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EXAMPLES OF THE NONEXISTENCE OF A SOLUTION IN THE PRESENCE OF UPPER AND LOWER SOLUTIONS

PATRICK HABETS[1] and RODRIGO L. POUSO[2]

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Abstract

Standard results for boundary value problems involving second-order ordinary differential equations ensure that the existence of a well-ordered pair of lower and upper solutions together with a Nagumo condition imply existence of a solution. In this note we introduce some examples which show that existence is not guaranteed if no Nagumo condition is satisfied.

1. Introduction

It is well-known that, if $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions, the periodic problem

$$u'' = f(t, u, u'), \quad u(a) = u(b), \quad u'(a) = u'(b),$$

as well as the separated boundary value problem

$$u'' = f(t, u, u'), \quad a_1 u(a) - a_2 u'(a) = 0, \quad b_1 u(b) + b_2 u'(b) = 0,$$

where $a_1, a_2, b_1, b_2 \in \mathbb{R}^+$, $|a_1| + |a_2| > 0$ and $|b_1| + |b_2| > 0$, have at least one solution provided that there exist lower and upper solutions α and β which are well-ordered, $\alpha \leq \beta$ and $f(t, u, u')$ satisfies a Nagumo condition.

The ideas of the lower and upper solution method can be traced back to E. Picard [6] in 1893 but the method was really grounded in 1931 by G. Scorza Dragoni [9]. This paper considers solutions which are $\mathscr{C}^2$ and in 1938 Scorza Dragoni [10] extended

[1]Institut de Mathématique Pure et Appliquée, Chemin du Cyclotron, 2, 1348 Louvain-La-Neuve, Belgium.
[2]Dept. de Análise Matemática, Univ. de Santiago de Compostela, 15706 Santiago de Compostela, Spain; e-mail: rodrigolp@usc.es.

the results to equations that satisfy Carathéodory conditions. *A priori* bounds on the derivative were considered by Bernstein [2] in 1904 but the now classical Nagumo conditions were introduced by Nagumo [4] in 1937. The method is today standard and can be found in several textbooks such as Bailey, Shampine and Waltman [1], Fučík [3], Piccinini, Stampacchia and Vidossich [7], Rouche and Mawhin [8].

In 1954, Nagumo [5] pointed out that the existence of well-ordered lower and upper solutions is not sufficient to ensure the existence of solutions of a Dirichlet problem. In this note, we generalise this remark and give examples for both the periodic and the separated boundary value problem. Although this is not essential, we present examples that use the mean curvature operator.

2. The periodic problem

Consider the problem

$$\frac{d}{dt}\left(\frac{u'}{\sqrt{1+u'^2}}\right) = u - p(t), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{2.1}$$

where $p(t) = -2$ on $[0, r]$, $p(t) = 2$ on $[T - r, T]$ and $0 < r < T/2$. Notice that $\alpha = -3$ is a lower solution and $\beta = 3$ an upper one of (2.1). Nevertheless, the following proposition holds.

PROPOSITION 2.1. *If* $r > \sqrt{2}$, *problem* (2.1) *has no solution.*

PROOF. Consider the equation

$$\frac{d}{dt}\left(\frac{u'}{\sqrt{1+u'^2}}\right) = u + 2, \tag{2.2}$$

and notice that the energy

$$\mathscr{E}_1(u, u') = \frac{1}{\sqrt{1+u'^2}} + \frac{(u+2)^2}{2}$$

is constant along the solutions of (2.2).

CLAIM 1. *Any solution* $u(t)$ *of* (2.2) *such that* $\mathscr{E}_1(u(t), u'(t)) = E \geq 2$ *cannot exist on an interval* $[a, b]$ *of length* $b - a > \sqrt{2}$.

Let $u(t)$ be such a solution. If $u(t) > -2$, we define t_0 to be such that u is minimal at $t = t_0$ and let $t > t_0$. We compute

$$t - t_0 = \int_{u(t_0)}^{u(t)} \frac{E - (u+2)^2/2}{\sqrt{1 - (E - (u+2)^2/2)^2}}\, du.$$

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Let $w = E - (u+2)^2/2 \in]0, 1]$. We have

$$t - t_0 = \int_{w(t_0)}^{w(t)} \frac{-w}{\sqrt{1-w^2}} \frac{dw}{u+2} \le \frac{1}{\sqrt{2}} \int_{w(t_0)}^{w(t)} \frac{-w}{\sqrt{1-w^2}} dw$$
$$\le \frac{1}{\sqrt{2}} \sqrt{1-w^2} \bigg|_1^0 = \frac{1}{\sqrt{2}}.$$

As a consequence, the claim follows. A similar argument holds for solutions $u(t) < -2$.

CLAIM 2. *Any solution $u(t)$ of $\frac{d}{dt}\left(u'/\sqrt{1+u'^2}\right) = u - 2$, such that*

$$\mathscr{E}_2(u(t_0), u'(t_0)) = \frac{1}{\sqrt{1+u'^2}} + \frac{1}{2}(u-2)^2 \ge 2,$$

cannot exist on an interval $[a, b]$ of length $b - a > \sqrt{2}$.

This claim follows from the same argument as that used for Claim 1.

Conclusion. Let $u(t)$ be a solution of (2.1) and assume $r > \sqrt{2}$. If $u(0) \ge 0$, we have $\mathscr{E}_1(u(t_0), u'(t_0)) \ge 2$. From Claim 1, such a solution cannot exist on $[0, r]$. On the other hand if $u(0) < 0$, we deduce from the boundary conditions $u(T) < 0$. This implies $\mathscr{E}_2(u(T), u'(T)) \ge 2$ and, using Claim 2, we deduce $u(t)$ cannot exist on $[T - r, T]$. Hence (2.1) has no solution.

REMARK. Notice that we can choose the function $p \in \mathscr{C}([0, T])$ so that solutions are $\mathscr{C}^2$.

3. The separated boundary value problem

Consider the problem

$$\frac{d}{dt}\left(\frac{u'}{\sqrt{1+u'^2}}\right) = u - p(t), \qquad \begin{array}{l} a_1 u(0) - a_2 u'(0) = 0, \\ b_1 u(T) + b_2 u'(T) = 0, \end{array} \tag{3.1}$$

where $a_1, a_2, b_1, b_2 \in \mathbb{R}^+$, $|a_1| + |a_2| > 0$, $|b_1| + |b_2| > 0$, $p(t) = -2$ on $[0, T/2]$ and $p(t) = 2$ on $]T/2, T]$. Notice that $\alpha = -3$ is a lower solution and $\beta = 3$ an upper one of (3.1). Nevertheless, if $T > 0$ is large enough, this problem has no solution.

PROPOSITION 3.1. *If $T > 2\sqrt{2}$, problem* (3.1) *has no solution.*

PROOF. From Claim 1 in the proof of Proposition 2.1, we have $\mathscr{E}_1(u(0), u'(0)) < 2$ which implies $u(T/2) < 0$. Hence,

$$\mathscr{E}_2\big(u(T/2), u'(T/2)\big) \ge \big(u(T/2) + 2\big)^2/2 \ge 2,$$

and the claim follows from Claim 2.

4. The Dirichlet problem

For the Dirichlet problem, we can find examples which are autonomous. Consider the problem

$$\frac{d}{dt}\left(\frac{u'}{\sqrt{1+u'^2}}\right) = u+2, \quad u(0)=0, \quad u(T)=0.$$

Here again $\alpha = -3$ is a lower solution and $\beta = 3$ an upper one. Notice that for this problem

$$\mathscr{E}_1(u, u') = \frac{1}{\sqrt{1+u'^2}} + \frac{(u+2)^2}{2} = \mathscr{E}_1(u(0), u'(0)) \geq 2.$$

From Claim 1 of Proposition 2.1, it is clear that solutions with such energy cannot exist if $T > \sqrt{2}$.

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