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THE COVER ASSOCIATED TO A (1,3)-POLARIZED BIELLIPTIC ABELIAN SURFACE AND ITS BRANCH LOCUS

by GIANFRANCO CASNATI*

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Let A be an abelian surface and let $|D|$ be a polarization of type $(1,3)$ on A. If $(A,|D|)$ is not a product of elliptic curves, such a polarization induces a finite morphism $\varrho: A \to \mathbb{P}^2_{\mathbb{C}}$ of degree 6. In this paper we describe the branch locus of ϱ when A is bielliptic in the sense of K. Hulek and S. H. Weintraub (see **[13]**), generalizing the results proved by Ch. Birkenhake and H. Lange in **[4]**.

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0. Introduction

Let $(A,|D|)$ be a $(1,d)$-polarized abelian surface. Here abelian surface means a surface A with $\omega_{A|\mathbb{C}} \cong \mathcal{O}_A$ and $q(A)=2$ over $\mathbb{C}$ and $|D|$ is an ample linear system of type $(1,d)$ up to translation in A. It is known (see e.g. **[14**, Lemma 10.1.1**]**) that if

(♡) $(A,|D|) \not\cong (E_1 \times E_2, p_1^*\mathcal{L}_1 \otimes p_2^*\mathcal{L}_2)$ where E_i is an elliptic curve, $p_i: E_1 \times E_2 \to E_i$ is the projection and $\mathcal{L}_i \in \mathrm{Pic}(E_i)$, $i=1,2$,

then $|D|$ is free from base components. Therefore it induces a quasi-finite rational map $\varrho: A \dashrightarrow \mathbb{P}^{d-1}_{\mathbb{C}}$ such that $2d = D^2 = \deg(\varrho)\deg(\varrho(A))$. Since $C^2 \geq 0$ for each irreducible curve C on A, ϱ is also finite.

There are many results about the behaviour of ϱ with respect to d. If $d=2$ then $|D|$ has four base points and the map ϱ has been studied in this case by W. Barth in **[1]**. If $d \geq 3$ then $|D|$ is base-point-free. When $d=4$ C. Birkenhake, H. Lange, D. van Straten in **[5]** and F. Tovena in **[22]** have dealt with the morphism ϱ. Finally the case $d \geq 5$ has been described by S. Ramanan in **[18]**.

In the case $d=3$ the map ϱ is surjective and, since both A and $\mathbb{P}^2_{\mathbb{C}}$ are smooth, it is also flat (see **[11**, Exercise III 10.9**]**), i.e. a cover in the sense of **[9]**. In **[4]** a family $\mathcal{H}_{BL}$ of dimension 1 of such kind of surfaces is studied in details. In particular the branch locus B_ϱ of $\varrho: A \to \mathbb{P}^2_{\mathbb{C}}$ is described for each such A.

Really the main property of the surfaces A corresponding to points in $\mathcal{H}_{BL}$ is the existence of a non-trivial involution $j_A: A \to A$. Hence they are bielliptic in the sense of

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[13]. Therefore there exists a double cover $\tau: A \to S$ onto a ruled surface S with invariant $e(S) = -1$ over an elliptic curve E, and the cover $\varrho: A \to \mathbb{P}^2_{\mathbb{C}}$ factors as $\varrho = \sigma \circ \tau$ where $\sigma: S \to \mathbb{P}^2_{\mathbb{C}}$ is a cover of degree 3.

The aim of this paper is to generalize to bielliptic abelian surfaces A the mentioned results of [4] about the branch locus B_ϱ of the cover $\varrho := \sigma \circ \tau: A \to \mathbb{P}^2_{\mathbb{C}}$ induced by $|D|$, using the theory of covers developed in [15, 21 and 9].

In Section 1 we study such a kind of cover σ, dealing with its branch locus and remification divisor. Moreover we describe the branch locus of the cover τ. Finally we show how to recover the family $\mathcal{H}_{BL}$ as a particular case. Section 2 is devoted to the proof of the following theorem.

Theorem 0.1. *There is a decomposition $B_\varrho = 2C' + C''$ into irreducible sextic curves birationally isomorphic to E.*

The singularities of C' are nine cusps of type A_2. The singularities of C'' are nine points of type A_1 (possibly three by three infinitely near, i.e. three points of type D_4), lying on the cuspidal tangent lines at C'.

In [9] a structure theorem for covers of degree d between smooth varieties has been proved. Such a result has been used in order to give a complete characterization of covers of low degree d, namely $3 \le d \le 5$ (see [9, 7, 8]).

More precisely if $\varrho: X \to Y$ is a cover of degree $d \ge 3$ and both X and Y are smooth, Theorem 2.1 of [9] asserts the existence of a locally free $\mathcal{O}_Y$-sheaf $\mathcal{E}$ of rank $d-1$, natural splittings

$$\begin{aligned} \varrho_* \omega_{X|Y} &\cong \mathcal{O}_Y \oplus \mathcal{E}, \\ \varrho_* \mathcal{O}_X &\cong \mathcal{O}_Y \oplus \check{\mathcal{E}}, \end{aligned} \tag{0.2}$$

and an embedding $i: X \hookrightarrow \mathbb{P} := \mathbb{P}(\mathcal{E})$ such that $\varrho = \pi \circ i$ ($\pi: \mathbb{P} \to Y$ is the projection) and $\mathcal{O}_{\mathbb{P}}(1)_{|X} \cong \omega_{X|Y}$. Following R. Miranda, $\check{\mathcal{E}}$ is called the *Tschirnhausen module of* ϱ (see [15]).

If in addition, $d \ge 4$, there also exists a locally free $\mathcal{O}_Y$-sheaf $\mathcal{F}$ fitting into a sequence of the form

$$0 \longrightarrow \mathcal{F} \xrightarrow{\eta} S^2\mathcal{E} \xrightarrow{\varphi} \varrho_* \omega^2_{X|Y} \longrightarrow 0. \tag{0.3}$$

Notice that $\mathcal{F}$ has rank $N_d := \frac{d(d-3)}{2}$.

If $d \ge 4$ (resp. $d = 3$), via $\Phi_d: H^0(Y, \check{\mathcal{F}} \otimes S^2\mathcal{E}) \xrightarrow{\sim} H^0(\mathbb{P}, \pi^*\check{\mathcal{F}}(2))$ (resp. $\Phi_3: H^0(Y, S^3\mathcal{E} \otimes \det \mathcal{E}^{-1}) \xrightarrow{\sim} H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(3) \otimes \pi^* \det \mathcal{E}^{-1})$) we obtain a morphism $\delta := \Phi_d(\eta): \pi^*\mathcal{F}(-2) \to \mathcal{O}_{\mathbb{P}}$ (resp. $\delta := \Phi_3(\eta): \mathcal{O}_{\mathbb{P}}(-3) \otimes \pi^* \det \mathcal{E} \to \mathcal{O}_{\mathbb{P}}$) and $X = D_0(\delta) \subseteq \mathbb{P}$. If $d \ge 5$ such a section η cannot be general since $\operatorname{codim}_{\mathbb{P}}(X) = d - 2 < N_d$. If $d = 5$, it can be proved (see [7] for the details) that such η's belong to the image of a natural quadratic map

$$H^0(Y, \Lambda^2\mathcal{F} \otimes \mathcal{E} \otimes \det \mathcal{E}^{-1}) \to H^0(Y, \check{\mathcal{F}} \otimes S^2\mathcal{E}).$$

Unfortunately for $d \ge 6$ there is not such a satisfactory theory.

Anyhow, given an arbitrary (1, 3)-polarized abelian surface $(A, |D|)$, it is interesting to compute the sheaves $\mathcal{E}$ and $\mathcal{F}$ of the associated cover $\varrho: A \to \mathbb{P}^2_{\mathbb{C}}$. In Section 3 we prove the following theorem.

Theorem 0.4. *Let $(A, |D|)$ be a $(1, 3)$-polarized abelian surface satisfying $(\heartsuit)$. Then the sheaves $\mathcal{E}$ and $\mathcal{F}$ corresponding to the cover $\varrho: A \to \mathbb{P}^2_{\mathbb{C}}$ are*

$$\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(3) \oplus \Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(3)^{\oplus 2}, \tag{0.4.1}$$

$$\mathcal{F} \cong S^2\Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(6)^{\oplus 3}. \tag{0.4.2}$$

Finally, in Section 4, we will give a complete description of the structure of the map ϱ for bielliptic surfaces.

Theorem 0.5. *Let $\varrho: A \to \mathbb{P}^2_{\mathbb{C}}$ a $(1, 3)$-polarized abelian surface and let $i: A \hookrightarrow \mathbb{P}$ the embedding above. Then A is bielliptic if and only if the restriction to A of the projection $\bar{\pi}: \mathbb{P}(\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(3) \oplus \Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(3) \oplus \Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(3)) \to \mathbb{P}(\Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(3))$ from $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(3) \oplus \Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(3))$ induced by the decomposition* 0.4.1 *is a morphism whose image is smooth.*

For all the notations and definitions used in the paper we refer to [**11**].

1. (1, 3)-Polarized bielliptic abelian surfaces

Let E be an elliptic curve and consider the unique ruled surface S over E with invariant $e(S) = -1$. Then there exists an indecomposable locally free $\mathcal{O}_E$-sheaf $\mathcal{H}$ of rank 2, fitting into the sequence

$$0 \to \mathcal{O}_E \to \mathcal{H} \to \mathcal{O}_E(P) \to 0$$

such that $S \cong \mathbb{P}(\mathcal{H}) \xrightarrow{e} E$. Fixing such an isomorphism, let $\{C_0\} = |\mathcal{O}_S(1)|$. Notice that $C_0^2 = 1$ and $\mathrm{Pic}(S) \cong \mathbb{Z}C_0 \oplus e^*\mathrm{Pic}(E)$.

Proposition 1.1. *If $Q \in E$, then $|C_0 + e^*Q|$ induces a cover $\sigma: S \to \mathbb{P}^2_{\mathbb{C}}$ of degree 3 with Tschirnhausen module $\Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}$.*

Proof. If $D \in |C_0 + Qf|$ then $D^2 = 3$ and $\dim |C_0 + e^*Q| = 2$. Moreover $|C_0 + e^*Q|$ is ample and base-point-free (see [**10**, Proposition 3.4 and 3.5]).

If $r \in \check{\mathbb{P}}^2_{\mathbb{C}}$ and $C := \sigma^{-1}(r)$ is smooth then the branch locus of $\sigma_{|C}$ has degree 6 by the theorem of Hurwitz. Thus the branch locus B_σ of σ has degree 6, hence $c_1(\mathcal{E}) = -3$ for the Tschirnhausen module $\mathcal{E}$ of σ. On the other hand computing $\chi(\mathcal{O}_S)$ for the triple cover σ (see [**17**, Section 8 or **15**, Section 10]) one obtains $c_2(\check{\mathcal{E}}) = 3$. Normalizing $\check{\mathcal{E}}$ we then get $c_1(\check{\mathcal{E}}_{norm}) = -1$, $c_2(\check{\mathcal{E}}_{norm}) = 1$. On the other hand

$$3 = h^0(E, \mathcal{H}(Q)) = h^0(S, \sigma^*\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(1)) = h^0(\mathbb{P}^2_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(1)) + h^0(\mathbb{P}^2_{\mathbb{C}}, \check{\mathcal{E}}_{norm}),$$

therefore $\check{\mathcal{E}}_{norm}$ is stable. We then conclude that $\check{\mathcal{E}}_{norm} \cong \Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(1)$ (see [**16**, p. 246]). □

Proposition 1.2. *The set H of sections of $H^0(\mathbb{P}^2_{\mathbb{C}}, \mathcal{S}^3\Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(6))$ inducing covers (as explained in the introduction) $\sigma: S \to \mathbb{P}^2_{\mathbb{C}}$ of degree 3 with a smooth S is open and dense.*

Each such cover σ is non-cyclic and there is an elliptic curve E, and a point $Q \in E$ such that S is the unique ruled surface over E with invariant $e(S) = -1$ and the pull back to S of the linear system of lines is $|C_0 + e^*Q|$.

Proof. The first statement follows from [**9**, Theorem 3.6], since $\mathcal{E} \cong \Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(3)$. Moreover σ is not cyclic since its Tschirnhausen module does not split.

Notice that $\chi(\mathcal{O}_S) = K_S^2 = 0$ (see [**17**, Section 8] or [**15**, Section 10]) hence S is minimal. Thus S is either a ruled surface $\mathbb{P}(\mathcal{H})$ over an elliptic curve E or a surface with Kodaira dimension $\kappa(D) \geq 0$.

Let $D \in |\sigma^*\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(1)|$. In the second case one would have $K_S \cdot D \geq 0$. On the other hand, using projection formula and the isomorphism $\sigma_*\omega^2_{S|\mathbb{C}} \cong \mathcal{S}^2\Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}$ (see [**17**, Section 8 or **9**, formula 5.1.2]), one has

$$2K_S \cdot D = \chi(\mathcal{O}_S) - \chi(\omega^2_{S|\mathbb{C}}) - \chi(\mathcal{O}_S(D)) + \chi(\omega^2_{S|\mathbb{C}}(D)) = -6. \tag{1.2.1}$$

Thus $S \cong \mathbb{P}(\mathcal{H})$ for some locally free $\mathcal{O}_E$-sheaf $\mathcal{H}$ of rank 2 on an elliptic curve E. Let $C_0 \subseteq S$ be a section of minimal self-intersection $C_0^2 = -e(S)$. Then $D \in |aC_0 + e^*\mathfrak{b}|$ where $a \geq 1$ and $\mathfrak{b}$ is a divisor on E. Moreover

$$3 = a^2C_0^2 + 2ab \tag{1.2.2}$$

where $b = \deg(\mathfrak{b})$. It follows that $e(S)$ is odd. If $e(S) > 0$, since D is ample then $b + aC_0^2 > 0$ ([**10**, Proposition 3.4]), hence $3 + a^2C_0^2 > 0$. It follows $C_0^2 = -1$, $a = 1$ and $b = 2$, which is absurd since then D would not be free from base points.

If $e(S) = -1$ then $\mathcal{H}$ fits into

$$0 \to \mathcal{O}_E \to \mathcal{H} \to \mathcal{O}_E(P) \to 0.$$

Since $2b = 3/a - a^2C_0^2$ then $a = 1, 3$. If $a = 3$ then $2K_S \cdot D = -6 - 4b = -6$ if and only if $b = 0$, contradicting formula 1.2.2 again. We conclude that $a = 1$ and $\mathfrak{b} = Q \in E$. □

Now we deal with the branch locus B_σ and the ramification divisor R_σ of the cover $\sigma: S \to \mathbb{P}^2_{\mathbb{C}}$ corresponding to a section $\vartheta \in H^0(\mathbb{P}^2_{\mathbb{C}}, \mathcal{S}^3\Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(6))$. We always refer to the careful description of the branch locus of a triple cover due to R. Miranda (see [**15**]).

More precisely B_σ is singular at $x \in \mathbb{P}^2_{\mathbb{C}}$ if and only if σ is totally ramified over x ([**15**, Lemma 4.8]). If $x \in \mathrm{Sing}(B_\sigma)$ then it is a double point with one tangent, and if it is also an isolated singularity, then x is a point of type A_{3k-1} for some $k \geq 1$ (see [**15**, Corollary 5.8 and its proof]).

Lemma 1.3. *Assume that B_σ is reduced. Then its singularities are a_2 points of type A_2 and a_5 points of type A_5. Moreover $2a_5 + a_2 = 9$.*

Proof. The first part of the statement follows from [**15**, Corollary 5.8], its proof and the fact that $\deg(B_\sigma) = 6$.

Let $\vartheta \in H^0(\mathbb{P}^2_{\mathbb{C}}, S^3\Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(6))$ correspond to $\sigma: S \to \mathbb{P}^2_{\mathbb{C}}$. As shown in Proposition 3.9 and Lemma 10.1 of [**15**] there is a natural map

$$\alpha: H^0(\mathbb{P}^2_{\mathbb{C}}, S^3\Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(6)) \to \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}}(\Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(-3), S^2\Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}})$$

inducing as an exact sequence

$$0 \longrightarrow \Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(-3) \xrightarrow{\alpha(\vartheta)} S^2\Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}} \longrightarrow \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}} \longrightarrow \mathcal{O}_T \longrightarrow 0,$$

where the support of T is the set of points of total ramification of σ. A Chern class computation shows that $\deg(T) = 9$.

Locally at $x \in T$

$$\mathfrak{J}_{T,x} = (a^2 - bd, ad - bc, d^2 - ac)\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}},x},$$

where $a, b, c, d \in \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}},x}$ are the local functions around x defining the cover σ (see Sections 3 and 4 of [**15**]). Let $\mathfrak{M}$ be the maximal ideal of $\mathcal{O}_{\mathbb{P}^3_{\mathbb{C}},x}$.

If $b, c \in \mathfrak{M}$ then also $a, d \in \mathfrak{M}$, thus the fibre of σ over x would be isomorphic to

$$\mathbb{C}[z, w]/(z^2, zw, w^2)$$

which is not Gorenstein, hence S could not be smooth over x.

Let $b \notin \mathfrak{M}$. Locally the equation of S is $z^3 + gz + h = 0$, where $h := 3abd - 2a^3 - b^2c$, $g := 3(bd - a^2)$ (see [**15**, Remark 2.8.1]). In particular $\mathfrak{J}_{T,x} = (g, h)\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}},x}$ and the local equation of B_σ around x is $27h^2 + 4g^3 = 0$. Since $h \in \mathfrak{M}\backslash\mathfrak{M}^2$ (see [**15**, Lemma 5.7]), thus x is of type A_2 (resp. A_5) if and only if $g \in \mathfrak{M}\backslash\mathfrak{M}^2$ (resp. $g \in \mathfrak{M}^2$) at x, that is if and only if x has degree 1 (resp. 2) inside T. □

Theorem 1.4. *B_σ is an irreducible sextic curve birationally isomorphic to the curve E. Its singularities are nine points of type A_2.*

Proof. Notice that each irreducible component $C \subseteq R_\sigma$ is mapped birationally onto $\sigma(C)$, since $\deg(\sigma) = 3$.

Since $R_\sigma \cdot (C_0 + e^*Q) = 6$ then $\deg(B_\sigma) = 6$. Assume that B_σ is reducible. Then it must be reduced, otherwise it would have at least a triple point, a contradiction by Lemma 4.8 and Corollary 5.8 of [**15**]. It follows that $R_\sigma \in |C_0 + e^*(P + 3Q)|$ is reducible too, hence R_σ contains a fibre e^*S.

Since $e^*S \cdot (C_0 + e^*Q) = 1$ then e^*S is mapped on a line $r \subseteq B_\sigma$. If B_σ contains another line r' then $r \neq r'$ hence B_σ has a node, an absurd. It follows that the residual divisor $B := B_\sigma - r$ must be irreducible.

The points of $B \cap r$ are simple on B and they are images of points of total ramification of σ. Since $e^*S \cdot (R_\sigma - e^*S) = 1$ it follows that $r \cap B$ is exactly one point x. Such a point is necessarily a flex on B and r is its inflectional tangent.

The other singularities of B_σ are also singularities of B, thus they must be points of type A_2. Since B is birationally isomorphic to E then B_σ must have exactly $a_2 = 5$ points of type A_2 and $a_5 = 1$ point of type A_5 (namely x). We conclude that in this case $2a_5 + a_2 = 7 \neq 9$.

Hence we have proved that B_σ is irreducible. If it is not reduced, then σ would be totally ramified, whence cyclic ([**21**, Proposition 3.1]).

We conclude that B_σ is irreducible and reduced, thus it is birationally equivalent to E. In particular the formula of Clebsch yields $1 = p_g(B_\sigma) = 10 - 3a_5 - a_2$, thus $a_2 + 3a_5 = a_2 + 2a_5 = 9$ which implies $a_5 = 0$. □

In Proposition 3.2 of [**10**], it is proved that the linear system $|\omega_{S|\mathbb{C}}^{-2}| = |4C_0 - 2e^*P|$ has dimension 1 and its generic member is a smooth irreducible curve. We now produce the two-dimensional family of bielliptic abelian surfaces with a $(1, 3)$-polarization.

Proposition 1.5. *Let $\sigma: S \to \mathbb{P}^2_{\mathbb{C}}$ be as in Proposition* 1.2 *and let $\tau: A \to S$ be the double cover branched along a general divisor $B_\tau \in |\omega_{S|\mathbb{C}}^{-2}|$. Then $(A, |\tau^*(C_0 + e^*Q)|)$ is a* $(1, 3)$-*polarized bielliptic abelian surface.*

Conversely each $(1, 3)$-*polarized bielliptic abelian surface* $(A, |D|)$ *arises in this way.*

Proof. The general element of $|\omega_{S|\mathbb{C}}^{-2}|$ is smooth and irreducible, thus A is smooth. Moreover $q(A) = 2$, $p_g(A) = 1$ and $\omega_{A|\mathbb{C}} \cong \mathcal{O}_A$ (see [**2**, Lemma 17.1 of Chapter I]), thus A is abelian.

The map $\varrho := \sigma \circ \tau: A \to \mathbb{P}^2_{\mathbb{C}}$ is a cover of degree 6, then $|\tau^*(C_0 + e^*Q)| = |\varrho^*\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(1)|$ is ample, hence it must be a $(1, 3)$-polarization. The map τ induces a non-trivial involution $j_A: A \to A$, thus A is bielliptic.

The converse follows trivially from (i) and (ii) of Proposition 4.4 in [**13**]. □

Remark 1.6. Since $(4C_0 - 2e^*Q)^2 = 0$ then we get a fibration $\varphi: S \to \mathbb{P}^1_{\mathbb{C}}$, whose generic fibre is a smooth elliptic curve.

Moreover $\chi(\mathcal{O}_S) = 0$ hence φ is isotrivial, its singular fibres are multiple of smooth curves (see Lemma 1.1 of [**20**]), and φ has exactly three double fibres by Proposition 3.2 of [**10**].

Consider the residual divisor $R_0 := \sigma^*B_\sigma - 2R_\sigma \in |4C_0 - 2e^*P|$. Then the restriction of σ to $R_0 \backslash \sigma^{-1}$ (Sing(B_σ)) is an isomorphism onto $B_\sigma \backslash$Sing (B_σ). Moreover Lemma 5.9 of [**15**] asserts the smoothness of R_0 also at the points of total ramification. We conclude that R_0 is globally smooth, hence irreducible, and $\sigma_{|R_0}$ is birational onto B_0. It follows that $R_0 \cong E$ and that all the smooth fibres of φ are isomorphic to E.

The fibres of the map $\varphi \circ \tau: A \to \mathbb{P}^1_{\mathbb{C}}$ are double étale covers of the curves in $|4C_0 - 2e^*P|$, since $B_\tau \cdot (4C_0 - 2e^*P) = 0$. It follows that they are not connected (see [**3**, Exercise IX 1]), hence the Stein factorization of $\varphi \circ \tau$ gives rise to a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\psi} & E' \\ \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\xi} \\ S & \xrightarrow{\varphi} & \mathbb{P}^1_{\mathbb{C}} \end{array}$$

where E' is a smooth elliptic curve and ξ is a double cover.

The map $\varphi_{|C_0}: C_0 \to \mathbb{P}^1_{\mathbb{C}}$ is a double cover since $C_0 \cdot (4C_0 - 2e^*P) = 2$. Its branch points are exactly the critical values of $\varphi \circ \tau$. Therefore they coincide with the branch points of ξ. In particular $E' \cong C_0 \cong E$, and we have an exact sequence of abelian varieties

$$0 \longrightarrow E \longrightarrow A \xrightarrow{\psi} E \longrightarrow 0$$

(see also [**3**, Example IX 4.3]).

Notice that $A \cong (E \times F)/G$ where $G := \mathbb{Z}_2 \times \mathbb{Z}_2$ (see [**13**, Proposition 4.1]) and $F/G \cong E$ (see [**20**, Theorem 1.2]).

In the following example, choosing $R_0 = B_\tau$, we obtain the family studied in [**4**].

Example 1.7. Let $\sigma: S \to \mathbb{P}^2_{\mathbb{C}}$ be as above. Since the Tschirnhausen module of σ does not split, then σ is not cyclic. According to [**21**] we can build the discriminant $D(S \mid \mathbb{P}^2_{\mathbb{C}})$ of σ and we have a commutative square

$$\begin{array}{cccc} A := & \hat{S} & \xrightarrow{\alpha} & D(S \mid \mathbb{P}^2_{\mathbb{C}}) \\ & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\beta} \\ & S & \xrightarrow{\sigma} & \mathbb{P}^2_{\mathbb{C}}. \end{array}$$

Theorem 1.4 above and [**21**, Proposition 3.4] give us the following results:

(i) β is a double cover branched along B_σ and $D(S \mid \mathbb{P}^2_{\mathbb{C}})$ is normal with 9 singular points of type A_2;

(ii) α is a cyclic triple cover of $D(S \mid \mathbb{P}^2_{\mathbb{C}})$ branched only at $\mathrm{Sing}(D(S \mid \mathbb{P}^2_{\mathbb{C}}))$ and $A := \hat{S}$ is smooth;

(iii) S is the quotient of $A := \hat{S}$ via an involution.

In particular $D(S \mid \mathbb{P}^2_{\mathbb{C}})$ is a singular $K3$-surface and A is a bielliptic abelian surface. Again $\varrho := \sigma \circ \tau$ is a cover of degree 6 and $\varrho^* \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(1)$ is a polarization of type $(1, 3)$ on A.

Notice that Lemma 1.4 of [**21**] implies that the reduced branch locus of ϱ is B_σ, thus the branch locus of ϱ satisfies $B_\varrho = 3B_\sigma$. It follows that $B_\tau = R_0$. Moreover ϱ is a Galois cover with Galois group $\mathfrak{S}_3$ (see again [**21**]).

2. The equation of the branch locus B_ϱ

In this section we will describe the branch locus B_ϱ of the cover $\varrho\colon A \to \mathbb{P}^2_{\mathbb{C}}$, when A is bielliptic. To this purpose we denote by $t_a : A \to A$ the translation by $a \in A$ and we set $|D| := |\varrho^* \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(1)|$.

Since D is a polarization of type $(1,3)$, the morphism $\varrho\colon A \to \mathbb{P}^2_{\mathbb{C}}$ is invariant with respect to the group

$$K(D) := \{a \in A \mid t_a^* D \in |D|\} \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \subseteq PGL_3.$$

Proposition 2.1. *There exists a decomposition into $K(D)$-invariant sextic curves $B_\varrho = 2B_\sigma + C_\varrho$.*

Proof. Obviously $B_\varrho = 2B_\sigma + \sigma_* B_\tau$. Let $C_\varrho := \sigma_* B_\tau$ and fix a general line $\ell \in \check{\mathbb{P}}^2_{\mathbb{C}}$. Then $C := \varrho^{-1}(\ell) \in |D|$ is a smooth irreducible curve of genus 4, by adjunction formula. Hence the theorem of Hurwitz applied to C yields $\deg(B_\varrho) = 18$, whence $\deg(C_\varrho) = 6$.

Since B_ϱ is $K(D)$-invariant, thus the two curves C_ϱ and B_σ must be invariant too. □

With a suitable choice of the coordinates x_0, x_1, x_2 in $\mathbb{P}^2_{\mathbb{C}}$, we can assume that $K(D) \subseteq PGL_3$ is generated by the classes of

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix}$$

where $\zeta \neq 1, \zeta^3 = 1$. The $K(D)$-orbit $O(x)$ of a point $x \in \mathbb{P}^2_{\mathbb{C}}$ contains at most nine distinct points. If $O(x)$ contains less than nine points then it coincides with one of the following:

$$O_0 := \{[1,0,0],[0,1,0],[0,0,1]\}, \quad O_1 := \{[1,1,1],[1,\zeta,\zeta^2],[1,\zeta^2,\zeta]\},$$

$$O_2 := \{[1,1,\zeta],[1,\zeta,1],[\zeta,1,1]\}, \quad O_3 := \{[1,1,\zeta^2],[1,\zeta^2,1],[\zeta^2,1,1]\}.$$

A simple computation shows that each $K(D)$-invariant sextic has an equation of the form

$$\begin{aligned} f(x_0,x_1,x_2) := {} & a(x_0^6+x_1^6+x_2^6) + b(x_0^3x_1^3 + x_0^3x_2^3 + x_1^3x_2^3) + \\ & + c x_0x_1x_2(x_0^3+x_1^3+x_2^3) + d x_0^2x_1^2x_2^2 = 0, \end{aligned} \tag{2.2}$$

for some $[a,b,c,d] \in \mathbb{P}^3_{\mathbb{C}}$. Let $C \subseteq \mathbb{P}^2_{\mathbb{C}}$ be the corresponding curve. We have a rational map

$$\varphi_f := \left(\frac{\partial f}{\partial x_0}, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right) : \mathbb{P}^2_{\mathbb{C}} \dashrightarrow \check{\mathbb{P}}^2_{\mathbb{C}},$$

and, computing $\partial f/\partial x_i$, one easily checks that φ_f is $K(D)$-equivalent. It follows that the dual curve $\check{C} \subseteq \check{\mathbb{P}}^2_{\mathbb{C}}$ is also $K(D)$-invariant. Let g be its equation in $\check{\mathbb{P}}^2_{\mathbb{C}}$ with coordinates y_0, y_1, y_2. We have

$$\psi_g := \left(\frac{\partial g}{\partial x_0}, \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}\right) : \check{\mathbb{P}}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^2_{\mathbb{C}},$$

and it is well known that the biduality $\psi_g \circ \varphi_f$ is the identity on C (in particular C and $\check{C}$ are birational).

Since B_σ has nine points of type A_2, the ordinary formula of Plücker implies that $\check{B}_\sigma$ is a smooth cubic. Therefore its equation is

$$y_0^3 + y_1^3 + y_2^3 - 3\lambda y_0 y_1 y_2 = 0, \qquad \lambda^3 \neq 1.$$

Taking into account Section 1 of [**4**], by biduality we get that the equation of B_σ with respect to the above system of coordinates is

$$\begin{aligned} f_\lambda(x_0, x_1, x_2) :=& (x_0^6 + x_1^6 + x_2^6) + 2(2\lambda^3 - 1)(x_0^3x_1^3 + x_0^3x_2^3 + x_1^3x_2^3) \\ &- 6\lambda^2 x_0x_1x_2(x_0^3 + x_1^3 + x_2^3) - 3\lambda(\lambda^3 - 4)x_0^2x_1^2x_2^2 = 0, \end{aligned} \tag{2.3}$$

where $\lambda^3 \neq 1$. Notice that $O_i \cap B_\sigma = \emptyset$ and that $\mathrm{Sing}(B_\sigma) = O([\lambda, 1, 1])$.

Remark 2.4. Let $\gamma, \delta \in PGL_3$ be classes of the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta^2 & \zeta \\ 1 & \zeta & \zeta^2 \end{pmatrix}.$$

The group $G := \langle \gamma, \delta \rangle$ is well-known to be isomorphic to the alternating group $\mathcal{A}_4$ of order 4 ([**6**, Section 7.3]). The elements of order two of G form a normal subgroup $G_0 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \trianglelefteq G$, γ generates a cyclic subgroup $G_1 \cong \mathbb{Z}_3$ and $G \cong G_0 \rtimes G_1$.

The polynomials f_λ and f_μ represent birationally isomorphic curves if and only if they lie in the same G-orbit, i.e. if and only if there is $g \in G$ such that $g(f_\lambda) = f_\mu$ (see [**4**, Section 1] and [**6**]). The group G induces an action on $\mathbb{C}\setminus\{1, \zeta, \zeta^2\}$ given by

$$\gamma(\lambda) := \zeta\lambda, \qquad \delta(\lambda) := \frac{\lambda + 2}{\lambda - 1}.$$

With respect to this action $g(f_\lambda) = f_{g(\lambda)}$, $g \in G$. Then f_λ and f_μ represent birationally isomorphic curves if and only if $g(\lambda) = \mu$. For such a pair (λ, μ), some easy computation shows the existence of $g \in G_0$, depending on (λ, μ), and sending (λ, μ) to (λ', μ') where $\mu' \in \{\lambda', \gamma(\lambda'), \gamma^2(\lambda'), \delta(\lambda')\}$.

Remark 2.5. We claim that if $y \in B_\tau \cap R_\sigma$ and $x := \sigma(y) \notin \mathrm{Sing}(B_\sigma)$, then the tangent space of C_ϱ at x contains the tangent space of B_σ at x.

Since the assertion is local we can consider $\sigma : \mathrm{spec}(B) \to \mathrm{spec}(A) \subseteq \mathbb{P}^2_{\mathbb{C}}$, where A is a ring with maximal ideal $\mathfrak{M}$ corresponding to x and $B \cong A[u]/p(u)$ where $p(u) = u^3 + \alpha u^2 + \beta u + \gamma, \alpha, \beta, \gamma \in A$. We can also assume that y corresponds to the ideal $(u) + \mathfrak{M}$ of B. In this setting R_σ has equations $3u^2 + 2\alpha u + \beta = p(u) = 0$. It follows that $y \in R_\sigma$ if and only if $\beta, \gamma \in \mathfrak{M}$. Moreover $\alpha \notin \mathfrak{M}$ since x is not a point of total ramification of σ. An easy local computation shows that $\gamma \notin \mathfrak{M}^2$, otherwise S would be singular at y. Eliminating the variable u we finally obtain an equation of B_σ of the form $b := 4\alpha^3\gamma + b_2$ where $b_2 \in \mathfrak{M}^2$.

Since $\mathrm{spec}(B) \cong \mathbb{A}^1_A \subseteq \mathbb{P}^1_A$ and $\mathcal{O}_{\mathbb{P}^1_A}(1)_{|\mathrm{spec}(B)} \cong \omega_{\mathrm{spec}(B)|\mathrm{spec}(A)}$ (see the introduction) and $|\omega_{S|\mathbb{P}^2_{\mathbb{C}}}| = |C_0 + e^*(P + 3Q)|$, a proper choice of u allows us to assume that $u = p(u) = 0$ are equations of C_0, thus $B_\tau \in |4C_0 - 2e^*Q|$ is given by $p(u) = q(u) = 0$, where $q(u)$ is a polynomial of degree 4. It follows that we can choose equations $p(u) = \delta u^2 + \varepsilon u + \eta = 0, \delta, \varepsilon, \eta \in A$, for B_τ. The condition $y \in B_\tau$ yields $\eta \in \mathfrak{M}$. Again eliminating the variable u we obtain the equation of C_ϱ of the form $c = (\alpha\delta\varepsilon - \varepsilon^3)\gamma + c_2$ where $c_2 \in \mathfrak{M}^2$.

Theorem 2.6. *C_ϱ is an irreducible sextic birationally isomorphic to E. Its singularities are either nine points of type A_1, possibly three by three infinitely near (i.e. three points of type D_4), or nine cusps of type A_2.*

Each cuspidal tangent lines at B_σ contains exactly one singular point of C_ϱ. C_ϱ has points of type D_4 if and only if $\mathrm{Sing}(C_\varrho) = O_i$ *for some* $i = 0, 1, 2, 3$. *C_ϱ has points of type A_2 if and only if $C_\varrho = B_\sigma$ and, in this case, ϱ is the cover described in Example* 1.7.

Finally C_ϱ and B_σ are tangent at each point of intersection.

Proof. If $C_\varrho \cap \mathrm{Sing}(B_\sigma) \neq \emptyset$ then B_τ contains at least one point of total ramification of σ, whence $R_0 \cap B_\tau \neq \emptyset$. Since $R_0 \cdot B_\tau = 0$ we get that $R_0 = B_\tau$, hence $C_\varrho = B_\sigma$ and ϱ is the cover described in Example 1.7.

For this reason, from now on, we will always assume that $B_\tau \neq R_0$, i.e. $C_\varrho \cap \mathrm{Sing}(B_\sigma) = \emptyset$. Notice that it follows from Remark 2.5 that C_ϱ and B_σ are tangent at each point of intersection.

Since B_τ is irreducible and $B_\tau \cdot (C_0 + e^*Q) = 6$ then C_ϱ is an irreducible sextic curve. If C_ϱ was not reduced then its reduced structure $(C_\varrho)_{red}$ would be either a conic or a cubic, thus $B_\tau \subseteq \sigma^{-1}((C_\varrho)_{red}) \in |n(C_0 + e^*Q)|$, where $n = 2, 3$, which is absurd since $B_\tau \in |4C_0 - 2e^*P|$.

It follows that $\sigma_{|B_\tau}: B_\tau \to C_\varrho$ is a resolution of singularities of C_ϱ, which is then birationally isomorphic to E. Therefore the formula of Clebsch becomes

$$\sum_{x \in C_\varrho} \frac{m_x(m_x - 1)}{2} = 9 \tag{2.6.1}$$

where m_x is the multiplicity of x. We also obtain that if the tangent lines at $x \in \mathrm{Sing}(C_\varrho)$ are not all distinct, then $x \in B_\sigma$. Since C_ϱ is $K(D)$-invariant, we get that $\mathrm{Sing}(C_\varrho)$ is union of $K(D)$-orbits.

Assume that $\mathrm{Sing}(C_\varrho)$ contains either a point of type A_k, with $k \geq 3$, or a non-ordinary point of multiplicity at least three. Such kind of points contribute at least two in the sum in formula 2.6.1. Then there is $i = 0, 1, 2, 3$ such that $\emptyset \neq \mathrm{Sing}(C_\varrho) \cap O_i \subseteq B_\sigma \cap O_i = \emptyset$, a contradiction.

If $\mathrm{Sing}(C_\varrho)$ contains an ordinary multiple point x of multiplicity at least three, 2.6.1 implies that x must be of type D_4 and $x \in O_i$, thus $\mathrm{Sing}(C_\varrho) = O_i$ for some $i = 0, 1, 2, 3$. Conversely if $a = 0$ in equation 2.2 then $O_0 \subseteq \mathrm{Sing}(C_\varrho)$. If also $c = 0$ then equality holds and each point is of type D_4. If $c \neq 0$ then the points of O_0 are of type A_1. Then by 2.6.1 we necessarily have $\mathrm{Sing}(C_\varrho) = \cup O_i$. As proved above the points in O_i must be ordinary, hence again by 2.6.1 they are all of type A_1 and $\mathrm{Sing}(C_\varrho) = O_0 \cup O_i \cup O_j$ $(i, j = 1, 2, 3, i \neq j)$.

Now let the singularities of C_ϱ be nine points of type A_2. Then the equation of C_ϱ is f_μ (see formula 2.3), hence $\mathrm{Sing}(C_\varrho) = O([\mu, 1, 1])$. Moreover C_ϱ is birationally isomorphic to E. Thus we can suppose that either $\lambda^3 = \mu^3$ or $\mu = (\lambda+2)/(\lambda-1)$ by remark 2.4. Moreover $\mathrm{Sing}(C_\varrho) \subseteq B_\sigma$, since ϱ is locally étale outside B_σ.

If $\lambda = \mu = 0$ then $C_\varrho = B_\sigma$. Assume that $\lambda\mu \neq 0$ and $\lambda^3 = \mu^3$. Since $C_\varrho \cdot B_\sigma = 36$, Remark 2.5 implies that the pencil Φ of sextic curves generated by C_ϱ and B_σ has at most 18 base points. On the other hand Φ contains a reducible curve $\overline{C}$ of equation $x_0x_1x_2(x_0^3 + x_1^3 + x_2^3 - 3mx_0x_1x_2) = 0$. It is not difficult to check that $\overline{C} \cap B_\sigma$ contains at least 27 points, which are base points of Φ.

Assume finally that $\mu = (\lambda+2)/(\lambda-1)$. By direct substitution one checks that the condition $\mathrm{Sing}(C_\varrho) \subseteq B_\sigma$ is equivalent to $\mathrm{Sing}(B_\sigma) \subseteq C_\varrho$. Thus in both these cases we obtain a contradiction.

The point $\bar{x} := [\bar{x}_0, \bar{x}_1, \bar{x}_2] \in \mathbb{P}^2_{\mathbb{C}}$ is singular on the curve C of equation 2.2, if and only if $[a, b, c, d] \in \mathbb{P}^3_{\mathbb{C}}$ is a solution of the homogeneous system

$$\begin{cases} 6\bar{x}_0^5 a + 3\bar{x}_0^2(\bar{x}_1^3 + \bar{x}_2^3)b + \bar{x}_1\bar{x}_2(4\bar{x}_0^3 + \bar{x}_1^3 + \bar{x}_2^3)c + 2\bar{x}_0\bar{x}_1^2\bar{x}_2^2 d = 0 \\ 6\bar{x}_1^5 a + 3\bar{x}_1^2(\bar{x}_0^3 + \bar{x}_2^3)b + \bar{x}_0\bar{x}_2(\bar{x}_0^3 + 4\bar{x}_1^3 + \bar{x}_2^3)c + 2\bar{x}_0^2\bar{x}_1\bar{x}_2^2 d = 0 \\ 6\bar{x}_2^5 a + 3\bar{x}_2^2(\bar{x}_0^3 + \bar{x}_1^3)b + \bar{x}_0\bar{x}_1(\bar{x}_0^3 + \bar{x}_1^3 + 4\bar{x}_2^3)c + 2\bar{x}_0^2\bar{x}_1^2\bar{x}_2 d = 0. \end{cases} \tag{2.6.2}$$

Let us denote by M the matrix of the system 2.6.2.

Obviously the system 2.6.2 has always ∞^1 solutions, corresponding to the unique curve of equation $(x_0^3 + x_1^3 + x_2^3 - 3mx_0x_1x_2)^2 = 0$ passing through $\bar{x}$.

Generically $\mathrm{rk}(M) = 3$. In order to have also solutions representing irreducible curves, we need $\mathrm{rk}(M) \leq 2$. Some easy computations show that the ideal I of 3×3-minors of M is generated by the three polynomials

$$\bar{x}_0^2\bar{x}_1^2\bar{x}_2^2 q(\bar{x}), \quad \bar{x}_0\bar{x}_1\bar{x}_2(\bar{x}_0^3 + \bar{x}_1^3 + \bar{x}_2^3)q(\bar{x}), \quad (\bar{x}_0^3 + \bar{x}_1^3 + \bar{x}_2^3)^2 q(\bar{x}),$$

where $q(\bar{x}) := (\bar{x}_0^3 - \bar{x}_1^3)(\bar{x}_0^3 - \bar{x}_2^3)(\bar{x}_1^3 - \bar{x}_2^3)$. It follows that $I \subseteq (q)$.

On the other hand if $\bar{x} \in \mathbb{P}^2_{\mathbb{C}}$ is singular on C but $q(\bar{x}) \neq 0$, then $\bar{x}_0\bar{x}_1\bar{x}_2 = 0$. If $\bar{x}_0 = 0$, we get $\bar{x}_1^3 + \bar{x}_2^3 = 0$. Assuming $\bar{x}_1 = 1$ and $\bar{x}_2 = -1$, then 2.6.2 becomes $2a - b = 0$. If $a = 0$ then C is reducible. Assume $a = 1$, then 2.2 becomes

$$(x_0^3 + x_1^3 + x_2^3)^2 + cx_0x_1x_2(x_0^3 + x_1^3 + x_2^3) + dx_0^2x_1^2x_2^2 = 0,$$

which is reducible too.

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We have proved that $\bar{x} \in \mathrm{Sing}(C_\varrho)$ if and only if $q(\bar{x}) = 0$. Notice that $q(x) = 0$ is the equation of the union of the cuspidal tangent lines at B_σ and it is easy to check that each cuspidal tangent line contains a singular point of C_ϱ. □

3. The sheaves $\mathcal{E}$ and $\mathcal{F}$

In this section we prove Theorem 0.4. Let $(A, |D|)$ be a $(1,3)$-polarized abelian surface satisfying (♡). One has for $n \geq 1$

$$h^i(A, \mathcal{O}_A(nD)) = \begin{cases} 3n^2 & \text{if } i = 0, \\ 0 & \text{if } i = 1, 2. \end{cases} \tag{3.1}$$

Since $K_A \sim 0$ then, by adjuction, $p_a(D) = 4$. If $C \in |D|$ is smooth then the map $\varrho_{|C}: C \to \mathbb{P}^1_{\mathbb{C}}$ is branched at 18 points hence ϱ is branched along a curve of degree 18. Finally

$$\omega_{A|\mathbb{P}^2_{\mathbb{C}}} \cong \omega_{A|\mathbb{C}} \otimes \varrho^* \omega^{-1}_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}} \cong \varrho^* \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(3) \cong \mathcal{O}_A(3D).$$

As usual one has the isomorphisms 0.2. Since

$$h^i(A, \mathcal{O}_A(nD)) = h^i(A, \varrho^*\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(n)) = h^i(\mathbb{P}^2_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(n)) + h^i(\mathbb{P}^2_{\mathbb{C}}, \check{\mathcal{E}}(n)),$$

using 3.1, Bott's formulas and Serre duality one easily checks that

$$h^i(\mathbb{P}^2_{\mathbb{C}}, \check{\mathcal{E}}(n)) = h^i(\mathbb{P}^2_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(n-3)) + 2h^i(\mathbb{P}^2_{\mathbb{C}}, \Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(n)), \tag{3.2}$$

for every $n \in \mathbb{Z}$ and $i = 0, 1, 2$.

Lemma 3.3. *Let $\mathcal{H}$ be a locally free $\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}$-sheaf of rank 4 such that $h^i(\mathbb{P}^2_{\mathbb{C}}, \mathcal{H}(p)) = 2h^i(\mathbb{P}^2_{\mathbb{C}}, \Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(p))$ for $i = 0, 1, 2$ and $p \in \mathbb{Z}$. Then $\mathcal{H} \cong \Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}} \oplus \Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}$.*

Proof. The only non-zero terms in the Beilinson's spectral sequence (see **[16]**) are $E_1^{-2,2} \cong \mathcal{O}^{\oplus 2}_{\mathbb{P}^2_{\mathbb{C}}}$ and $E_1^{0,1} \cong \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(-1)^{\oplus 6}$. It follows $E_1^{0,1} \cong E_2^{0,1}$ and $E_1^{-2,2} \cong E_2^{-2,2}$, hence a complex

$$0 \to E_2^{-2,2} \xrightarrow{d_2^{-2,2}} E_2^{0,1} \to 0 \tag{3.3.1}$$

is defined. Moreover $E_r^{0,1} \cong E_3^{0,1}$ and $E_r^{-2,2} \cong E_3^{-2,2}$ for any $r \geq 3$. Since $E_\infty^{0,1} = 0$ then $d_2^{-2,2}$ is surjective. On the other hand $E_\infty^{-2,2} \cong \mathcal{H}$ hence the complex 3.3.1 yields the following exact sequence

$$0 \to \mathcal{H} \to \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(-1)^{\oplus 6} \xrightarrow{s} \mathcal{O}^{\oplus 2}_{\mathbb{P}^2_{\mathbb{C}}} \to 0.$$

The matrix S of s is of the form

$$S = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ b_0 & b_1 & b_2 & b_3 & b_4 & b_5 \end{pmatrix}$$

where $a_i, b_i \in H^0(\mathbb{P}^2_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(1))$. Since $\mathrm{rk}(S) = 2$ then $H^0(\mathbb{P}^2_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(1))$ is generated by the a_i's, otherwise there exists a point $x \in \mathbb{P}^2_{\mathbb{C}}$ such that $\mathrm{rk}(s_x) \leq 1$. In particular, up to a proper choice of a basis of $\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(-1)^{\oplus 6}$ which corresponds to a proper sequence of elementary operations on the columns of S, one can assume that

$$S = \begin{pmatrix} x_0 & x_1 & x_2 & 0 & 0 & 0 \\ b_0 & b_1 & b_2 & b_3 & b_4 & b_5 \end{pmatrix}.$$

If b_3, b_4, b_5 were linearly dependent then $\mathcal{H}$ would contain $\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(-1)$ as direct summand. Hence

$$0 \neq h^0(\mathbb{P}^2_{\mathbb{C}}, \mathcal{H}(1)) = 2h^0(\mathbb{P}^2_{\mathbb{C}}, \Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(1)) = 0.$$

We conclude that, up to a proper choice of a basis of $\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(-1)^{\oplus 6}$, one gets

$$S = \begin{pmatrix} x_0 & x_1 & x_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_0 & x_1 & x_2 \end{pmatrix}$$

hence $\mathcal{H} \cong \Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}} \oplus \Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}$. □

Proof of isomorphism 0.4.1. The isomorphism

$$\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}} \oplus \mathcal{E} \cong \varrho_* \omega_{A|\mathbb{P}^2_{\mathbb{C}}} \cong \varrho_* \varrho^* \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(3) \cong \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(3) \oplus \check{\mathcal{E}}(3),$$

gives rise to a factorization of the identity on $\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}$ as

$$\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}} \xrightarrow{i} \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(-3) \oplus \mathcal{E}(-3) \xrightarrow{p} \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}.$$

Since $h^0(\mathbb{P}^2_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(-3)) = 0$ one can split both i and p through $\mathcal{E}(-3)$ hence $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(3) \oplus \mathcal{E}_0$. Thus identities 3.2 imply that

$$h^i(\mathbb{P}^2_{\mathbb{C}}, \check{\mathcal{E}}_0(n)) = 2h^i(\mathbb{P}^2_{\mathbb{C}}, \Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(n)), \quad i = 0, 1, 2, n \in \mathbb{Z}.$$

It follows from Lemma 3.3 that $\check{\mathcal{E}}_0 \cong \Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}} \oplus \Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}$. □

Proof of isomorphism 0.4.2. Consider the exact sequence 0.3. Since $\omega_{A|\mathbb{C}} \cong \mathcal{O}_A$ then $\omega_{A|\mathbb{P}^2_{\mathbb{C}}} \cong \varrho^* \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(3)$. Thus, taking into account the decomposition of $\mathcal{E}$ and the natural splitting $\Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}} \otimes \Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}} \cong S^2\Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}} \oplus \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(-3)$, then we can identify $\varphi: S^2\mathcal{E} \to \varrho_* \omega^2_{A|\mathbb{P}^2_{\mathbb{C}}}$ (see sequence 0.3) with

$$\varphi: \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(3) \oplus \Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(6)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(6) \oplus S^2\Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(6)^{\oplus 3} \twoheadrightarrow \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(3) \oplus \Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(6)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(6).$$

We want to prove that φ has a section. To this purpose note that φ induces two morphisms

$$\varphi_1 \colon \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(3) \oplus \Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(6)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(6) \to \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(3) \oplus \Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(6)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(6),$$

$$\varphi_2 \colon \mathcal{S}^2\Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(6)^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(3) \oplus \Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(6)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(6).$$

There exists a natural monomorphism $i \colon \varrho_*\omega^2_{A|\mathbb{P}^2_{\mathbb{C}}} \to \mathcal{S}^2\mathcal{E}$ such that $\varphi \circ i = \varphi_1$. We now prove that φ_1 is an isomorphism. Let $r \subseteq \mathbb{P}^2_{\mathbb{C}}$ be a line. We claim that $\varphi_{1|r}$ is an isomorphism. If this is the case then

$$\det(\varphi_1) \in H^0(\mathbb{P}^2_{\mathbb{C}}, \det(\varrho_*\omega^2_{A|\mathbb{P}^2_{\mathbb{C}}}) \otimes \det(\varrho_*\omega^2_{A|\mathbb{P}^2_{\mathbb{C}}})^{-1}) \cong \mathbb{C}.$$

Since, by the claim, $\det(\varphi_{1|r}) \neq 0$ then $\det(\varphi_1) \neq 0$ too. Let $\psi := i \circ \varphi_1^{-1}$. ψ is a section of φ hence

$$\mathcal{F} \cong \mathcal{S}^2\mathcal{E}/\mathrm{im}(\psi) \cong (\mathcal{S}^2\Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(6))^{\oplus 3}.$$

Now we prove the claim. Assume that $C_r := \varrho^* r$ is smooth: set $\varrho_r := \varrho_{|r}$, $\mathbb{P}_r := \pi^{-1}(r)$, $\mathcal{E}_r := \mathcal{E}_{|r}$, fix an identification $\Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(6)_{|r} \cong \mathcal{O}_r(4) \oplus \mathcal{O}_r(5)$ and take non-zero sections

$$s, t \in H^0(\mathbb{P}_r, \mathcal{O}_{\mathbb{P}_r} \otimes \varrho_r^*\mathcal{O}_r(-1)) \cong H^0(r, \mathcal{E}_r(-1)),$$

$$v, w \in H^0(\mathbb{P}_r, \mathcal{O}_{\mathbb{P}_r} \otimes \varrho_r^*\mathcal{O}_r(-2)) \cong H^0(r, \mathcal{E}_r(-2)),$$

$$u \in H^0(\mathbb{P}_r, \mathcal{O}_{\mathbb{P}_r} \otimes \varrho_r^*\mathcal{O}_r(-3)) \cong H^0(r, \mathcal{E}_r(-3)).$$

The matrices M of $\varphi_{|r}$ and M_i of $\varphi_{i|r}$ satisfy $M = (M_1 \mid M_2)$. Moreover

$$M_1 = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ b_1 & c_{1,1} & c_{1,2} & 0 & 0 & 0 \\ b_2 & c_{2,1} & c_{2,2} & 0 & 0 & 0 \\ d_1 & e_{1,1} & e_{1,2} & f_{1,1} & f_{1,2} & 0 \\ d_2 & e_{2,1} & e_{2,2} & f_{2,1} & f_{2,2} & 0 \\ g & h_1 & h_2 & m_1 & m_2 & n \end{pmatrix}$$

whose elements have degrees

$$\begin{pmatrix} 0 & -1 & -1 & -2 & -2 & -3 \\ 1 & 0 & 0 & -1 & -1 & -2 \\ 1 & 0 & 0 & -1 & -1 & -2 \\ 2 & 1 & 1 & 0 & 0 & -1 \\ 2 & 1 & 1 & 0 & 0 & -1 \\ 3 & 2 & 2 & 1 & 1 & 0 \end{pmatrix}.$$

The first row of M is the matrix associated to $\vartheta_r: S^2\mathcal{E}_r \to \mathcal{O}_r(3)$.

It follows that if $a = 0$, then the first row of M would be 0 which is absurd since φ is surjective.

If $a \neq 0$ and $\mathrm{rk}(M_1) \leq 5$, then the system $Mx = 0$ has a solution $\bar{x} := (0, \alpha_2, \ldots, \alpha_6, 0, \ldots, 0) \neq 0$ hence

$$C_r \subseteq Q = V_+(u(\alpha_2 s + \alpha_3 t + \alpha_4 v + \alpha_5 w + \alpha_6 u)) \subseteq \mathbb{P}_r.$$

Since C_r is irreducible this is absurd.

Hence the claim is proved. □

Remark 3.4. It is not difficult to check that if a section $\eta \in H^0(\mathbb{P}^2_{\mathbb{C}}, \check{\mathcal{F}} \otimes S^2\mathcal{E})$ defines the smooth surface $A := D_0(\Phi_6(\eta)) \subseteq \mathbb{P}$ then A is an abelian surface and $\varrho := \pi_{|A}$ is a cover of degree 6.

Unfortunately, by dimensional reasons, the generic section η does not define a surface.

4. Biellipitic abelian surfaces in $\mathbb{P}(\mathcal{E})$

In this last section we characterize (1, 3)-polarized bielliptic abelian surfaces with respect to the behaviour of the embedding $i: A \hookrightarrow \mathbb{P}(\mathcal{E})$.

Let $(A, |D|)$ be a (1, 3)-polarized abelian surface satisfying condition (♡). Let $\varrho: A \to \mathbb{P}^2_{\mathbb{C}}$ be the corresponding cover of degree 6. It follows from the previous section and Theorem 2.1 of **[9]** applied to ϱ the existence of a unique embedding $i: A \hookrightarrow \mathbb{P}$ such that $\varrho = \pi \circ i$ and the scheme-theoretic fibre $A_y := \varrho^{-1}(y) \subseteq \mathbb{P}^4_{k(y)} \cong \mathbb{P}_y := \pi^{-1}(y)$ is an arithmetically Gorenstein subscheme.

Moreover such embedding is induced by the composition of $\varrho^*\mathcal{E} \hookrightarrow \varrho^*(\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}} \oplus \mathcal{E}) \xrightarrow{\sim} \varrho^*\varrho_*\omega_{A|\mathbb{P}^2_{\mathbb{C}}}$, see 0.2, followed by $\varrho^*\varrho_*\omega_{A|\mathbb{P}^2_{\mathbb{C}}} \to \omega_{A|\mathbb{P}^2_{\mathbb{C}}}$.

We fix a decomposition $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(3) \oplus \Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(3) \oplus \Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(3)$. The two projections (on the sum of the first two summands and on the third one), allow us to define two subbundles $\mathbb{P}(\Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(3)) \cong U \subseteq \mathbb{P}$ and $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(3) \oplus \Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(3)) \cong V \subseteq \mathbb{P}$. Let $\bar{\pi}: \mathbb{P} \dashrightarrow U$ be the projection from V.

Let S be the closure of $\bar{\pi}(A)$. There exists a dominant rational map $\tau: A \dashrightarrow S$ and we define $\sigma := \pi_{|S}$, so that $\varrho = \sigma \circ \tau$. Since $S \subseteq U$ is a divisor then it is locally Gorenstein, hence σ is a Gorenstein cover.

Since $\deg(\varrho) = 6$ then $\deg(\tau) = 1, 2, 3, 6$, and if $\deg(\tau) = 6$ then $\deg(\sigma) = 1$ and the map $\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}} \to \sigma_*\mathcal{O}_S$ is an isomorphism, thus the same is true for σ. If $x \in S$ is a general point then $\varrho^{-1}(\sigma(x)) = \tau^{-1}(x) \subseteq \langle x, V \cap \pi^{-1}(x) \rangle \cong \mathbb{P}^3_{k(x)} \subseteq \pi^{-1}(x)$, which is absurd since $\varrho^{-1}(\sigma(x)) \subseteq \pi^{-1}(x)$ is arithmetically Gorenstein (see **[9**, Theorem 2.1] and **[19**, Lemma 4.2]).

Proposition 4.1. *Let S be smooth and assume that τ is a morphism. Then A is bielliptic and the maps σ and τ coincide with the ones defined in Propositions* 1.1 *and* 1.5 *respectively.*

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Proof. Since $\deg(\varrho)=6$ then $\deg(\tau)=1,2,3,6$ and the case $\deg(\tau)=6$ is impossible as shown above. Moreover the smoothness of S yields that τ is actually a cover.

If $\deg(\tau)=1$ the map $\mathcal{O}_S\to\tau_*\mathcal{O}_A$ is an isomorphism, thus the same is true for τ. The surjective map $\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(1)^{\oplus 3}\twoheadrightarrow\Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(3)$, yields

$$A\subseteq\mathbb{P}(\Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(3))\subseteq\mathbb{P}(\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(1)^{\oplus 3})\cong X:=\mathbb{P}^2_{\mathbb{C}}\times\mathbb{P}^2_{\mathbb{C}}.$$

Let $p_i\colon X\to\mathbb{P}^2_{\mathbb{C}}$ be the projection onto the i-th factor and as usual set $\mathcal{O}_X(a,b):=p_1^*\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(a)\otimes p_2^*\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(b)$. If h_1 and h_2 are the classes of $\mathcal{O}_X(1,0)$ and $\mathcal{O}_X(0,1)$ respectively in the Chow ring $A(X)$, then there are $\alpha,\beta,\gamma\in\mathbb{Z}$ such that the class of A is $\alpha h_1^2+\beta h_2^2+\gamma h_1\cdot h_2$. It is proved in Section 2 of **[12]**, that $\alpha=6,\beta=0$, hence γ is a solution of $\gamma^2-9\gamma-18=0$ which has not integral solutions.

Thus $\deg(\tau)=2,3$. Assume that $\deg(\tau)=3$ and let $\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(n)$ and $\mathcal{F}$ be the Tschirnhausen modules of σ and τ respectively. Since, in this case, $B_\varrho=3B_\sigma+\sigma_*B_\tau$ then $\deg(B_\sigma)\le 6$, hence $n=-1,-2,-3$. From the isomorphisms

$$\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}\oplus\check{\mathcal{E}}\cong\varrho_*\mathcal{O}_A\cong\sigma_*\tau_*\mathcal{O}_A\cong\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}\oplus\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(n)\oplus\sigma_*\mathcal{F},$$

and formula 0.4.1, we obtain a factorization of the identity

$$\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(n)\rightarrowtail\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}\oplus\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(-3)\oplus\Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}\oplus\Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}\twoheadrightarrow\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(n).$$

On the other hand $h^0(\mathbb{P}^2_{\mathbb{C}},\Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(1))=0$, thus only the case $n=-3$ is possible. In this case S is a $K3$ surface hence τ is étale. Therefore $0=\chi(\mathcal{O}_A)=3\chi(\mathcal{O}_S)=6$ (see **[15]** or **[17]**).

Assume now that $\deg(\tau)=2$. Fix a line $\ell\in\mathbb{P}^2_{\mathbb{C}}$ such that both $E:=\sigma^{-1}(\ell)$ and $C:=\varrho^{-1}(\ell)$ are smooth. Let p be the geometric genus of E and define $t:=\tau_{|C}$, $s:=\Sigma_{|E}$, $r:=\varrho_{|C}$. The branch loci of r,s,t satisfy $B_r=2B_s+s_*B_t$ and $\deg(B_r)=18$. The formula of Hurwitz applied to s and t implies that either $\deg(B_s)=\deg(s_*B_t)=6$ and $p=1$ or $\deg(B_s)=8$, $\deg(s_*B_t)=2$ and $p=2$.

Since σ factors through $\mathbb{P}(\Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(3))$, then s factors through $\mathbb{P}(\mathcal{O}_\ell(1)\oplus\mathcal{O}_\ell(2))$. In any case the Tschirnhausen module of s, $\mathcal{E}_s$, is dual to $\mathcal{O}_\ell(1+h)\oplus\mathcal{O}_\ell(2+h)$ for some $h\in\mathbb{Z}$. Since $B_s\in|\det(\mathcal{E}_s)^{-2}|$ we get that $p=1$ and $h=0$. In particular the dual of the Tschirnhausen module of S is $\Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(3)$. It follows from Proposition 1.2 that S is ruled with invariant $e(S)=-1$ over an elliptic curve.

On the other hand if $\mathcal{L}\in Pic(S)$ is the Tschirnhausen module of τ the $\mathcal{O}_A\cong\omega_{A|\mathbb{C}}\cong\omega_{S|\mathbb{C}}\otimes\mathcal{L}$, thus τ is induced by a smooth and irreducible element of $|\omega_{S|\mathbb{C}}^{-2}|$. ☐

Conversely let A be bielliptic and let $\varrho\colon A\to\mathbb{P}^2_{\mathbb{C}}$ be the corresponding cover factorizing as $\pi\circ i$. The double cover τ factors through $A\hookrightarrow\mathbb{V}(\omega_{S|\mathbb{C}})\subseteq\mathbb{P}(\mathcal{O}_S\oplus\omega_{S|\mathbb{C}})$ followed by the projection onto S.

In order to simplify notations, we will set $\overline{\mathbb{P}}:=\mathbb{P}(\Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(3))$. Let $\mathcal{F}:=\mathcal{O}_{\overline{\mathbb{P}}}\oplus(\mathcal{O}_{\overline{\mathbb{P}}}(1)\otimes p^*\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(-3))$ and $q\colon\mathbb{F}:=\mathbb{P}(\mathcal{F})\to\overline{\mathbb{P}}$ be the projection. Since $\omega_{S|\mathbb{C}}\cong(\mathcal{O}_{\overline{\mathbb{P}}}(1)\otimes p^*\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(-3))_{|S}$ then $\mathbb{P}(\mathcal{O}_S\oplus\omega_{S|\mathbb{C}})\cong\mathbb{F}\times_{\overline{\mathbb{P}}}S$. Define $\mathcal{M}:=\mathcal{O}_{\mathbb{F}}(1)\otimes q^*p^*\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(3)$. The general morphism $q^*p^*\mathcal{E}\to\mathcal{M}$ is surjective, thus we get $f\colon\mathbb{F}\to\mathbb{P}$ inducing

$$u: U \cong \mathbb{P}(\mathcal{O}_{\bar{\mathbb{P}}}) \subseteq \mathbb{F} \to \mathbb{P},$$

$$f': \mathbb{P}(\mathcal{O}_{\bar{\mathbb{P}}}(1) \otimes p^*\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(-3)) \subseteq \mathbb{F} \to \mathbb{P}.$$

Fix $x \in \mathbb{P}^2_{\mathbb{C}}$. Then u embeds linearly $U_x := p^{-1}(x) \cong \mathbb{P}^1_{k(x)} \subseteq \mathbb{P}_x := \pi^{-1}(x) \cong \mathbb{P}^4_{k(x)}$ and f is the natural embedding $\mathbb{F}_x := (p \circ q)^{-1}(x) \cong \mathbb{F}_1 \subseteq \mathbb{P}_x \cong \mathbb{P}^4_{k(x)}$ as a cubic scroll. In particular f is actually an embedding. By construction $A_x := \varrho^{-1}(x)$ generates a subscheme $\Sigma_x \subseteq \mathbb{F}_x$ which is exactly the pull back of $S_x := \sigma^{-1}(x)$ via $q_{|\mathbb{F}_x} : \mathbb{F}_x \to U_x$.

Each subscheme $A' \subseteq A_x$ of degree at least 5 generates Σ_x. On the other hand each hyperplane $H \subseteq \mathbb{P}_x$ intersect all the fibres of $\mathbb{F}_x$ and $H \cap \mathbb{F}_x$ is a cubic curve, thus $\Sigma_x \not\subseteq H$. It follows that $A' \not\subseteq H$. Hence $A_x \subseteq \mathbb{P}_x$ is an arithmetically Gorenstein subscheme (see [**19**, Lemma 4.2]).

We then obtain that the induced embedding $i: A \hookrightarrow \mathbb{P}$ coincides with the embedding given by the canonical factorization of ϱ in the sense of Theorem 2.1 of [**9**].

In this case q is induced by the projection of $\mathbb{P}$ onto U from the subbundle V generated by $\operatorname{im} f'$. Necessarily there exists a locally free $\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}$-sheaf $\mathcal{G}$ of rank 3 such that $V \cong \mathbb{P}(\mathcal{G})$.

Since U and V generate fibrewise $\mathbb{P}$ and $U \cap V = \emptyset$, then $\mathcal{E} \cong \mathcal{G} \oplus \Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(3)$. Notice that such an isomorphism gives rise to a factorization

$$\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(3) \rightarrowtail \mathcal{G} \oplus \Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(3) \twoheadrightarrow \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(3)$$

of the identity on $\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}$. Since $h^0(\mathbb{P}^2_{\mathbb{C}}, \Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}) = 0$, one can split the above sequence through $\mathcal{G}$, hence $\mathcal{G} \cong \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(3) \oplus \mathcal{H}$. As in the proof of Proposition 1.1 one easily checks that $\mathcal{H} \cong \Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(3)$.

Thus we have proved the following converse of Proposition 4.1.

Proposition 4.2. *If A is bielliptic, then there are subbundles $\mathbb{P}(\Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(3)) \cong U \subseteq \mathbb{P}$ and $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(3) \oplus \Omega^1_{\mathbb{P}^2_{\mathbb{C}}|\mathbb{C}}(3)) \cong V \subseteq \mathbb{P}$ such that*

(i) *$A \cap V = U \cap V = \emptyset$;*

(ii) *let $\bar{\pi}: \mathbb{P} \dashrightarrow U$ be the projection from V, and identify S with its image inside U: then $S = \bar{\pi}(A)$, $\tau = \bar{\pi}_{|A}$ and $\sigma = \bar{\pi}_{|S}$.*

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Dipartimento di Matematica Pura ed Applicata
Università degli Studi di Padova
via Belzoni 7
I-35131 Padova
Italy
E-mail address: casnati@math.unipd.it