THE LEAST COMMUTATIVE CONGRUENCE ON A SIMPLE REGULAR ω-SEMIGROUP†

by C. BONZINI, A. CHERUBINI and B. PIOCHI

(Received 22 March, 1988; revised 17 October, 1988)

Introduction. Piochi in **[10]** gives a description of the least commutative congruence λ of an inverse semigroup in terms of congruence pairs and generalizes to inverse semigroups the notion of solvability. The object of this paper is to give an explicit construction of λ for simple regular ω-semigroups exploiting the work of Baird on congruences on such semigroups. Moreover the connection between the solvability classes of simple regular ω-semigroups and those of their subgroups is studied.

As usual σ indicates the least group congruence. $\mathbb{H}$ and $\mathbb{D}$ the Green's relations, $\mathbb{N}$ the set of non negative integers, $\mathbb{Z}$ the additive group of the integers. For notations and definitions not given in this paper the reader is referred to **[9]**.

1. Preliminary results.

Definition 1. An *ω-semigroup* S is a semigroup whose set E of idempotents form an ω-chain

$$e_0 > e_1 > \ldots > e_n > \ldots$$

under the natural order defined on E by the rule $e \geq f$ if and only if $ef = f = fe$.

For a regular ω-semigroup. Munn in **[6]** proved the following result.

Theorem A. *Let S be a regular ω-semigroup.*

If S has no kernel, then it is the union of an ω-chain of groups.

If the kernel of S coincides with S, then S is a simple regular ω-semigroup.

If S has a proper kernel, then S is a (retract) ideal extension of a simple regular ω-semigroup K by a finite chain of groups with 0 adjoined H^0. Moreover this extension is determined by means of a homomorphism of H into the group of units of K.

Piochi in **[10]** characterized (by means of congruence pairs) the least commutative congruence of an inverse semigroup, proving

Theorem B (**[10]**, Th. 2.4 and Th. 2.6). *Let S be an inverse semigroup and E its semilattice of idempotents. Define on E the relation $e \sim f$ if and only if there exist $a, b \in S$ such that $e = abb^{-1}a^{-1}$, $f = baa^{-1}b^{-1}$ and denote by λ_E the transitive closure of $\sim$. Denote by S' the subsemigroup of S generated by the elements $[a, b] = aba^{-1}b^{-1}$ with $a, b \in S$ and put*

$$\partial(S) = \{a \in S \mid a^{-1}a\lambda_E e \text{ for some } e \in E \text{ and } ae \in S'\}.$$

Then $(\lambda_E, \partial(S))$ is a congruence pair and the congruence associated with it is the least commutative congruence on S.

Henceforward the least commutative congruence on a semigroup S will be denoted by λ_S (or simply by λ). We remark that the congruence λ is denoted γ in **[10]**; here we changed notation, in order to avoid confusion with the mappings γ_i of Theorem C below.

† Work supported by M.P.I.

The main aim of this paper is to give an explicit construction of λ for a simple regular ω-semigroup. The construction of λ for the non-simple case is a result of a routine nature but adds to the technical problems, hence here is deleted; however it can be found in **[4]**.

2. The least commutative congruence on a simple regular ω-semigroup. Several authors, e.g. Kocin **[5]** and Munn **[6]**, gave structure theorems for simple regular ω-semigroups. The one given by Munn is the following

THEOREM C. *Let d be a positive integer and let $\{G_i \mid i=0, \ldots, d-1\}$ be a set of d pairwise disjoint groups. Let γ_{d-1} be a homomorphism of G_{d-1} into G_0 and, if $d>1$, let γ_i be a homomorphism of G_i into G_{i+1} $(i=0, \ldots, d-2)$. For every $n \in \mathbb{N}$ let $\bar{n}$ denote the integer equivalent to n modulo d, belonging to $\mathbb{N}$ and less than d and let $\gamma_n = \gamma_{\bar{n}}$. For $m, n \in \mathbb{N}$ and $m<n$ write*

$$\alpha_{m,n} = \gamma_m \gamma_{m+1} \cdots \gamma_{n-1}$$

and for all $n \in \mathbb{N}$ let $\alpha_{n,n}$ denote the identity automorphism of $G_{\bar{n}}$. Let S be the set of the ordered triples (m, a_i, n), where $m, n \in \mathbb{N}$, $0 \le i \le d-1$ and $a_i \in G_i$. Define a multiplication in S by the rule that

$$(m, a_i, n)(p, b_j, q) = (m+p-r, (a_i \alpha_{u,w})(b_j \alpha_{v,w}), n+q-r)$$

where $r = \min\{n, p\}$, $u = nd+i$, $v = pd+j$ and $w = \max\{u, v\}$. Denote the so formed groupoid by $S(d, G_i, \gamma_i)$. Then $S(d, G_i, \gamma_i)$ is a simple regular ω-semigroup with exactly d $\mathbb{D}$-classes and any simple regular ω-semigroup is isomorphic to a semigroup $S(d, G_i, \gamma_i)$. For $n \in \mathbb{N}$ and $i=0, \ldots, d-1$ write $e_i^n = (n, e_i, n)$, where e_i is the identity of the group G_i. The elements e_i^n are the idempotents of $S(d, G_i, \gamma_i)$ and we have

$$e_0^0 > e_1^0 > \ldots > e_{d-1}^0 > e_0^1 > \ldots > e_{d-1}^1 > e_0^2 > \ldots$$

NOTATION. In the remainder of the paper $\bar{n}$ will denote, as in previous theorem, the integer equivalent to n modulo d, belonging to $\mathbb{N}$ and less than d, and, for every $i \in \mathbb{N}$, the endomorphism $\alpha_{i,i+d}$ of $G_{\bar{i}}$ will be indicated by α_i.

REMARK 2.1. For every $i, j \in \mathbb{N}$ with $i<j$ we have obviously $\alpha_i = \alpha_{\bar{i}}$ and, putting $i = md+\bar{i}$, $j = nd+\bar{j}$, $\alpha_{i,j} = \alpha_{\bar{i},\bar{j}} \alpha_{\bar{j}}^{n-m}$ if $\bar{i} \le \bar{j}$ and $\alpha_{i,j} = \alpha_{\bar{i},\bar{j}+d} \alpha_{\bar{j}}^{n-m-1}$ if $\bar{i} > \bar{j}$.

REMARK 2.2. For fixed i satisfying $0 \le i \le d-1$, put

$$S_i = \{(m, a_i, n) \mid m, n \in \mathbb{N}, a_i \in G_i\}.$$

Then S_i is a bisimple inverse semigroup of S (**[2]**, p. 462) and the Reilly multiplication (**[7]**, formula (1)) applies with $\alpha = \alpha_i$.

LEMMA 2.3. *Let $S = S(d, G_i, \gamma_i)$ be a simple regular ω-semigroup. Then its least commutative congruence λ is a group congruence contained in $\sigma \vee \mathbb{H}$.*

Proof. First we recall that the congruence λ_E defined in Th. B is, by Th. 2.2 of **[3]**, a uniform congruence of E. Moreover, for every $m, n \in \mathbb{N}$ and for every i such that $0 \le i \le d-1$, we have $e_i^m \lambda_E e_i^n$. In fact, putting $a = (m, e_i, n)$, $b = (n, e_i, m)$ we have $e_i^m = abb^{-1}a^{-1}$ and $e_i^n = baa^{-1}b^{-1}$. Hence, by the remarks preceding Lemma 2.1 of **[3]**, we immediately deduce that λ_E is the universal congruence ω_E on E, hence λ is a group congruence. Finally we remark that, since $S/\sigma \vee \mathbb{H} \cong \mathbb{Z}$ (see **[2]**, Corollary 3.1), $\sigma \vee \mathbb{H}$ is a commutative congruence; so λ is contained in $\sigma \vee \mathbb{H}$.

Corollary 2.4. *If ρ is a commutative congruence on $S = S(d, G_i, \gamma_i)$, then ρ is a group congruence.*

Definition 2.5 ([**2**], p. 463). Let $S = S(d, G_i, \gamma_i)$. A subset A of $G = G_0 \times G_1 \times \ldots \times G_{d-1}$ which satisfies the conditions

(i) $A = A_0 \times A_1 \times \ldots \times A_{d-1}$ for some $A_i \subseteq G_i$, $i = 0, \ldots, d-1$,

(ii) $A_i \trianglelefteq G_i$, $i = 0, \ldots, d-1$,

(iii) $A_{d-1}\gamma_{d-1} \subseteq A_0$ and $A_i\gamma_i \subseteq A_{i+1}$, $i = 0, \ldots, d-2$,

is called a *γ-admissible subset* of G.

Γ^* denotes the set of the γ-admissible subsets of G satisfying the condition

(iv) $\operatorname{rad} A = A$ where $\operatorname{rad} A = \operatorname{rad} A_0 \times \ldots \times \operatorname{rad} A_{d-1}$ and

$$\operatorname{rad} A_i = \{a_i \in G_i \mid a_i\alpha_i^n \in A_i \text{ for some nonnegative integer } n\}.$$

Lemma 2.6 ([**2**], Lemma 3.2, Lemma 3.4 and Lemma 3.5). *Let $S = S(d, G_i, \gamma_i)$ and let ρ be a congruence on S such that $\rho \in [\sigma, \sigma \vee \mathbb{H}]$. Put $A^\rho = A_0^\rho \times \ldots \times A_{d-1}^\rho$ where $A_i^\rho = \{a_i \in G_i \mid (0, a_i, 0) \in \ker \rho\}$. Then the following conditions hold.*

(i) $A^\rho \in \Gamma^*$.

(ii) $\ker \rho = \{(m, a_i, m) \mid m \in \mathbb{N}, a_i \in A_i^\rho, i = 0, \ldots, d-1\}$.

(iii) *Let $x = (m, g_i, n)$, $y = (p, h_j, q)$ be two elements of S, then we have $x\rho y$ if and only if $m - n = p - q$ and $(g_i\alpha_{u,w})(h_j^{-1}\alpha_{v,w}) \in A_k^\rho$ where $u = nd + i$, $v = qd + j$, $w = \max\{u, v\}$, $k = \bar{w}$.*

Conversely, for every $A \in \Gamma^*$, the relation ρ defined by (iii) *is a congruence on S belonging to $[\sigma, \sigma \vee \mathbb{H}]$ such that $A^\rho = A$.*

Remark 2.7. By Lemma 2.6 it follows that a group congruence ρ contained in $\sigma \vee \mathbb{H}$ is completely determined by the subset A^ρ of G. Hence, by Lemma 2.3, λ can be described by means of A^λ.

Remark 2.8. We recall that for every two congruences ρ and τ on an inverse semigroup, we have $\rho \leq \tau$ if and only if $\operatorname{tr} \rho \leq \operatorname{tr} \tau$ and $\ker \rho \subseteq \ker \tau$. Hence, if $S = S(d, G_i, \gamma_i)$ and $\rho, \tau \in [\sigma, \sigma \vee \mathbb{H}]$ then $\rho \leq \tau$ if and only if $A^\rho \subseteq A^\tau$.

Definition 2.9. Let H be a group and ϕ an endomorphism of H. For every $a, b \in H$ we call

$$(a\phi^r)(b\phi^s)(a^{-1}\phi^t)(b^{-1}\phi^u) \qquad (r, s, t, u \in \mathbb{N})$$

a *ϕ-commutator* of a and b, and, if it is unambiguous, we put

$$(a\phi^r)(b\phi^s)(a^{-1}\phi^t)(b^{-1}\phi^u) = [a, b]_\phi.$$

We denote by H'_ϕ the subgroup of H generated by the ϕ-commutators of H and we call it the *ϕ-derivate* of H.

Lemma 2.10. *The following properties hold.*

(i) *$H'_\phi \supseteq H'$ where H' indicates the derivate of the group H.*

(ii) *If $g\phi^k \in H'_\phi$ for some non negative integer k, then $g \in H'_\phi$.*

(iii) $H'_\phi \trianglelefteq H$.

https://doi.org/10.1017/S0017089500009022 Published online by Cambridge University Press

Proof. Property (i) is obvious, property (ii) easily follows because $g(g^{-1}\phi^k)$ is a ϕ-commutator of g and of the identity of H. To prove property (iii), let $a, b, g \in H$ and consider a ϕ-commutator

$$[a, b]_\phi = (a\phi^r)(b\phi^s)(a^{-1}\phi^t)(b^{-1}\phi^u).$$

If $t < r$,

$$\begin{aligned} g[a, b]_\phi g^{-1} &= g((a\phi^t)\phi^{r-t})g^{-1}(a^{-1}\phi^t)(a\phi^t)g(b\phi^s)gg^{-1}(a^{-1}\phi^t)g^{-1}(b^{-1}\phi^s)(b\phi^s)g(b^{-1}\phi^u)g^{-1} \\ &= [g, a\phi^t]_\phi[(a\phi^t)g, (b\phi^s)g]_\phi[b, g]_\phi. \end{aligned}$$

If, on the contrary $t \geq r$,

$$\begin{aligned} g[a, b]_\phi g^{-1} &= g(a\phi^r)(b\phi^s)((a^{-1}\phi^r)\phi^{t-r})(g^{-1}\phi^{t-r})b^{-1}b(g\phi^{t-r})(b^{-1}\phi^u)g^{-1} \\ &= [g(a\phi^r), b]_\phi[b, g]_\phi. \end{aligned}$$

LEMMA 2.11. *Let $S = S(d, G_i, \gamma_i)$ and λ its least commutative congruence. Then $A^\lambda = G'_\alpha$ where $G'_\alpha = (G_0)'_{\alpha_0} \times \ldots \times (G_{d-1})'_{\alpha_{d-1}}$.*

Proof. First we prove that $G'_\alpha \in \Gamma^*$. In fact condition (i) of Definition 2.5 obviously holds; conditions (ii) and (iv) follow by Lemma 2.10. Moreover $\alpha_i\gamma_i = \gamma_i\alpha_{i+1}$ for every $i = 0, \ldots, d-2$, $\alpha_{d-1}\gamma_{d-1} = \gamma_{d-1}\alpha_0$. Let $0 \leq j \leq d-1$. For every $a_j, b_j \in G_j$ and for every α_j-commutator of a_j, b_j we have

$$[a_{d-1}, b_{d-1}]_{\alpha_{d-1}}\gamma_{d-1} = [a_{d-1}\gamma_{d-1}, b_{d-1}\gamma_{d-1}]_{\alpha_0}$$

and $[a_j, b_j]_{\alpha_j}\gamma_j = [a_j\gamma_j, b_j\gamma_j]_{\alpha_{j+1}}$ with $j \leq d-2$, i.e. condition (iii) of Definition 2.5 holds. Now, let ρ be the group congruence induced by G'_α following Lemma 2.6. We will show that ρ is commutative, i.e. that for every $x, y \in S$ we have $xy\,\rho\,yx$. Put

$$x = (m, g_i, n), \qquad y = (p, h_j, q), \ (g_i \in G_i, h_j \in G_j; 0 \leq i, j \leq d-1; m, n, p, q \in \mathbb{N}),$$

then

$$xy = (m + p - r, (g_i\alpha_{u,w})(h_j\alpha_{v,w}), q + n - r)$$

and

$$yx = (p + m - s, (h_j\alpha_{a,c})(g_i\alpha_{b,c}), q + n - s)$$

where $r = \min\{n, p\}$, $s = \min\{q, m\}$, $u = nd + i$, $v = pd + j$, $w = \max\{u, v\}$, $a = qd + j$, $b = md + i$, $c = \max\{a, b\}$.

Obviously condition

$$(m + p - r) - (q + n - r) = (p + m - s) - (q + n - s) \tag{1}$$

holds. Now consider the element

$$g = (((g_i\alpha_{u,w})(h_j\alpha_{v,w}))\alpha_{l,t})(((h_j\alpha_{a,c})(g_i\alpha_{b,c}))^{-1}\alpha_{k,t})$$

with

$$l = (n + q - r)d + \bar{w}, \qquad k = (q + n - s)d + \bar{c}, \qquad t = \max\{l, k\}.$$

We have

$$\begin{aligned} g &= ((g_i\alpha_{u,w})\alpha_{l,t})((h_j\alpha_{v,w})\alpha_{l,t})((g_i^{-1}\alpha_{b,c})\alpha_{k,t})((h_j^{-1}\alpha_{a,c})\alpha_{k,t}) \\ &= ((g_i\alpha_{i,z})\alpha_z^{i_1})((h_j\alpha_{j,z})\alpha_z^{i_2})((g_i\alpha_{i,z})^{-1}\alpha_z^{i_3})((h_j\alpha_{j,z})^{-1})\alpha_z^{i_4}) \end{aligned} \tag{2}$$

https://doi.org/10.1017/S0017089500009022 Published online by Cambridge University Press

where, if $\bar{t} \geq \max\{i, j\}$, $z = \bar{t}$, otherwise $z = \bar{t} + d$ and i_1, i_2, i_3, i_4 are suitable non negative integers (Remark 2.1). By (2) it follows that $g \in (G_{\bar{t}})'_{\alpha_{\bar{t}}}$; hence by (1) and by condition (iii) of Lemma 2.6, $xy\,\rho\,yx$.

Now, let τ be a commutative congruence on S; we shall prove that $\rho \leq \tau$. The congruence $\tau' = \tau \wedge (\sigma \vee \mathbb{H})$ is commutative; hence it is a group congruence by Corollary 2.4. Let $A^{\tau'}$ be the γ-admissible subgroup of G induced by τ' following Lemma 2.6 and let

$$(g_i\alpha_i^p)(h_i\alpha_i^n)(g_i^{-1}\alpha_i^q)(h_i^{-1}\alpha_i^m) \quad (i = 0, \ldots, d-1; g_i, h_i \in G_i; p, n, q, m \in \mathbb{N}\}$$

be an α_i-commutator of G_i. Put $x = (m+k, g_i, n+k)$, $y = (p+k, h_i, q+k)$ where $k = \min\{r, s\}$, $r = \min\{p, n\}$, $s = \min\{q, m\}$. Since τ' is a commutative congruence we have $xy\,\tau'\,yx$, hence recalling condition (iii) of Lemma 2.6 and Remark 2.2 it follows that

$$[(g_i\alpha_i^{p+k-t})(h_i\alpha_i^{n+k-t})]\alpha_{u,w}[(g_i^{-1}\alpha_i^{q+k-j})(h_i^{-1}\alpha_i^{m+k-j})]\alpha_{v,w} \in A_i^{\tau'}$$

where $t = \min\{n+k, p+k\}$, $j = \min\{q+k, m+k\}$, $u = (n+q+2k-t)d+i$, $v = (n+q+2k-j)d+j$ and $w = \max\{u, v\}$, whence

$$(g_i\alpha_i^p)(h_i\alpha_i^n)(g_i^{-1}\alpha_i^q)(h_i^{-1}\alpha_i^m) \in A_i^{\tau'};$$

thus $(G_i)'_{\alpha_i} \subseteq A_i^{\tau'}$ and $G'_\alpha \subseteq A^{\tau'}$. So $\rho \leq \tau' \leq \tau$, hence $\rho = \lambda$.

By previous Lemmas we can deduce the following description of the least commutative congruence for a simple regular ω-semigroup.

Theorem 2.12. *Let $S = S(d, G_i, \gamma_i)$ be a simple regular ω-semigroup and λ its least commutative congruence. Then*

$$(m, g_i, n)\,\lambda\,(p, h_j, q) \text{ if and only if } m - n = p - q$$

and

$$(g_i\alpha_{u,w})(h_j^{-1}\alpha_{v,w}) \in (G_z)'_{\alpha_z}$$

with

$$u = nd + i, \qquad v = qd + j, \qquad w = \max\{u, v\}, \qquad z = \bar{w}.$$

Remark 2.13. λ is a group congruence such that

$$\ker\lambda = \{(m, g_i, m) \mid m \in \mathbb{N}, g_i \in (G_i)'_{\alpha_i}; i = 0, \ldots, d-1\}.$$

Clearly $\ker\lambda$ is an ω-chain of groups.

Theorem 2.14. *Let $S = S(d, G_i, \gamma_i)$ be a simple regular ω-semigroup. Denote by $\mathcal{G}$ the direct product $G_0/(G_0)'_{\alpha_0} \times \ldots \times G_{d-1}/(G_{d-1})'_{\alpha_{d-1}}$ and consider the subgroup K of $\mathcal{G}$ defined by*

$$\mathcal{K} = \{(g_i\alpha_{i,d}(G_0)'_{\alpha_0}, \ldots, g_i\alpha_{i,d+s}(G_s)'_{\alpha_s}, \ldots, g_i\alpha_{i,2d-1}(G_{d-1})'_{\alpha_{d-1}}) \mid g_i \in G_i; i, s = 0, \ldots, d-1\}.$$

The mapping f of S/λ onto $\mathcal{K} \times \mathbb{Z}$ defined by

$$f : (m, g_i, n)\lambda \to ((g_i\alpha_{i,d}(G_0)'_{\alpha_0}, \ldots, g_i\alpha_{i,2d-1}(G_{d-1})'_{\alpha_{d-1}}), m-n)$$

is an isomorphism.

Proof. Consider $(p, h_j, q)\lambda \in S/\lambda$ and its image

$$((h_j\alpha_{j,d}(G_0)'_{\alpha_0}, \ldots, h_j\alpha_{j,2d-1}(G_{d-1})'_{\alpha_{d-1}}), p-q).$$

First we prove that $(m, g_i, n)\lambda = (p, h_j, q)\lambda$ iff

$$((g_i\alpha_{i,d}(G_0)'_{\alpha_0}, \ldots, g_i\alpha_{i,2d-1}(G_{d-1})'_{\alpha_{d-1}}), m-n) = ((h_j\alpha_{j,d}(G_0)'_{\alpha_0}, \ldots, h_j\alpha_{j,2d-1}(G_{d-1})'_{\alpha_{d-1}}), p-q).$$

In fact suppose $(m, g_i, n)\lambda(p, h_j, q)$; then from Theorem 2.12 it follows that $m-n = p-q$ and $g_i\alpha_{u,w}h_j^{-1}\alpha_{v,w} \in (G_{\bar{w}})'_{\alpha_{\bar{w}}}$ with $u = nd+i$, $v = qd+j$ and $w = \max\{u, v\}$. Hence, by Remark 2.1 $(g_i\alpha_{i,\bar{w}+d})\alpha_{\bar{w}}^{i_1}(h_j^{-1}\alpha_{j,\bar{w}+d})\alpha_{\bar{w}}^{i_2} \in (G_{\bar{w}})'_{\alpha_{\bar{w}}}$ for some nonnegative integers i_1, i_2 and, since we have

$$(g_i\alpha_{i,\bar{w}+d})\alpha_{\bar{w}}^{i_1}(h_j^{-1}\alpha_{j,\bar{w}+d})\alpha_{\bar{w}}^{i_2}(g_i^{-1}\alpha_{i,\bar{w}+d})(h_j\alpha_{j,\bar{w}+d}) \in (G_{\bar{w}})'_{\alpha_{\bar{w}}}$$

we deduce $(g_i^{-1}\alpha_{i,\bar{w}+d})(h_j\alpha_{j,\bar{w}+d}) \in (G_{\bar{w}})'_{\alpha_{\bar{w}}}$. Hence for every nonnegative integer k

$$[(g_i^{-1}\alpha_{i,\bar{w}+d})(h_j\alpha_{j,\bar{w}+d})]\alpha_{\bar{w}+d,\bar{w}+\bar{k}+d} \in (G_{\bar{w}})'_{\alpha_{\bar{w}}}\alpha_{\bar{w}+d,\bar{w}+\bar{k}+d} \subseteq (G_{\overline{w+k}})'_{\alpha_{\overline{w+k}}}$$

and, by Remark 2.1,

$$[(g_i^{-1}\alpha_{i,\overline{w+k}+d})(h_j\alpha_{j,\overline{w+k}+d})]\alpha_{\overline{w+k}+d}^{i_3} \in (G_{\overline{w+k}})'_{\alpha_{\overline{w+k}}}$$

for some nonnegative integer i_3. So (ii) of Lemma 2.10 gives

$$(g_i^{-1}\alpha_{i,\overline{w+k}+d})(h_j\alpha_{j,\overline{w+k}+d}) \in (G_{\overline{w+k}})'_{\alpha_{\overline{w+k}}}$$

whence, since k is an arbitrary nonnegative integer, $(g_i^{-1}\alpha_{i,t+d})(h_j\alpha_{j,t+d}) \in (G_t)'_{\alpha_t}$ for every integer t with $0 \le t \le d-1$. Thus, we deduce that $(m, g_i, n)\lambda(p, h_j, q)$ implies $h_j\alpha_{j,t+d} \in g_i\alpha_{i,t+d}(G_t)'_{\alpha_t}$ for every $t = 0, \ldots, d-1$ and $m-n = p-q$ whence

$$((g_i\alpha_{i,d}(G_0)'_{\alpha_0}, \ldots, g_i\alpha_{i,2d-1}(G_{d-1})'_{\alpha_{d-1}}), m-n) = ((h_j\alpha_{j,d}(G_0)'_{\alpha_0}, \ldots, h_j\alpha_{j,2d-1}(G_{d-1})'_{\alpha_{d-1}}), p-q).$$

Conversely, let

$$((g_i\alpha_{i,d}(G_0)'_{\alpha_0}, \ldots, g_i\alpha_{i,2d-1}(G_{d-1})'_{\alpha_{d-1}}), m-n) = ((h_j\alpha_{j,d}(G_0)'_{\alpha_0}, \ldots, h_j\alpha_{j,2d-1}(G_{d-1})'_{\alpha_{d-1}}), p-q).$$

This implies $m-n = p-q$ and $g_i^{-1}\alpha_{i,\bar{k}+d}h_j\alpha_{j,\bar{k}+d} \in (G_{\bar{k}})'_{\alpha_{\bar{k}}}$ for every nonnegative integer k whence

$$(g_i\alpha_{i,\bar{k}+d})\alpha_{\bar{k}}^{j_1}(h_j^{-1}\alpha_{j,\bar{k}+d})\alpha_{\bar{k}}^{j_2} \in (G_{\bar{k}})'_{\alpha_{\bar{k}}}$$

for every nonnegative integers j_1, j_2. Let $u = nd+i$, $v = qd+j$ and $w = \max\{u, v\}$. Then for every j_1, j_2 such that

$$\min\{\bar{k} + (j_1+n+1)d, \bar{k} + (j_2+q+1)d\} \ge w,$$

we have

$$\alpha_{i,\bar{k}+d}\alpha_{\bar{k}}^{j_1} = \alpha_{i,\bar{k}+(1+j_1)d} = \alpha_{u,\bar{k}+(j_1+n+1)d} = \alpha_{u,w}\alpha_{w,\bar{k}+(j_1+n+1)d}$$

and

$$\alpha_{j,\bar{k}+d}\alpha_{\bar{k}}^{j_2} = \alpha_{v,\bar{k}+(j_2+q+1)d} = \alpha_{v,w}\alpha_{w,\bar{k}+(j_2+q+1)d}.$$

Now let $w = \bar{w} + td$; if $n \ge q$, choosing $\bar{k} = \bar{w}$ and $j_1 = t$, $j_2 = t+n-q$, we have

$$\bar{k} + (j_1+n+1)d = \bar{w} + td + nd + d = w + (n+1)d = \bar{k} + (j_2+q+1)d.$$

Hence, by Remark 2.1, $(g_i\alpha_{u,w}h_j^{-1}\alpha_{v,w})\alpha_{\bar{w}}^{n+1} \in (G_{\bar{w}})'_{\alpha_{\bar{w}}}$. Analogously, if $n<q$, putting $j_1 = t+q-n$, $j_2 = t$, we obtain

$$(g_i\alpha_{u,w}h_j^{-1}\alpha_{v,w})\alpha_{\bar{w}}^{q+1} \in (G_{\bar{w}})'_{\alpha_{\bar{w}}}.$$

In both the cases from (ii) of Lemma 2.10 it follows that $(g_i\alpha_{u,w}h_j^{-1}\alpha_{v,w}) \in (G_{\bar{w}})'_{\alpha_{\bar{w}}}$ with $u = nd+i$, $v = qd+j$ and $w = \max\{u, v\}$, so we have $(m, g_i, n)\lambda = (p, h_j, q)\lambda$. Thus f is well defined and injective.

The mapping f is obviously onto and finally it is a homomorphism; in fact consider

$$[(m, g_i, n)\lambda][(p, h_j, q)\lambda] = (m+p-r, (g_i\alpha_{nd+i,w})(h_j\alpha_{pd+j,w}), n+q-r)\lambda,$$

with $w = \max(nd+i, pd+j)$, $r = \min(n, p)$.

$$\begin{aligned} f((m+p-r, (g_i\alpha_{nd+i,w})(h_j\alpha_{pd+j,w}), n+q-r)\lambda) \\ = (((g_i\alpha_{nd+i,w}h_j\alpha_{pd+j,w})\alpha_{\bar{w},d}(G_0)'_{\alpha_0}, \ldots, \\ (g_i\alpha_{nd+i,w}h_j\alpha_{pd+j,w})\alpha_{\bar{w},2d-1}(G_{d-1})'_{\alpha_{d-1}}), m+p-n-q). \end{aligned}$$

Also

$$\begin{aligned} ((g_i\alpha_{i,d}(G_0)'_{\alpha_0}, \ldots, g_i\alpha_{i,2d-1}(G_{d-1})'_{\alpha_{d-1}}, m-n)((h_j\alpha_{j,d}(G_0)'_{\alpha_0}, \ldots, h_j\alpha_{j,2d-1}(G_{d-1})'_{\alpha_{d-1}}), p-q) \\ = ((g_i\alpha_{i,d}h_j\alpha_{j,d}(G_0)'_{\alpha_0}, \ldots, g_i\alpha_{i,2d-1}h_j\alpha_{j,2d-1}(G_{d-1})'_{\alpha_{d-1}}), m-n+p-q). \end{aligned}$$

Moreover

$$g_i\alpha_{i,d+\bar{h}}h_j\alpha_{j,d+\bar{h}}(h_j^{-1}\alpha_{pd+j,w})\alpha_{\bar{w},d+\bar{h}}(g_i^{-1}\alpha_{nd+i,w})\alpha_{\bar{w},d+\bar{h}} \in (G_{\bar{h}})'_{\alpha_{\bar{h}}};$$

in fact, if $w = pd+j$, $(h_j\alpha_{pd+j,w})\alpha_{\bar{w},d+\bar{h}} = h_j\alpha_{j,d+\bar{h}}$ and, by Remark 2.1, $g_i\alpha_{i,d+\bar{h}}(g_i^{-1}\alpha_{nd+i,w})\alpha_{\bar{w},d+\bar{h}} \in (G_{\bar{h}})'_{\alpha_{\bar{h}}}$; if $w = nd+i$, $g_i\alpha_{i,d+\bar{h}} = (g_i\alpha_{nd+i,w})\alpha_{\bar{w},d+\bar{h}}$ and by Remark 2.1, $h_j\alpha_{j,d+\bar{h}}(h_j^{-1}\alpha_{pd+j,w})\alpha_{\bar{w},d+\bar{h}} \in (G_{\bar{h}})'_{\alpha_{\bar{h}}}$, hence the result follows because $(G_{\bar{h}})'_{\alpha_{\bar{h}}}$ is a normal subgroup of $G_{\bar{h}}$. Thus, we have

$$f([(m, g_i, n)\lambda](p, h_j, q)\lambda]) = f((m, g_i, n)\lambda)f((p, h_j, q)\lambda).$$

When $d = 1$, S is a bisimple ω-semigroup, usually denoted by $S = S(G, \alpha)$. Thus we have the following result.

Corollary 2.15. *The kernel of the least commutative congruence λ on a bisimple ω-semigroup S is the commutator subsemigroup S', which is an ω-chain of groups S_i isomorphic to $(G')_\alpha$ for every integer i. The semigroup S/λ is a commutative group which is isomorphic to the direct product $G/(G')_\alpha \times \mathbb{Z}$.*

Proof. Let x be an element of $\ker\lambda$, thus from Remark 2.13 it follows that

$$\begin{aligned} x = (m, a\alpha^r b\alpha^s a^{-1}\alpha^t b^{-1}\alpha^v, m) = \{(m, a\alpha^r, m)(m, b\alpha^s, m)(m, a^{-1}\alpha^r, m)(m, b^{-1}\alpha^s, m)\} \\ . \{(m, b\alpha^s, m)(m, a\alpha^r, m)(m, a^{-1}\alpha^t, m)(m, b^{-1}\alpha^s, m)\}\{(m, b\alpha^s, m)(m, b^{-1}\alpha^v, m)\}. \end{aligned}$$

Moreover, denoting by e the identity of G, for every $b \in G$ and s, v nonnegative integers, if $s \le v$ we have

$$(m, b\alpha^s, m)(m, b^{-1}\alpha^v, m) = [(m, b\alpha^s, 0), (0, e, v-s)] \in S'$$

and if $s > v$ we have

$$\begin{aligned} (m, b\alpha^s, m)(m, b^{-1}\alpha^v, m) = \{(m, b\alpha^v, m)(m, b^{-1}\alpha^s, m)\}^{-1} \\ = [(m, b\alpha^v, 0), (0, e, s-v)]^{-1} \in S'. \end{aligned}$$

Now, we consider

$$(m, b\alpha^s, m)(m, a\alpha^r, m)(m, a^{-1}\alpha^t, m)(m, b^{-1}\alpha^s, m) = (m, b\alpha^s a\alpha^r a^{-1}\alpha^t b^{-1}\alpha^s, m).$$

If $r \le t$, by a simple calculus, we have

$$(m, b\alpha^s a\alpha^r a^{-1}\alpha^t b^{-1}\alpha^s, m) = [(m, b\alpha^s a\alpha^r b^{-1}\alpha^s, 0), (0, b\alpha^s b^{-1}\alpha^{s+t-r}, t-r)] \\ .\, (m, b\alpha^{s+m} b^{-1}\alpha^{s+m+t-r}, m)(m, b\alpha^{s+t-r} b^{-1}\alpha^s, m) \in S',$$

if $r > t$

$$(m, b\alpha^s a\alpha^r a^{-1}\alpha^t b^{-1}\alpha^s, m) = (m, b\alpha^s a\alpha^t a^{-1}\alpha^r b^{-1}\alpha^s, m)^{-1} \in S'.$$

Hence $x \in S'$ and $\ker \lambda = S'$, moreover, from Remark 2.13 it follows that S' is an ω-chain of groups S_i isomorphic to $(G')_\alpha$. Finally from Lemma 2.3 it follows that λ is a group congruence and from Theorem 2.14 we deduce that S/λ is isomorphic to the direct product $[G/(G')_\alpha] \times \mathbb{Z}$.

For a direct proof see [**11**], Theorem 2.5.

3. Solvability of simple regular ω-semigroups. In [**10**] the following definition was introduced for inverse semigroups:

Definition 3.1. Let S be an inverse semigroup. Denote $\delta_0(S) = S$, $\lambda_0 = \omega_S$, the universal congruence on S, and for $i \ge 1$, let $\lambda_{i,S}$ (or simply λ_i) be the least commutative congruence on $\delta_{i-1}(S) = \ker \lambda_{i-1}$ (trivially $\lambda_1 = \gamma$). S is called *solvable of solvability class c* or c-solvable if c is the least index i such that $\lambda_c = \mathrm{id}_{\delta_{c-1}(S)}$, the identity map on $\delta_{c-1}(S)$.

Lemma 3.2 ([**10**], 3.3). *S is solvable of class c if and only if c is the least index i such that $\delta_{i-1}(S)$ is commutative.*

Since any simple regular ω-semigroup is trivially inverse, it makes sense to ask about its solvability. We want to prove that there is a strict connection between the solvability of S and that of the groups G_i.

We remark that a simple regular ω-semigroup is a Bruck–Reilly semigroup over T where T is a chain $G_0 > G_1 > \ldots > G_{d-1}$ of groups [**6**, Structure Theorem], hence we state the result for Bruck–Reilly semigroups. We recall the following definition.

Definition 3.3. Let T be a monoid, α be a homomorphism of T into its group of units. The *Bruck–Reilly semigroup* over T is the semigroup $B(T, \alpha)$ of the triplets (m, a, n): m, n are nonnegative integers, $a \in T$ and the multiplication is defined as follows:

$$(m, a, n)(p, b, q) = (m + p - r, (a\alpha^{p-r})(b\alpha^{n-r}), n + q - r),$$

where $r = \min(n, p)$ and α^0 is the identity map on T.

It is well-known (see [**9**], e.g.) that $B(T, \alpha)$ is a simple monoid for each T and α, and that it is inverse if and only if T is inverse.

Theorem 3.4. *Let $S = B(T, \alpha)$ be a Bruck-Reilly semigroup over an inverse monoid T. Then S is solvable if and only if T is solvable. If S is solvable of class n then T is solvable of class n or $n-1$.*

Proof. If S is a solvable semigroup of class n, then, by Theorem 3.5 of [**10**], T is immediately seen to be solvable, and its solvability class is less than or equal to n, since T is (isomorphic to) a subsemigroup of S.

To prove that the condition is sufficient, consider firstly a commutator of elements of S:

$$c=[(m,a,n),(p,b,q)].$$

Let $r=\min(n,p)$, $t=\min(m,q)$, $v=\min(n+q-r,n+q-t)$. Then

$$c=(m+p-r+n+q-t-v,(a^{-1}\alpha^{p-r}b^{-1}\alpha^{n-r})\alpha^{n+q-t-v}(a\alpha^{q-t}b\alpha^{m-t})\alpha^{n+q-r-v},m+p-t+n+q-r-v).$$

Remark that:

$$n+q-r-t-v=\left\langle\begin{array}{l}=-t\Leftrightarrow v=n+q-r\Leftrightarrow t\le r\Leftrightarrow t=\min(m,q)\le\min(n,p)\\=-r\Leftrightarrow v=n+q-t\Leftrightarrow r\le t\Leftrightarrow r=\min(n,p)\le\min(m,q)\end{array}\right..$$

If we denote $k=\min(m,n,p,q)$, then we have proved that:

$$c=[(m,a,n),(p,b,q)]=(m+p-k,a\alpha^{p-k}b\alpha^{n-k}a^{-1}\alpha^{q-k}b^{-1}\alpha^{m-k},m+p-k).$$

Since the product of two elements (m,a,m) and (p,b,p) is again of type (n,x,n) then:

$$(m,a,n)\in S' \quad \text{implies that } m=n.$$

Let

$$(m,a,n)(p,e,p)=(m+p-\min(n,p),\ldots,n+p-\min(n,p)),$$

where $(p,e,p)\in E_S$. Such a product belongs to S' only if $m+p-\min(n,p)=n+p-\min(n,p)$, i.e. only if $m=n$. Then

$$(m,a,n)\in\delta(S) \quad \text{implies that } m=n.$$

Now, since $(m,a,m)\in\delta(S)$ implies trivially that $a\in T=\delta_0(T)$, suppose, by induction, that we have proved:

$$\text{for}\quad i\ge 2,\ (m,a,m)\in\delta_{i-1}(S)\quad\text{implies that } a\in\delta_{i-2}(T).$$

Consider $c=[(p,a,p),(q,b,q)]\in(\delta_{i-1}(S))'$; for $k=\min(p,q)$, one has:

$$\begin{aligned}c&=(p+q-k,a\alpha^{q-k}b\alpha^{p-k}a^{-1}\alpha^{q-k}b^{-1}\alpha^{p-k},p+q-k)\\&=(p+q-k,[a\alpha^{q-k},b\alpha^{p-k}],p+q-k).\end{aligned}$$

Since the commutator subsemigroup and the derivative of any inverse semigroup T are trivially closed with respect to powers of any endomorphism of T, then we get:

$$(m,a,m)\in(\delta_{i-1}(S))'\quad\text{implies that } a\in(\delta_{i-2}(T))'. \tag{3}$$

Let (m,e,m) and $(p,f,p)\in E_S$, and $(m,e,m)\lambda_{\delta_{i-1}(S)}(p,f,p)$. Then, there exists a sequence:

$$(n_0,a_0,n_0),(q_0,b_0,q_0),\ldots,(n_h,a_h,n_h),(q_h,b_h,q_h)$$

of elements of $\delta_{i-1}(S)$ such that

$$(m,e,m)=(n_0,a_0,n_0)(q_0,b_0,q_0)(q_0,b_0^{-1},q_0)(n_0,a_0^{-1},n_0),$$
$$(q_h,b_h,q_h)(n_h,a_h,n_h)(n_h,a_h^{-1},n_h)(q_h,b_h^{-1},q_h)=(p,f,p),$$

https://doi.org/10.1017/S0017089500009022 Published online by Cambridge University Press

and, for every j such that $0 \le j \le h$, if $u_j = \min(m_j, q_j)$ we get

$$(q_j, b_j, q_j)(n_j, a_j, n_j)(n_j, a_j^{-1}, n_j)(q_j, b_j^{-1}, q_j) = (n_{j+1}, a_{j+1}, n_{j+1})(q_{j+1}, b_{j+1}, q_{j+1})(q_{j+1}, b_{j+1}^{-1}, q_{j+1})(n_{j+1}, a_{j+1}^{-1}, n_{j+1});$$

$$n_j + q_j - u_j = n_{j+1} + q_{j+1} - u_{j+1}$$

and

$$b_j\alpha^{n_j-u_j}a_j\alpha^{q_j-u_j}a_j^{-1}\alpha^{q_j-u_j}b_j^{-1}\alpha^{n_j-u_j} = a_{j+1}\alpha^{q_{j+1}-u_{j+1}}b_{j+1}\alpha^{n_{j+1}-u_{j+1}}b_{j+1}^{-1}\alpha^{n_{j+1}-u_{j+1}}a_{j+1}^{-1}\alpha^{q_{j+1}-u_{j+1}}.$$

Also

$$m = n_0 + q_0 - u_0,$$
$$p = n_h + q_h - u_h,$$
$$e = a_0\alpha^{q_0-u_0}b_0\alpha^{n_0-u_0}b_0^{-1}\alpha^{n_0-u_0}a_0^{-1}\alpha^{q_0-u_0},$$
$$f = b_h\alpha^{n_h-u_h}a_h\alpha^{q_h-u_h}a_h^{-1}\alpha^{q_h-u_h}b_h^{-1}\alpha^{n_h-u_h}.$$

At last:

$$(m, e, n)\lambda_{\delta_{i-1}(S)}(p, f, p) \quad \text{implies} \quad m = p \quad \text{and} \quad e\lambda_{\delta_{i-2}(T)}f. \tag{4}$$

Let

$$(m, a, m) \in \delta_i(S) = \delta(\delta_{i-1}(S));$$

then there exists an idempotent $(m, e, m)\lambda_{\delta_{i-1}(S)}(m, a^{-1}a, m)$ such that:

$$(m, ae, m) \in (\delta_{i-1}(S))'.$$

By (3) and (4), this implies that $e\lambda_{\delta_{i-2}(T)}a^{-1}a$ and $ae \in (\delta_{i-2}(T))'$. Hence:

$$\text{for every } i \geqslant 1, \quad (m, a, m) \in \delta_i(S) \quad \text{implies} \quad a \in \delta_{i-1}(T).$$

If T is solvable of class n, then $\delta_{n-1}(T)$ is the first derivate subsemigroup which is commutative; thus one can easily see that $\delta_n(S)$ must be commutative, too, and S is solvable of solvability class less than or equal to $n+1$.

Now, from Theorem 3.4 of **[10]** we can deduce the announced result on a simple regular ω-semigroup:

Corollary 3.5. *Let $S = S(d, G_i, \gamma_i)$ be a simple regular ω-semigroup. Then S is solvable if and only if all the groups G_i are solvable. If S is n-solvable, then the greatest solvability class of the groups is n or $n-1$.*

Remark 3.6. When $S = S(G, \alpha)$ is a bisimple ω-semigroup, then it is solvable of class n if and only if G is solvable of class n or $n-1$. Both of these possibilities may occur. In fact consider the two following special cases of endomorphism α of the group G:

If the endomorphism α is nilpotent, that is if $\alpha^n(G) = 1$ for some $n \geq 1$, then $(G')_\alpha = G$. As $\delta(S)$ is a Clifford semigroup, we have that if G is solvable of class $n-1$, then $\delta(S)$ is solvable of class $n-1$. Thus S is solvable of class n.

If $\alpha = \mathrm{id}_G$, then $(G)'_\alpha = G'$. Now, if G is a solvable group of class n, then G' is solvable of class $n-1$. Hence $\delta(S)$ is solvable of class $n-1$ and S is solvable of class n.

Remark 3.7. It follows from Theorem 3.11 of **[10]** that the maximum group homomorphic image of a solvable inverse semigroup is solvable. The converse is not true in general; actually, here we have a new family of counter-examples.

In [7], Munn and Reilly proved that if α is nilpotent, then $S(G, \alpha)/\sigma$ is isomorphic to the additive group of integers. Hence, one can easily build up bisimple ω-semigroups which are not solvable, if G is not solvable, but where S/σ is solvable, since it is isomorphic to a commutative group.

REFERENCES

1. J. E. Ault, Group congruences on a bisimple ω-semigroup, *Semigroup Forum* **10** (1975), 351–366.

2. G. R. Baird, On a sublattice of the lattice of congruences on a simple regular ω-semigroup, *J. Austral. Math. Soc.* **13** (1972), 461–471.

3. G. R. Baird, Congruences on simple regular ω-semigroups, *J. Austral. Math. Soc.* **14** (1972), 155–167.

4. C. Bonzini and A. Cherubini, The least commutative congruence on a regular ω-semigroup, *Quaderno* n. **21**/1987, *Dipartimento di Matematica dell'Universita' di Milano.*

5. B. P. Kocin, The structure of inverse ideally simple ω-semigroups, *Vestnik Leningrad Univ.* **237** (1968), 41–50.

6. W. D. Munn, Regular ω-semigroups, *Glasgow Math. J.* **9** (1968), 46–66.

7. W. D. Munn and N. R. Reilly, Congruences on a bisimple ω-semigroup, *Proc. Glasgow Math. Ass.*, **7** (1966), 184–192.

8. M. Petrich, Congruences on simple ω-semigroups, *Glasgow Math. J.* **20** (1979), 87–101.

9. M. Petrich, *Inverse semigroups* (Wiley & Sons, 1984).

10. B. Piochi, Solvability in inverse semigroups, *Semigroup Forum* **34** (1987), 287–303.

11. B. Piochi, The least commutative congruence on bisimple ω-semigroup, *Rapp. Dip. Mat. Univ. Siena,* **158** (1987), 1–14.

Dipartimento di Matematica,
Università,
Via Saldini, 50
20133 Milano.

Dipartimento di Matematica,
Università,
Via del Capitano, 15
53100 Siena.

Dipartimento di Matematica,
Politecnico,
Piazza L. da Vinci, 32
20133 Milano.

https://doi.org/10.1017/S0017089500009022 Published online by Cambridge University Press