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REACTION WAVES AND NON-CONSTANT DIFFUSIVITIES

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Abstract

A reaction-diffusion equation with non-constant diffusivity,

$$u_t = (D(x,t)u_x)_x + F(u),$$

is studied for $D(x,t)$ a continuous function. The conditions under which the equation can be reduced to an equivalent constant diffusion equation are derived. Some exact forms for $D(x,t)$ are given. For $D(x,t)$ a stochastic function, an explicit finite difference method is used to numerically determine the effect of randomness in $D(x,t)$ upon the speed of the reaction wave solution to Fisher's equation. The extension to two spatial dimensions is considered.

1. Introduction

The patterns and waves that arise in a study of reaction-diffusion equations are becoming quite well-known; for example, through the monograph by Grindrod [2]. Applications are occurring in many areas, one of particular interest being the spread of wildfires [9]. One of the basic problems in the application of reaction-diffusion equations is to predict the features of any propagating wave that may occur, especially in two or more spatial dimensions. While the basic results were established in 1978 by Aronson and Weinberger [1], interest in complex patterns has continued, for example Tyson and Keener [8].

Diffusivities which vary with time, and possibly space, have not been considered by any of these workers. A primary reason for this is that the laboratory experiments under consideration have reasonably constant conditions. However, when the "laboratory" becomes a wildfire in a forest (or similar), and the diffusion process is due to turbulent eddies of varying scales [5], variability of the diffusivity becomes of interest.

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The present paper will focus on simple waves in one and two spatial dimensions. The effect of variable diffusivity on the speed of such waves will be considered by analytical methods and by numerical methods. In particular, the functional dependencies for diffusivity for which analytical progress is possible, will be derived. Time-dependent diffusivity will be treated numerically, as the analytical methods we develop do not work in this case.

2. Background

Grishin [3] was the first to attempt to describe a wildfire with a system of reaction-diffusion-advection equations. The possibilities and limitations of using a single such equation as a modelling tool have since been discussed [9]. In either case, the basic idea is to begin from the conservation of energy (and mass) in order to write an equation for the temperature of the combusting materials. While it is an oversimplification to reduce this to a single equation of the form

$$u_t + w_i \partial_i u = \partial_i D_{ij} \partial_j u + F(u), \tag{1}$$

many realistic features are retained. In this equation, u represents a (scaled) temperature, w_i is the i^{th} component of the wind vector, D_{ij} is the ij-th component of the diffusivity tensor, and $F(u)$ is the reaction function. As the strength of any wind is known to vary with time (both the mean speed and direction, and fluctuations about these), we anticipate that w_i and D_{ij} will be functions of time. This is what we shall consider in the next section.

To illustrate the type of behaviour described by (1), we now summarise some of the findings when w_i and D_{ij} are assumed constant [1, 9]. If w_i is constant a translation in x_i will remove the advection term from (1). (Any $w_i(t)$ can be removed by a translation. In the rest of this paper it will be assumed that such a translation has been carried out, and advection will not be considered further.) Rescaling of x_i will remove D_{ii}, providing it is constant and diagonal. Then (1) is simplified to

$$u_t = \nabla^2 u + F(u). \tag{2}$$

For suitable $F(u)$, (2) admits travelling wave solutions, or "fronts". Necessary restrictions on $F(u)$ are that $F(0) = F(1) = 0$, $F'(0) > 0$ and $F(u) > 0$ for $0 < u < 1$. An example would be

$$F(u) = u(1 - u), \tag{3}$$

much studied by others. The fronts can be planar, circular, or obey more general rules [2, 8].

Planar fronts are found by assuming

$$u = u(z) \quad \text{with} \quad z = \mathbf{x} \cdot \mathbf{n} - ct, \tag{4}$$

where the unit vector $\mathbf{n} = (\cos\theta, \sin\theta)$, θ is a parameter and c is a constant to be found. Substituting this into (2) yields

$$u'' + cu' + F(u) = 0, \tag{5}$$

where $' \equiv d/dz$, the solution of which determines c, provided appropriate boundary conditions are used.

For example, for the Fisher-Kolmogorov equation (where (3) is used in (5)), an exact solution is

$$u(x,t) = \frac{1}{\left[1 + \exp\left(x/\sqrt{6} - 5t/6\right)\right]^2}, \tag{6}$$

with a speed $c = 5/\sqrt{6} \cong 2.04$. In fact for this particular problem, any $c \geq 2$ is a solution of the eigenvalue problem (5), but not all of them yield an explicit representation for $u(x,t)$.

Prediction of the speed at which the front moves is the primary objective. If the speed is c in dimensionless coordinates, then it will be $c\sqrt{D}$ in the original variables. This suggests that any variability in the diffusivity will manifest itself directly in the speed at which the front moves. In particular, a stochastic diffusivity will result in stochastic motion of the front. In the following sections we shall examine the extent to which this is true.

3. Diffusivity as a function of x and t

In this section, let us consider a one-dimensional reaction-diffusion equation of the form

$$u_t = (D(x,t)u_x)_x + F(u). \tag{7}$$

If we are going to make use of the existing results for a one-dimensional version of (2), it will be necessary to find a mapping

$$T : (x,t) \rightarrow (X,T), \tag{8}$$

such that (7) is transformed into (2). The mapping

$$X = f(x,t), \tag{9}$$

$$T = t, \tag{10}$$

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yields two relations involving $f(x, t)$ and $D(x, t)$, namely

$$Df_x^2 = 1, \tag{11}$$

$$f_t - (Df_x)_x = 0. \tag{12}$$

From these equations, we can see there are two approaches. The first approach is to investigate the equations

$$f = \int D^{-\frac{1}{2}}\, dx + g(t), \tag{13}$$

$$0 = D_t - \tfrac{1}{2}D_x^2 + DD_{xx}. \tag{14}$$

The second approach is to investigate the equations

$$D = 1/f_x^2, \tag{15}$$

$$f_x^2 f_t + f_{xx} = 0. \tag{16}$$

3.1. The first approach

3.1.1. $D(t)$ We can see that if D is a function of time only then from (14), $D_t = 0$, and thus D is a constant. Therefore there is no suitable transformation of the form (9) which allows us to solve the equation analytically when $D = D(t)$. This case will be considered numerically in Section 4.

3.1.2. $D(x)$ Let D be a function of x only. From (14), this gives

$$-\tfrac{1}{2}D_x^2 + DD_{xx} = 0, \tag{17}$$

which is an ordinary differential equation for $D(x)$.

The general solution to this is

$$D = c_1^2(x + c_2)^2, \tag{18}$$

where c_1 and c_2 are constants of integration.

By substituting this into (13) and satisfying (12), we find the transformation function, f, is

$$f = (1/c_1)\log(x + c_2) + c_1 t + c_3, \tag{19}$$

where c_3 is a constant of integration.

By substituting this into (93) from Appendix A, we find that the asymptotic wave speed is

$$c \sim -c_1(c_1 - 2)e^{-c_1 c_3}e^{-c_1(c_1-2)t}. \tag{20}$$

Note here that if $c_1 = 2$, then the wave should stop. For $c_1 > 2$, the wave should actually turn around, then slow down exponentially to zero.

3.1.3. $D(x,t)$ For D a function of x and t, finding solutions for (14) is more difficult.

A straightforward solution can be found by assuming a travelling wave solution exists. Thus (14) becomes

$$kD'(\eta) - \tfrac{1}{2}D'(\eta)^2 + D(\eta)D''(\eta) = 0, \tag{21}$$

where $\eta = x + kt$ and k is an arbitrary constant.

The general solution to this is

$$D(x,t) = \frac{4k^2}{c_1}\left(\omega\left(-\frac{1}{2k}\exp\left(-\frac{(x+kt)\,c_1^2 + 4\,k + c_2\,c_1^2}{4\,k}\right)\right)+1\right)^2, \tag{22}$$

where c_1 and c_2 are constants and $\omega(x)$ is Lambert's ω-function, defined to be the solution of $\omega(x)e^{\omega(x)} = x$ which is analytic at $x = 0$. Substituting this into (13) and satisfying (12), we find

$$f(x,t) = -\frac{2}{c_1}\ln\left(\omega\left(-\frac{1}{2k}\exp\left(-\frac{c_1^2(x+kt) + 4\,k + c_2\,c_1^2}{4\,k}\right)\right)\right). \tag{23}$$

By substituting this into (93) from Appendix A, we find that the asymptotic wave speed is

$$c \sim \frac{4\,k}{c_1} + \frac{4\,k}{c_1\,e^{tc_1}} - k. \tag{24}$$

As the general solution is rather complex, a linear function solution can be found to be

$$D(x,t) = 2k(x+kt) + c_1. \tag{25}$$

Substituting this into (13) and satisfying (12), we find

$$f(x,t) = (1/k)(2k(x+kt)+c_1)^{\frac{1}{2}} + c_2, \tag{26}$$

where c_1 and c_2 are constants.

By substituting this into (93) from Appendix A, we find that the asymptotic wave speed is

$$c \sim 4kt - 2kc_2 - k. \tag{27}$$

We have solved (7) numerically for D as in (25). The wave speed as a function of time is shown in Figure 1. Note that there is an arbitrary shift in the wave speed, (27). Therefore $4kt$ and the numerically computed speed need only be parallel, as is the case in Figure 1.

Separation of variables is also possible, as is a similarity solution of the form

$$D = t^{-\alpha}\Phi(x/t^{\beta}), \tag{28}$$

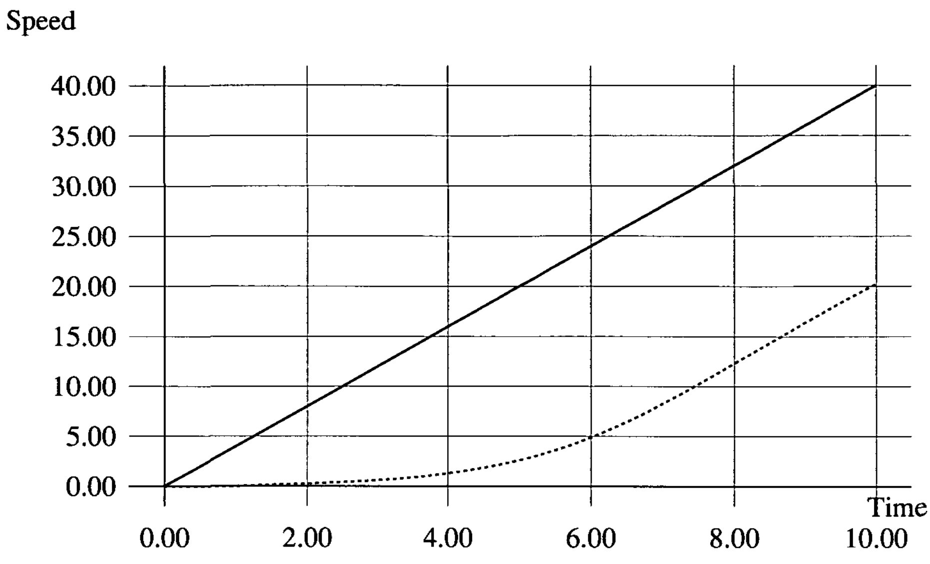

FIGURE 1. A comparison of the analytically and numerically obtained wave speeds for the diffusivity $D(x, t) = 2(x + t)$, where the analytical wave speed is the solid line (———) and the numerical wave speed is the dotted line (······).

where Φ must satisfy

$$\Phi\Phi'' - \tfrac{1}{2}\Phi'^2 - \beta\eta\Phi' - \alpha\Phi = 0, \tag{29}$$

α is a constant and $\beta = \frac{1}{2}(1 - \alpha)$, and $\eta = x/t^{\beta}$. Note that (29) with $\beta = 0$ is the same equation one encounters in separation of variables. The case $\alpha = \beta = \frac{1}{3}$ is required for a source solution. Unfortunately, these methods do not lead to any explicit solutions and are hence of little value here.

If we look for Lie symmetries of (7), we find that the transformations η, ξ and ζ satisfy

$$\zeta = 0, \tag{30}$$

$$\xi = \xi(x), \tag{31}$$

$$\xi_{xx} = 0, \tag{32}$$

$$\eta = \eta(t), \tag{33}$$

$$\eta_t = 2\xi_x. \tag{34}$$

From this we see that the similarity variable is

$$s = (x + c_1)/\sqrt{2t + c_2}. \tag{35}$$

As (14) is time and space invariant, the similarity variable s is equivalent to

$$s' = x/\sqrt{t}, \tag{36}$$

which is a special case of (28).

Thus the Lie group analysis confirms that we have exhausted the symmetries of (14).

Transformation techniques should also be considered. Let $D = w^2$. Then $w(x,t)$ satisfies

$$w_t + w^2 w_{xx} = 0. \tag{37}$$

Hill [4] considers this equation for normal diffusion. There one finds that the implicit transformation

$$w(x,t) = \frac{\partial x}{\partial s}(s,t) \tag{38}$$

results in the linear heat equation, but with negative diffusivity,

$$x_t + x_{ss} = 0. \tag{39}$$

This merely revisits all solutions already found. Hence, we need not consider this equation further.

3.2. The second approach

3.2.1. From (16), we can see that if f is a function of t only, then f can be an arbitrary function of t. However, from (15) this means that the diffusivity is undefined. The reason for this is that as f is the transformation, if there is no x dependence then the space dimension has been taken away from the problem.

3.2.2. Let f be a function of x only. Thus (16) simplifies to

$$f_{xx} = 0. \tag{40}$$

The solution of this is

$$f = c_1 x + c_2. \tag{41}$$

Substituting this into (15) gives D as a constant.

3.2.3. *Suppose* $f = f(x,t)$. We could assume transformations of the form

$$f = c_1 \log(x + c_2) + c_3 t + c_4 \tag{42}$$

and

$$f = h(x + kt). \tag{43}$$

These would give us (18) and (25) respectively. So let us try two other methods.

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Our first method is to assume that there exists a similarity solution of the form

$$f = t^a h(\eta), \tag{44}$$

where $\eta = xt^b$.

By substituting this into (16), we find $a = 1/2$ and h satisfies

$$\tfrac{1}{2}h'^2 h + b\eta h'^3 + h'' = 0. \tag{45}$$

For $b = -1/2$, we find that

$$f = c_1 t^{1/2}\eta = c_1 x. \tag{46}$$

The corresponding diffusivity is a constant. This case has already been considered.

For $b = 1/2$, we find that (45) can be integrated once to get

$$hh'\eta - 1 = kh', \tag{47}$$

where k is a constant.

This gives us that

$$f = 2\sqrt{t\log\left(kx\sqrt{t}\right)}. \tag{48}$$

On substituting this into (15), we find that

$$D = \frac{x^2}{t}\log\left(kx\sqrt{t}\right). \tag{49}$$

By substituting f into (93) from Appendix A, we find that the wave speed is

$$c \sim \frac{e^t(2t-1)}{2kt^{3/2}}. \tag{50}$$

Note that as k increases, the diffusivity also increases, but the wave speed decreases.

Our second method is to try a Lie group analysis. We find that the transformations η, ξ and ζ satisfy

$$\eta = \eta(t), \tag{51}$$

$$\xi_{xx} = 0, \tag{52}$$

$$\zeta = \zeta(f, t), \tag{53}$$

$$\xi_t + \xi_{ff} = 0, \tag{54}$$

$$\zeta_t + \zeta_{ff} = 2\xi_{xf}, \tag{55}$$

$$\eta_t + \eta_{ff} = 2\zeta_f. \tag{56}$$

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We can solve these equations to find that

$$\eta = \alpha t^2 + \beta t, \tag{57}$$

$$\xi = \frac{\alpha}{4} f^2 x + \frac{\epsilon}{2} f x - \frac{\alpha}{2} t x + \delta x, \tag{58}$$

$$\zeta = \alpha t f + \frac{\beta}{2} f + \epsilon t, \tag{59}$$

where α, β, δ and ϵ are constants.

Substituting these equations into (99), we should be able to find the relevant similarity variables. In trying to solve (99), we find that the similarity variable explicitly depends on the solution of (99).

However, we did find another similarity solution, by making the assumptions that $\alpha = 0$, $\beta = 2$, $\delta = 0$ and $\epsilon = 1$, of the form

$$f = h(s), \tag{60}$$

where $s = xe^{-t/4}$. This similarity solution does not, in fact, come from the classical Lie analysis described previously.

We find that h satisfies

$$h'' - 4h'^3 s = 0. \tag{61}$$

The solution to this is

$$h = \pm 2 \arcsin\left(\frac{s}{c_1}\right) + c_2. \tag{62}$$

Thus

$$f = \pm 2 \arcsin\left(\frac{x}{c_1 e^{t/4}}\right) + c_2, \tag{63}$$

where c_1 and c_2 are constants.

By substituting this into (15), we find

$$D = \frac{1}{4} c_1^2 e^{t/4} - \frac{1}{4} x^2. \tag{64}$$

Substituting f into (93) from Appendix A, we find that the wave speed is

$$c \sim \frac{1}{4} c_1 e^{t/4} \left[4 \cos\left(t - \frac{c_2}{2}\right) + \sin\left(t - \frac{c_2}{2}\right)\right]. \tag{65}$$

Note that this says that the wave will move backwards half of the time and forwards the other half.

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4. $D(x,t)$ a random function

The success of the Euler method in rapidly obtaining accurate solutions to the pure initial-value problem for reaction-diffusion equations with compact initial data [6] entices one to consider applying the method to random diffusivities.

Specifically, we shall solve (7) with the explicit finite difference scheme

$$u_i^{k+1} = u_i^k + \frac{h}{\delta^2} D(t) \left(u_{i+1}^k - 2u_i^k + u_{i-1}^k \right) + hF(u_i^k), \tag{66}$$

where u_i^k denotes the computed value $u(i\delta, kh)$, δ is the grid spacing and h the time step. For any $D(t)$, (66) provides a rapid numerical algorithm. The extension to $D(x,t)$ is straightforward.

To get stability conditions for this scheme, we linearise the equation about an unstable equilibrium, u_0. This gives bounds on both the space and time steps

$$\delta^2 < \frac{2D_{\min}}{F'(u_0)}, \tag{67}$$

$$h < \frac{\delta^2}{2D_{\max} - \delta^2 F'(u_0)}. \tag{68}$$

Of primary interest is the effect of non-constant diffusivity upon the speed of the reaction wave. Notice that for the Fisher equation (with constant diffusivity)

$$u_t = Du_{xx} + u(1-u), \tag{69}$$

the minimal wave speed is

$$c = 2\sqrt{D}. \tag{70}$$

Thus one is tempted to infer that any time dependence of the diffusivity will immediately affect the speed of the reaction wave. However, as we are dealing with a nonlinear system, and diffusion will act to smooth any variation (over some time scale), we should not hastily infer anything.

In Figure 2 we present the results of numerical calculation of the speed of the reaction wave, as it develops with time, for three cases, namely

$$D = 1, \tag{71}$$

$$D = 1 + 0.1 \sin t, \tag{72}$$

$$D = 1 + \text{RNG}. \tag{73}$$

Here RNG denotes a random number generator with mean zero, amplitude ± 0.1, and following a rectangular probability distribution. Note that all diffusivities are

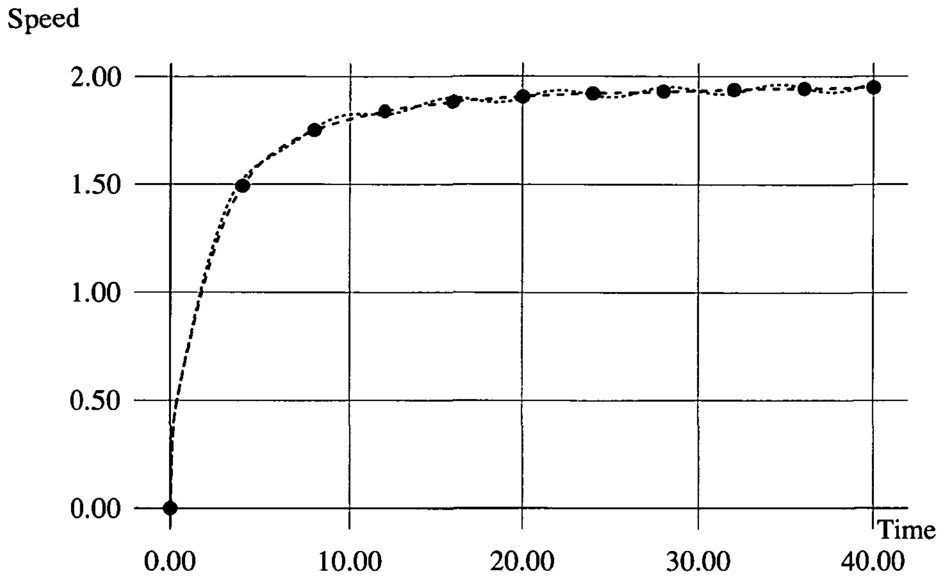

FIGURE 2. Speed of the reaction wave as a function of time, using an explicit finite difference scheme and tracking the average position of the front, where $D = 1$ is the circles (•), $D = 1+$ RNG is the dashed line (- - - - -) and $D = 1 + 0.1 \sin t$ is the dotted line (······). Diffusivities $D = 1$ and $D = 1$+RNG, where RNG denotes a random noise generated number between ± 0.1, are indistinguishable. The speed from $D = 1 + 0.1 \sin t$ exhibits small amplitude oscillations about the $D = 1$ result. Note that all diffusivities are perfectly correlated in space.

perfectly correlated in space. It is quite clear that the constant case (71) and the randomly varying case (73) are very similar. The difference in speeds is shown in Figure 3, and it is much less than one might naively infer from (70). The smoothly varying case (72) does exhibit a smooth variation in speed with the same period, but with a smaller amplitude than one might naively infer from (70). It would appear that the diffusion process results in an insensitivity to changes in diffusivity below some time scale. A theoretical analysis of the appropriate time scales would be of considerable interest, but is beyond the scope of the present paper. Note that the general analysis of stochastic reaction diffusion equations is proving to be a difficult task; for example, witness the effort required to determine the changes in equilibria by Tuckwell [7].

5. $D_1(x, y, t)$ and $D_2(x, y, t)$ continuous functions

Consider the equation in two spatial dimensions

$$u_t = (D_1 u_x)_x + (D_2 u_y)_y + F(u). \tag{74}$$

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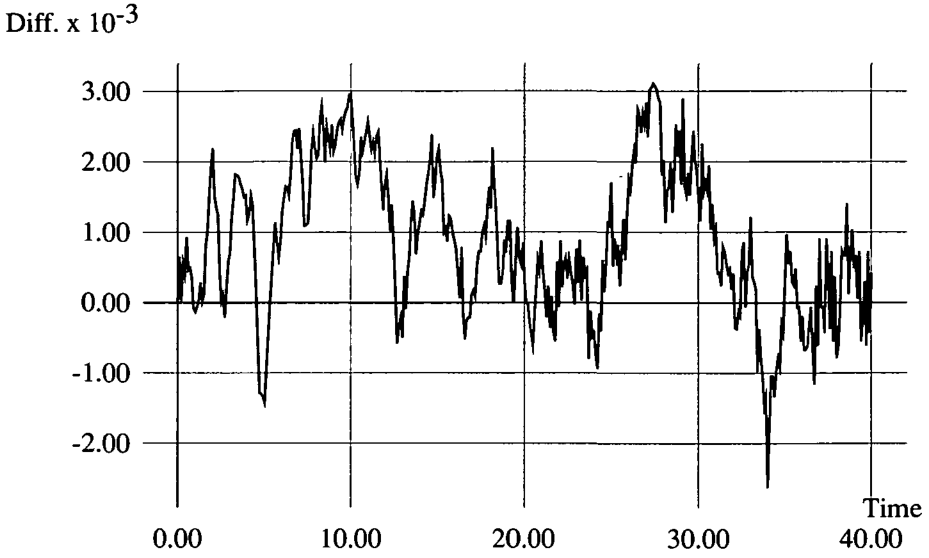

FIGURE 3. Difference between the speed versus time results for $D = 1$ and $D = 1+$ RNG. This highlights just how close the results remain.

We seek a mapping

$$X = f(x, y, t), \tag{75}$$

$$Y = g(x, y, t), \tag{76}$$

$$T = t, \tag{77}$$

under which (74) is transformed into

$$u_T = u_{XX} + u_{YY} + F(u). \tag{78}$$

This requires us to solve a system of five simultaneous partial differential equations:

$$D_1 f_x^2 + D_2 f_y^2 = 1, \tag{79}$$

$$D_1 g_x^2 + D_2 g_y^2 = 1, \tag{80}$$

$$D_1 f_x g_x + D_2 f_y g_y = 0, \tag{81}$$

$$(D_1 f_x)_x + (D_2 f_y)_y - f_t = 0, \tag{82}$$

$$(D_1 g_x)_x + (D_2 g_y)_y - g_t = 0. \tag{83}$$

For $D_1(x, t)$ and $D_2(y, t)$, (79)-(83) uncouple and the analytical results of Section 3 apply to each of $D_1(x, t)$ and $D_2(y, t)$ separately. In the general case, we have

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not been able to find any nontrivial solutions. The numerical method of Section 4 could easily be extended to the present case, and the results are likely to mimic the one-spatial-dimension results already reported.

6. Conclusions

Four special forms for $D(x,t)$ have been found. With these forms it is possible to reduce the reaction-diffusion equation with $D(x,t)$ to an equivalent constant diffusivity equation, for arbitrary $F(u)$. Although the special forms for $D(x,t)$ for which reduction was possible do not seem to be of intrinsic physical interest, they do provide valuable examples for testing other analytical techniques and computational algorithms. The same will hold in two spatial dimensions for $D_1(x,t)$ and $D_2(y,t)$. However, when D_1 and D_2 are both functions of x, y and t, nontrivial solutions of the five coupled partial differential equations have eluded us.

Stochastic diffusivity was treated numerically. It appears that the mean value of the speed of the reaction wave is unchanged by variations in diffusivity about its mean value. The variability of the speed about the mean is much less that the variability of the diffusivity about its mean.

Appendix A: Calculation of the wave speed

If our original system is of the form (7) and we can reduce it to an equivalent equation

$$u_T = u_{XX} + F(u), \tag{84}$$

then it is possible to find an asymptotic approximation to the minimal speed of the wave front, which for the class of functions given in Section 2, the waves move from the unstable equilibrium $u = 0$ to the stable $u = 1$.

Assume there exists an unstable equilibrium point, $\overline{u}$, of (7). Thus we require

$$F(\overline{u}) = 0, \tag{85}$$

$$F'(\overline{u}) > 0. \tag{86}$$

Now, let us expand (84) about $\overline{u}$, using a Taylor Series for $F(u)$ about $\overline{u}$

$$F(u) \approx F(\overline{u}) + F'(\overline{u})u^*, \tag{87}$$

where $u^* = u - \overline{u}$.

Thus

$$u_t^* = u_{xx}^* + F'(\overline{u})u^*. \tag{88}$$

For the class of functions mentioned, $\overline{u} = 0$.
Thus

$$u_t^* = u_{xx}^* + F'(0)u^*. \tag{89}$$

The fundamental solution of this equation is related to the source solution. Thus

$$u^* = \frac{1}{\sqrt{t}} e^{-X^2/4t} e^{F'(0)t}. \tag{90}$$

As we are assuming that there exists a travelling wave front, which is well behaved, we want u^* to also be well behaved. To do this, there must be a balance between the two exponentials (diffusion and reaction). Thus we require

$$\frac{X^2}{4t} \sim F'(0)t, \tag{91}$$

or

$$X \sim 2F'(0)t. \tag{92}$$

What we want to do now is to substitute in $X = f(x,t)$ for X and obtain the speed in the original variables; namely, $\frac{dx}{dt}$. We then get an asymptotic estimate of the speed:

$$c \sim \frac{dx}{dt} \sim (X^{-1})'(t). \tag{93}$$

Appendix B: Lie group analysis

Suppose we want to solve a general partial differential equation (PDE) of the form

$$N(x, t, c, c_t, c_x, c_{xx}) = 0, \tag{94}$$

using a Lie group analysis. We suppose a general transform of variables as one-parameter groups of the form

$$\overline{c} = D + \epsilon\zeta(x,t,c) + \mathrm{O}(\epsilon^2), \tag{95}$$

$$\overline{x} = x + \epsilon\xi(x,t,c) + \mathrm{O}(\epsilon^2), \tag{96}$$

$$\overline{t} = t + \epsilon\eta(x,t,c) + \mathrm{O}(\epsilon^2). \tag{97}$$

The new PDE becomes

$$N(\overline{x}, \overline{t}, \overline{c}, \overline{c}_{\overline{t}}, \overline{c}_{\overline{x}}, \overline{c}_{\overline{x}\overline{x}}) = 0, \tag{98}$$

and we want this PDE to be the same as the original one.

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Once we have found the functions $\zeta(x,t,c)$, $\xi(x,t,c)$ and $\eta(x,t,c)$, the similarity variable and functional form of the solution are obtained from the first order PDE; this is called the "invariant surface condition":

$$\xi c_x + \eta c_t = \zeta. \tag{99}$$

Without going into any of the messy details (see for example Hill [4]), the derivatives in the transformed variables are

$$\bar{c}_{\bar{x}} = c_x + \epsilon\left\{\zeta_x + (\zeta_c - \xi_x)c_x - \eta_x c_t - \xi_D c_x^2 - \eta_c c_t c_x\right\}, \tag{100}$$

$$\begin{aligned}\bar{c}_{\bar{x}\bar{x}} = c_{xx} + \epsilon\big\{&\zeta_{xx} + (2\zeta_{xc} - \xi_{xx})c_x - \eta_{xx}c_t \\ &+(\zeta_{cc} - 2\xi_{xc})c_x^2 - 2\eta_{xc}c_t c_x - \xi_{cc}c_x^3 - \eta_{cc}c_t c_x^2 \\ &+[(\zeta_c - 2\xi_x) - 3\xi_c c_x - \eta_c c_t]\, c_{xx} - 2(\eta_x + \eta_c c_x)c_{xt}\big\},\end{aligned} \tag{101}$$

$$\bar{c}_{\bar{t}} = c_t + \epsilon\left\{\zeta_t + (\zeta_c - \eta_t)c_t - \xi_t c_x - \eta_c c_t^2 - \xi_c c_x c_t\right\}. \tag{102}$$

Substituting (100)-(102) into (98), and reverting (94) in terms of c_{xx} and replacing this for c_{xx} in (98), we get a multinomial in terms of c, c_t, c_x and c_{xt}, to first order in ϵ. By equating the various coefficients to zero, we get a set of equations involving η, ξ and ζ. These must then be satisfied to find η, ξ and ζ which are subsequently used in the invariant surface condition, (99), to determine the transformation under which the PDE, (94), is invariant.

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