

Therefore, the application of this bound and (6-4) yield

$$\int_{\mathfrak{N}\setminus\mathfrak{P}} |W(\alpha)|^{11}|f(81\alpha)|^{46}\,d\alpha \ll (HM)^{11}P^{34}(\log P)^{-11}R^{-6}\sum_{q\leq R} q^{\varepsilon}w_4(q)^7 + (HM)^{11}P^{34}(\log P)^{-11}\sum_{q>R} q^{\varepsilon}w_4(q)^7.$$

Consequently, a succinct application of Lemma 4.1 and (3-5) implies that this integral is $O((HM)^{11}P^{34}(\log P)^{-11-\delta})$.

In what follows, we briefly describe the singular series involved in the problem. There might be other approaches that would lead to more precise asymptotic formulae, but for the sake of simplicity we avoid including the sums of three cubes in the singular series. Recalling (2-1), it is convenient to consider, for an integer $m \in \mathbb{N}$ and a prime p, the sums

$$S_m(q) = q^{-46}\sum_{\substack{a=1\\(a,q)=1}}^{q} S_{12}(q,81a)^{46}e_q(-a(n-m)),\quad \sigma_m(p) = \sum_{h=0}^{\infty} S_m(p^h).$$

Observe that whenever $3 \nmid q$ we can make a change of variables to rewrite $S_m(q)$ as

$$S_m(q) = q^{-46}\sum_{\substack{a=1\\(a,q)=1}}^{q} S_{12}(q,a)^{46}e_q(-a\overline{81}^{-1}(n-m)),$$

where $\overline{81}^{-1}$ denotes the inverse of 81 (mod q). Note as well that Lemma 3 of [14] yields the bound $S_m(q) \ll q\tau_{12}(q)^{46}$, which implies that $\sigma_m(p) = 1 + O(p^{-22})$ and delivers the convergence of the singular series

$$\mathfrak{S}_m(n) = \sum_{q=1}^{\infty} S_m(q)$$

and its upper bound $\mathfrak{S}_m(n) \ll 1$. Here, we implicitly use the multiplicativity of $S_m(q)$ and the expression of the singular series as the product

$$\mathfrak{S}_m(n) = \prod_p \sigma_m(p).$$

The estimate $S_m(q) \ll q^{-17/6}$, which follows trivially via an application of [16, Theorem 4.2], also delivers, for $Q \geq 1$, the bound

$$\sum_{q>Q} |S_m(q)| \ll Q^{-\alpha} \tag{7-4}$$

for some $\alpha > 0$. Observe that by Lemmas 2.12, 2.13 and 2.15 of [16] one gets for every prime $p \neq 3$ the lower bound $\sigma_m(p) \geq p^{-45\gamma}$, where $\gamma = 3$ when $p = 2$ and $\gamma = 1$ otherwise. Likewise, note that when $m \equiv n \pmod{81}$ and $h \geq 5$, orthogonality yields

$$\sum_{l=0}^{h} S_m(3^l) = 3^{-45h} M_{n,m}(3^h),$$

where $M_{n,m}(3^h)$ denotes the number of solutions of the congruence

$$x_1^{12} + \cdots + x_{46}^{12} \equiv (n-m)/81 \pmod{3^{h-4}}$$

with $1 \leq x_i \leq 3^h$. Therefore, the application of Lemmas 2.13 and 2.15 of [16] gives $\sigma_m(3) \geq 3^{-86}$. Consequently, combining the preceding lower bounds with the favourable fact that $\sigma_m(p) - 1 = O(p^{-22})$ we obtain $\mathfrak{S}_m(n) \gg 1$. Observe as well that the above discussion yields $\mathfrak{S}_m(n) \geq 0$ for every $m \in \mathbb{N}$.

Before determining a lower bound of the expected size for the contribution of the set of narrow arcs, we introduce for convenience the weighted exponential sum

$$w(\beta) = \sum_{P^{12\eta} < x \leq n} \frac{1}{12} x^{-11/12} \rho\Big(\frac{\log x}{12\eta \log P}\Big) e(\beta x),$$

where ρ denotes the Dickman function, defined for real x by

$$\rho(x) = 0 \quad \text{when } x < 0,$$

$$\rho(x) = 1 \quad \text{when } 0 \leq x \leq 1,$$

$$\rho \text{ continuous for } x > 0,$$

$$\rho \text{ differentiable for } x > 1,$$

$$x\rho'(x) = -\rho(x-1) \quad \text{when } x > 1.$$

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For the sake of simplicity, we define the auxiliary function $f^*(\alpha)$ by putting $f^*(\alpha) = q^{-1}S(q, 81a)w(81(\alpha - a/q))$ when $\alpha \in \mathfrak{P}(a, q) \subset \mathfrak{P}$, and $f^*(\alpha) = 0$ for $\alpha \in \mathfrak{p}$. Then, it is a consequence of [15, Lemma 5.4] (where the condition $(a, q) = 1$ can be relaxed to $(a, q) = C$ for some constant C) that for $\alpha \in \mathfrak{P}(a, q) \subset \mathfrak{P}$ one has $f(81\alpha) - f^*(\alpha) = O(PR^{-3})$ and $f^*(\alpha) \ll q^{-1/12}P(1 + n|\beta|)^{-1/12}$. Moreover, by the methods of [16, Lemma 2.8] and the monotonicity of ρ it follows that

$$w(\beta) \ll \frac{P}{(1 + n|\beta|)^{1/12}}. \tag{7-5}$$

Finally, when $m \in \mathbb{N}$ it is convenient to introduce $K(m)$, defined as the number of solutions of the equation

$$m = T(p_1\mathbf{x}_1)^4 + \cdots + T(p_{11}\mathbf{x}_{11})^4$$

for $\mathbf{x}_i \in \mathcal{W}$ and $M/2 \le p_i \le M$. Combining the estimates mentioned before (7-5), we obtain that

$$\int_{\mathfrak{P}} W(\alpha)^{11} f(81\alpha)^{46} e(-\alpha n)\, d\alpha = \sum_{m \le 11n} K(m) \int_{\mathfrak{P}} f^*(\alpha)^{46} e(-\alpha(n - m))\, d\alpha \\ + O((HM)^{11} P^{34} (\log P)^{-11-\delta}).$$

Observe that the main term on the right can be written as

$$\sum_{m \le 11n} K(m) \sum_{q \le R} S_m(q) \int_{|\beta| \le n^{-1}R} w(81\beta)^{46} e(-\beta(n - m))\, d\beta. \tag{7-6}$$

By (7-5) we obtain that the integral in the above expression over the range $|\beta| > n^{-1}R$ is $O(P^{34}(\log P)^{-\delta})$. Therefore, an application of this observation and (7-4) gives that the contribution of the set of narrow arcs $\mathfrak{P}$ is

$$\sum_{m \le 11n} K(m)\mathfrak{S}_m(n) \int_0^1 w(81\beta)^{46} e(-\beta(n - m))\, d\beta + O((HM)^{11} P^{34} (\log P)^{-11-\delta}).$$

We further note that whenever $P^{12\eta} < x \le n$ we have

$$\rho\Big(\frac{\log x}{12\eta \log P}\Big) \gg 1;$$

so, combining the positivity of $\mathfrak{S}_m(n)$, orthogonality and the lower bound $\mathfrak{S}_m(n) \gg 1$ when $m \equiv n \pmod{81}$, we obtain that (7-6) is bounded below by

$$\sum_{\substack{m \le 11n/12 \\ m \equiv n \ (\mathrm{mod}\ 81)}} K(m)(n - m)^{17/6}.$$

One can check via an application of Hensel's Lemma (because the set of sums of three cubes modulo 27 comprises the residue classes not congruent to 4 nor 5 modulo 9) and Lemma 2.14 of [16] that the set of numbers of the shape $T(p_1\mathbf{x}_1)^4 + \cdots + T(p_{11}\mathbf{x}_{11})^4$

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with $p_i, \mathbf{x}_i \le 81$ covers all the residue classes modulo 81. Consequently, by the preceding discussion we find that

$$\int_{\mathfrak{P}} W(\alpha)^{11} f(81\alpha)^{46} e(-\alpha n)\, d\alpha \gg (HM)^{11} P^{34} (\log P)^{-11},$$

which, combined with the estimates obtained through the pruning process, yields $R_4(n) \gg (HM)^{11} P^{34} (\log P)^{-11}$.

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