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REAL HYPERSURFACES OF A COMPLEX PROJECTIVE SPACE

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We study real hypersurfaces M of a complex projective space and show that a condition on the derivative of the Ricci Tensor of M implies M is locally homogeneous with two or three principal curvatures.

0. Introduction.

Let $P^n(\mathbb{C})$ be an n-dimensional complex projective space with Fubini-Study metric of constant holomorphic sectional curvature 4. We consider a real hypersurface M of $P^n(\mathbb{C})$. Let (ϕ,ξ,η,g) be an almost contact metric structure induced from the complex structure on $P^n(\mathbb{C})$ (§1). If the Ricci transformation of M satisfies

$$SX = aX + b\eta(X)\xi \ . \tag{0.1}$$

where a and b are constant, we call M a pseudo-Einstein hypersurface [3]. Pseudo-Einstein real hypersurfaces in $P^n(\mathbb{C})$ are completely classified by Kon [3] (see [4]). This result shows that if the Ricci tensor of M has a nice form, then M is determined (see [5]). In this paper, we consider the following problem: If the derivative of the Ricci tensor of M has a nice form, what can we say about M?

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We have the following

THEOREM 1. *Let* M *be a real hypersurface of* $P^n(\mathbb{C})$. *If the Ricci transformation* S *of* M *satisfies*

$$(0.2) \qquad (\nabla_X S)Y = c\{g(\phi AX, Y)\xi + \eta(Y)\phi AX\},$$

where c *is a non-zero constant, and* A *denotes the shape operator* (§1). *Then* M *is locally congruent to a homogeneous hypersurface with two or three distinct principal curvatures.*

We note that pseudo-Einstein hypersurfaces satisfy (0.2). Moreover,

THEOREM 2. *There are no real hypersurfaces with parallel Ricci tensor on which* ξ *is principal.*

1. Preliminaries.

Let M be a real hypersurface of $P^n(\mathbb{C})$. In a neighbourhood of each point, we choose a unit normal vector field N in $P^n(\mathbb{C})$. The Riemannian connections $\bar{\nabla}$ in $P^n(\mathbb{C})$ and ∇ in M are related by the following formulas for arbitrary vector fields X and Y on M :

$$(1.1) \qquad \bar{\nabla}_X Y = \nabla_X Y + g(AX, Y)N,$$

$$(1.2) \qquad \bar{\nabla}_X N = -AX,$$

where g denotes the Riemannian metric on M induced from the Fubini-Study metric $\bar{g}$ on $P^n(\mathbb{C})$ and A is the shape operator of M in $P^n(\mathbb{C})$.

An eigenvector X of the shape operator A is called a <u>principal curvature vector</u>. Also an eigenvalue λ of A is called a <u>principal curvature</u>.

It is known that M has an almost contact metric structure induced from the complex structure J on $P^n(\mathbb{C})$, (see [6]), that is, we define a tensor field ϕ of type (1.1), a vector field ξ and a 1-form η on M by

$$g(\phi X, Y) = \bar{g}(JX, Y) \quad \text{and} \quad g(\xi, X) = \eta(X) = \bar{g}(JX, N).$$

Then we have

$$(1.3) \qquad \phi^2 X = -X + \eta(X)\xi, \; g(\xi, \xi) = 1, \; \phi\xi = 0 .$$

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From (1.1), we easily have

$$(1.4)\qquad (\nabla_X\phi)Y = \eta(Y)AX - g(AX, Y)\xi ,$$

$$(1.5)\qquad \nabla_X\xi = \phi AX .$$

Let $\bar{R}$ and R be the curvature tensors of $P^n(\mathbb{C})$ and M respectively. Since the curvature tensor $\bar{R}$ has a nice form, we have the following Gauss and Codazzi equations.

$$(1.6)\qquad \begin{aligned} g(R(X, Y)Z, W) = {} & g(Y, Z)g(X, W) - g(X, Z)g(Y, W) \\ & + g(\phi Y, Z)g(\phi X, W) - g(\phi X, Z)g(\phi Y, Z) \\ & - 2g(\phi X, Y)g(\phi Z, W) + g(AY, Z)g(AX, W) \\ & - g(AX, Z)g(AY, W) \end{aligned}$$

and

$$(1.7)\qquad (\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi.$$

Using (1.3), (1.5), (1.6) and (1.7), we get

$$(1.8)\qquad SX = (2n+1)X - 3\eta(X)\xi + hAX - A^2X ,$$

$$(1.9)\qquad \begin{aligned} (\nabla_X S)Y = {} & -3\{g(\phi AX, Y)\xi + \eta(Y)\phi AX\} + (Xh)AY + (h - A)(\nabla_X A)Y \\ & - (\nabla_X A)AY , \end{aligned}$$

where h = trace A and S denotes the Ricci tensor on M .

2. Proof of Theorems.

First, we determine the hypersurface M satisfying (0.2). Using (1.9), we see that (0.2) is equivalent to

$$(2.1)\qquad \begin{aligned} & (c+3)\{\eta(W)g(\phi AX, Y) + \eta(Y)g(\phi AX, W)\} - (Xh)g(AY, W) \\ & + g((A - h)(\nabla_X A)Y + (\nabla_X A)AY, W) = 0 . \end{aligned}$$

Contraction with respect to Y and W , together with (1.3), yields

$$(2.2)\qquad -(Xh)h + \mathrm{trace}(\nabla_X A)(2A - h) = 0 .$$

It follows that h^2 - trace A^2 is constant. Next, using (1.7), we see that (2.1) becomes

$$(2.3)\qquad (c+3)\{\eta(W)g(\phi AX, Y) + \eta(Y)g(\phi AX, W)\} - (Xh)g(AY, W)$$
$$+ g((\nabla_Y A)X + \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi, (A - h)W)$$
$$+ g((\nabla_{AY} A)X + \eta(X)\phi AY - \eta(AY)\phi X - 2g(\phi X, AY)\xi, W) = 0.$$

Contraction with respect to X and W yields

$$(2.4)\qquad (c+3)g(\phi A\xi, Y) - (AY)h + \operatorname{trace}(A - h)(\nabla_Y A) + \eta((A - h)\phi Y)$$
$$+ 2g(\phi Y, (A - h)\xi) + \operatorname{trace}(\nabla_{AY} A) + 3\eta(\phi AY) = 0 .$$

since ϕ and ϕA are skew-symmetric. (1.3) and commutativity of contraction and covariant differentiation imply

$$(2.5)\qquad -cg(A\xi, \phi Y) + \operatorname{trace}(\nabla_Y A)A - h \operatorname{trace}(\nabla_Y A) = 0 .$$

and

$$(2.6)\qquad -cg(A\xi, \phi Y) + \frac{1}{2}Y(\operatorname{trace} A^2 - h^2) = 0 .$$

so that $g(A\xi, \phi Y) = 0$. Consequently, ξ is principal. Let $A\xi = \mu\xi$. Then Lemma 2.4 of [5] implies μ is locally constant. If we replace Y by ξ , (2.1) becomes

$$(2.7)\qquad (c+3)\phi AX - (Xh)A\xi + (A - h + \mu)(\nabla_X A)\xi = 0 .$$

by (1.3). From (1.5), we have $(\nabla_X A)\xi = \nabla_X(A\xi) - A\nabla_X\xi = (\mu - A)\phi AX$. Then (2.7) gives

$$(2.8)\qquad \{(A - h + \mu)(\mu - A) + (c + 3)\}\phi AX - \mu(Xh)\xi = 0 .$$

Since $A\phi(T_x M)$ is orthogonal to ξ , both first term and second term are zero, so that $\mu(Xh) = 0$.

Let X be a principal vector with principal curvature λ , which is orthogonal to ξ . Then Lemma 2.2 of [5] implies that ϕX is a principal vector with principal curvature $(\lambda\mu + 2)/(2\lambda - \mu)$, and $2\lambda - \mu \neq 0$. Hence (2.8) gives

$$(2.9)\qquad \lambda\{ (\frac{\lambda\mu + 2}{2\lambda - \mu} - h + \mu)(\mu - \frac{\lambda\mu + 2}{2\lambda - \mu}) + c + 3\} = 0 .$$

If $\mu = 0$, then $\lambda \neq 0$, $A\phi X = \lambda^{-1}\phi X$ and $(\lambda h - 1) + \lambda^2(c + 3) = 0$. Let A_0 be the restriction of A to the orthogonal complement

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$\xi^{\perp}$ $(= \phi T_x M)$ of ξ . Then A_0 has at most two distinct eigenvalues, so that M has at most three distinct principal curvalures. From the proof of Theorem 4 of [1], M is a homogeneous hypersurface with 2 or 3 distinct principal curvatures.

If $\mu \neq 0$, then h is constant. From (2.9), A_0 has at most three distinct constant eigenvalues so that M has at most four distinct constant principal curvatures. Since ξ is principal, Theorem 1 and Theorem 4 in [2] implies that M is a homogeneous hypersurface with 2 or 3 distinct principal curvatures. Thus Theorem 1 is proved.

The same argument implies that if M has parallel Ricci tensor and ξ is principal, then M is a homogeneous hypersurface with 2 or 3 distinct principal survatures. But there is no homogeneous hypersurface with parallel Ricci tensor. Hence Theorem 2 is proved.

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