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CONSTRUCTION OF LIE SUPERALGEBRAS $D(2,1;\alpha)$, $G(3)$ AND $F(4)$ FROM SOME TRIPLE SYSTEMS

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Abstract We have constructed Lie superalgebras $D(2,1;\alpha)$, $G(3)$ and $F(4)$ from $U(-1,-1)$-balanced Freudenthal–Kantor triple systems in a natural way.

Keywords: Lie superalgebras; triple system; octonions

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1. Introduction

Lie algebras and Lie superalgebras play important roles in many mathematical and physical subjects, e.g. differential geometry, Yang–Baxter equations, etc.

It is well known that a characterization of these algebras can be given in terms of the root system and Cartan matrix. However, there also exist other ways of constructing them as follows. First, Benkart and Zelmanov [**3**] and Benkart and Elduque [**2**] extended the classical Tits method (e.g. [**17**]) for constructions of exceptional Lie superalgebras $D(2,1;\alpha)$, $G(3)$ and $F(4)$ in the Kac classification [**7**]. On the other hand, it is known (see [**8**, **11**, **12**, **20**] and references cited therein) that both Lie algebras and Lie superalgebras can also be constructed from so-called (ϵ,δ) Jordan triple systems as well as from more general $U(\epsilon,\delta)$ Freudenthal–Kantor triple systems, where ϵ and δ assume values of $+1$ and -1. In particular, all finite-dimensional simple Lie algebras have been constructed in this way (see [**1**, **9**]).

In this paper, we will show that we can also construct, in a very natural way, exceptional Lie superalgebras $D(2,1;\alpha)$, $G(3)$ and $F(4)$ from a restricted class of $U(-1,-1)$-balanced Freudenthal–Kantor triple systems (hereafter abbreviated as $U(-1,-1)$-BFKTSs). We note that the results of [**1**, **9**] mentioned above are, in contrast, based essentially upon $U(1,1)$-BFKTSs, which are analogues of $U(-1,-1)$-BFKTSs.

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Our paper is organized as follows. We introduce the notion of the $U(-1,-1)$-BFKTS in § 2 and give constructions of exceptional Lie superalgebras $D(2,1;\alpha)$, $G(3)$ and $F(4)$ in § 3.

2. $U(-1,-1)$-balanced Freudenthal–Kantor triple systems

Let V be a finite-dimensional vector space with a symmetric bilinear non-degenerate form $\langle\cdot|\cdot\rangle$ over the field F of characteristic not 2. Suppose that the triple product

$$V \otimes V \otimes V \to V$$

denoted by the juxtaposition xyz for $x, y, z \in V$ satisfies

$$xxy = \langle x|x\rangle y = xyx, \tag{2.1 a}$$

$$uv(xyz) = (uvx)yz - x(vuy)z + xy(uvz). \tag{2.1 b}$$

We say that the triple system is a $U(-1,-1)$-BFKTS. Note that the present definition for $\langle\cdot|\cdot\rangle$ differs by a factor of 2 from the one given in [**11**].

The $U(-1,-1)$-BFKTS is intimately related to the orthogonal triple system introduced in [**13**] as follows. In the same vector space V, we introduce the second triple product by

$$x \cdot y \cdot z \doteq xyz - \langle x|y\rangle z, \tag{2.2}$$

which will then satisfy

$$x \cdot y \cdot z = -y \cdot x \cdot z, \tag{2.3 a}$$

$$x \cdot y \cdot z + x \cdot z \cdot y = -2\langle y|z\rangle x + \langle x|z\rangle y + \langle x|y\rangle z, \tag{2.3 b}$$

$$u \cdot v \cdot (x \cdot y \cdot z) = (u \cdot v \cdot x) \cdot y \cdot z + x \cdot (u \cdot v \cdot y) \cdot z + x \cdot y \cdot (u \cdot v \cdot z). \tag{2.3 c}$$

Conversely, Equations (2.3) with Equation (2.2) imply Equations (2.1). The left multiplication operator $\ell(x,y)$ defined by

$$\ell(x,y)z \doteq x \cdot y \cdot z = xyz - \langle x|y\rangle z \tag{2.4}$$

is then a derivation of $x \cdot y \cdot z$ with Lie relation

$$\ell(x,y) = -\ell(y,x), \tag{2.5 a}$$

$$[\ell(u,v), \ell(x,y)] = \ell(\ell(u,v)x, y) + \ell(x, \ell(u,v)y). \tag{2.5 b}$$

As we will see shortly, this will play a significant role in our construction of Lie superalgebras $D(2,1;\alpha)$, $G(3)$ and $F(4)$.

The reason for considering the $U(-1,-1)$-BFKTS is as follows. We first set

$$W = V \oplus V \tag{2.6}$$

and define a new triple product in W by

$$\left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right] = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in W, \tag{2.7 a}$$

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where

$$w_1 = x_1 y_2 z_1 + y_1 x_2 z_1 - 2\langle x_1|y_1\rangle z_2, \tag{2.7 b}$$
$$w_2 = 2\langle x_2|y_2\rangle z_1 - y_2 x_1 z_2 - x_2 y_1 z_2, \tag{2.7 c}$$

for $x_j, y_j, z_j \in V$ $(j = 1, 2)$ with

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in W,$$

etc. As a special case of a more general $U(\epsilon, \delta)$ Freudenthal–Kantor supertriple system (see, for example, [**8**, **11**, **20**]), we have

$$[X, Y, Z] = [Y, X, Z], \tag{2.8 a}$$
$$[X, Y, Z] + [Y, Z, X] + [Z, X, Y] = 0, \tag{2.8 b}$$
$$[U, V, [X, Y, Z]] = [[U, V, X], Y, Z] + [X, [U, V, Y], Z] + [X, Y, [U, V, Z]] \tag{2.8 c}$$

for $X, Y, Z, U, V \in W$. We can also readily verify the validity of Equations (2.8) directly from Equations (2.1) and (2.7). In other words, W is an anti-Lie triple system [**5**] or a special Lie supertriple system in which the even subspace $W_{\bar{0}}$ of W is set to be null with $W = W_{\bar{1}}$ consisting only of the odd parity space. In any case, we can now construct a Lie superalgebra from W canonically as follows. First we introduce the left multiplication operator as

$$L(X, Y)Z \doteq [X, Y, Z], \tag{2.9}$$

which satisfies Lie algebra relations

$$L(X, Y) = L(Y, X), \tag{2.10 a}$$
$$[L(U, V), L(X, Y)] = L(L(U, V)X, Y) + L(X, L(U, V)Y). \tag{2.10 b}$$

We now introduce a superspace L by

$$L = L_{\bar{0}} \oplus L_{\bar{1}}, \tag{2.11 a}$$
$$L_{\bar{0}} \doteq L(W, W), \tag{2.11 b}$$
$$L_{\bar{1}} \doteq W = V \oplus V, \tag{2.11 c}$$

and define Lie supercommutators in L by Equation (2.10 *b*) and

$$[X, Y] \doteq L(X, Y), \tag{2.12 a}$$
$$[L(X, Y), Z] \doteq -[Z, L(X, Y)] \doteq [X, Y, Z] \tag{2.12 b}$$

for $X, Y, Z \in W$. L is then a Lie superalgebra (e.g. [**8**, **16**, **20**]).

As we will demonstrate in the next section, this construction gives Lie superalgebras $D(2, 1; \alpha)$, $G(3)$ and $F(4)$ naturally from orthogonal triple systems found in [**13**]. To this

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end, we will study the structure of the Lie superalgebra L in some detail. First of all,

$$\operatorname{Dim} L_{\bar{1}} = 2 \operatorname{Dim} V, \tag{2.13}$$

is obvious, although $\operatorname{Dim} L_{\bar{0}}$ depends upon the nature of the underlying $U(-1,-1)$-BFKTS. Next, from Equations (2.7), we find

$$L\left(\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} y \\ 0 \end{pmatrix}\right) = -2\langle x|y\rangle Q, \tag{2.14 a}$$

$$L\left(\begin{pmatrix} 0 \\ x \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix}\right) = +2\langle x|y\rangle \bar{Q}, \tag{2.14 b}$$

$$L\left(\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix}\right) = L\left(\begin{pmatrix} 0 \\ y \end{pmatrix}, \begin{pmatrix} x \\ 0 \end{pmatrix}\right) = \langle x|y\rangle K + \Sigma(x,y), \tag{2.14 c}$$

where we have set

$$Q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \bar{Q} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.15 a}$$

$$\Sigma(x,y) \doteq \begin{pmatrix} \ell(x,y) & 0 \\ 0 & \ell(x,y) \end{pmatrix}. \tag{2.15 b}$$

Here, $\ell(x,y)$ is defined by Equation (2.4). Noting that Q, $\bar{Q}$ and K satisfy the $su(2)$ relations

$$[K,Q] = 2Q, \qquad [K,\bar{Q}] = -2\bar{Q}, \qquad [Q,\bar{Q}] = K, \tag{2.16}$$

and that they commute with $\Sigma(x,y)$, we see that the Lie algebra $L_{\bar{0}}$ is isomorphic to

$$L_{\bar{0}} = su(2) \oplus g_0, \tag{2.17}$$

where g_0 is the Lie algebra specified by Equations (2.5). The odd parity part $L_{\bar{1}}$ of L is a $L_{\bar{0}}$ module with

$$\left[L\left(\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} y \\ 0 \end{pmatrix}\right), \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right] = -2\langle x|y\rangle \begin{pmatrix} z_2 \\ 0 \end{pmatrix}, \tag{2.18 a}$$

$$\left[L\left(\begin{pmatrix} 0 \\ x \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix}\right), \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right] = 2\langle x|y\rangle \begin{pmatrix} 0 \\ z_1 \end{pmatrix}, \tag{2.18 b}$$

$$\left[L\left(\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix}\right), \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right] = \langle x|y\rangle \begin{pmatrix} z_1 \\ -z_2 \end{pmatrix} + \begin{pmatrix} \ell(x,y)z_1 \\ \ell(x,y)z_2 \end{pmatrix}. \tag{2.18 c}$$

Therefore, the problem is reduced to determining the Lie algebra g_0 and its action on V.

Concluding this section, we note the following simplicity criteria for L due to Kamiya [**10**]. Let us introduce the left and right multiplication operators in V by

$$L(x,y)z \doteq xyz, \qquad R(x,y)z \doteq zxy, \tag{2.19}$$

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and set

$$\gamma(x,y) = \mathrm{Tr}(R(x,y) + R(y,x) + \tfrac{1}{2}(L(x,y) + L(y,x))). \tag{2.20}$$

We then have the following proposition [**10**].

Proposition 2.1.

(i) $\gamma(x,y) = (4 - \mathrm{Dim}\, V)\langle x|y\rangle$.

(ii) *The resulting Lie superalgebra L is simple if $\gamma(x,y)$ is non-degenerate.*

Remark 2.2. From Proposition 2.1 (i), we see that the case of $\mathrm{Dim}\, V = 4$ is special. As we will see in the next section, we may indeed have non-simple Lie superalgebras for $\mathrm{Dim}\, V = 4$ in some cases.

In [**10**], an attempt has been made to find $U(-1,-1)$-BFKTS which may lead to Lie superalgebras $G(3)$ and $F(4)$. This paper offers such a construction.

3. Constructions of $D(2,1;\alpha)$, $G(3)$ and $F(4)$

Let V again be the vector space with a symmetric bilinear non-degenerate form $\langle\cdot|\cdot\rangle$ over a field F of characteristic not 2, and set

$$N = \mathrm{Dim}\, V. \tag{3.1}$$

A triple product given by

$$xyz \doteq -\langle y|z\rangle x + \langle x|z\rangle y + \langle x|y\rangle z \tag{3.2}$$

is then a $U(-1,-1)$-BFKTS, as has already been noted in [**11**]. Since this $U(-1,-1)$-BFKTS will play some role in our construction, it will be instructive to study its resulting Lie superalgebra L in some detail. First, we see immediately that

$$\ell(x,y)z = \langle x|z\rangle y - \langle y|z\rangle x,$$

so that Equation (2.5 b) gives the $so(N)$ Lie algebra relation

$$[\ell(u,v), \ell(x,y)] = \langle u|x\rangle \ell(v,y) - \langle v|x\rangle \ell(u,y) + \langle u|y\rangle \ell(x,v) - \langle v|y\rangle \ell(x,u).$$

It is then easy to see that L is the Lie superalgebra $osp(N,2)$ in the standard notation [**6**, **18**].

In [**13**] we considered a triple product $[x,y,z]$ in V, satisfying

$$[x,y,z] \text{ is totally antisymmetric in } x,y,z, \tag{3.3 a}$$

$$\langle w|[x,y,z]\rangle \text{ is totally antisymmetric in } w,x,y,z, \tag{3.3 b}$$

$$\langle [x,y,z]|[u,v,w]\rangle = \sum_P (-1)^P \langle x|u\rangle\langle y|v\rangle\langle z|w\rangle + \tfrac{1}{4}\beta \sum_P \sum_{P'} (-1)^P (-1)^{P'} \langle x|u\rangle \langle y|[z,v,w]\rangle, \tag{3.3 c}$$

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for a constant $\beta \in F$, where P and P' refer to 3! permutations of x, y, z and of u, v, w, respectively. In view of Equation (3.3 b), we can rewrite Equation (3.3 c) as

$$\begin{aligned}[u,v,[x,y,z]] = {} & \{\langle y|v\rangle\langle z|u\rangle - \langle y|u\rangle\langle z|v\rangle - \beta\langle u|[v,y,z]\rangle\}x \\ & + \{\langle z|v\rangle\langle x|u\rangle - \langle z|u\rangle\langle x|v\rangle - \beta\langle u|[v,z,x]\rangle\}y \\ & + \{\langle x|v\rangle\langle y|u\rangle - \langle x|u\rangle\langle y|v\rangle - \beta\langle u|[v,x,y]\rangle\}z \\ & - \beta\{\langle x|v\rangle[u,y,z] + \langle y|v\rangle[u,z,x] + \langle z|v\rangle[u,x,y] \\ & \quad + \langle x|u\rangle[v,z,y] + \langle y|u\rangle[v,x,z] + \langle z|u\rangle[v,y,x]\}. \end{aligned} \tag{3.4}$$

We remark that we have set $\alpha = 1$ here for the parameter α in [**13**] without loss of generality. Also, this triple system is intimately related to the one studied by Shaw [**19**] as has been noted in [**4**].

It has been proved in [**13**] (see also [**4**]) that Equations (3.3), (3.4) are only possible for

$$N = 8 \quad \text{and} \quad \beta = \pm 1, \tag{3.5 a}$$

$$N = 4 \quad \text{and} \quad \beta = 0. \tag{3.5 b}$$

Moreover, for the case of $N = 8$ a bilinear product in V defined by

$$x \cdot y \doteq [x,y,e] + \langle x|e\rangle y + \langle y|e\rangle x - \langle x|y\rangle e \tag{3.6}$$

for any $e \in V$ satisfying $\langle e|e\rangle = 1$ defines an octonion algebra with e as its unit element. Conversely, the original triple product can be expressed as

$$[x,y,z] = -(x \cdot \bar{y}) \cdot z + \langle y|z\rangle x - \langle x|z\rangle y + \langle x|y\rangle z \quad (\beta = -1), \tag{3.7 a}$$

$$[x,y,z] = -x \cdot (\bar{y} \cdot z) + \langle y|z\rangle x - \langle x|z\rangle y + \langle x|y\rangle z \quad (\beta = +1), \tag{3.7 b}$$

in terms of the octonionic bilinear products. Also, Equations (3.6) and (3.7) still hold for the case of $N = 4$ and $\beta = 0$ with $x \cdot y$ now being the associative quaternion product, although we will not go into detail.

In [**13**], we have constructed three orthogonal triple systems which can be converted into $U(-1,-1)$-BFKTSs by Equation (2.2), when the triple product is renormalized suitably. In this way, we obtain the following proposition.

Proposition 3.1. *The following are $U(-1,-1)$-BFKTSs.*

(i) $N = 8$ *with* $\beta = \pm 1$,

$$xyz \doteq \tfrac{1}{3}\beta[x,y,z] - \langle y|z\rangle x + \langle x|z\rangle y + \langle x|y\rangle z, \tag{3.8}$$

where we assumed that the underlying field F is of characteristic not 3.

(ii) $N = 4$ *with* $\beta = 0$,

$$xyz \doteq \sigma[x,y,z] - \langle y|z\rangle x + \langle x|z\rangle y + \langle x|y\rangle z \tag{3.9}$$

for arbitrary $\sigma \in F$.

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(iii) $N = 7$.

Let A be an octonion algebra with the bilinear product $x \cdot y$ and set

$$V = \{x \mid \langle e|x\rangle = 0,\ x \in A\} \tag{3.10 a}$$

so that V with $\operatorname{Dim} V = 7$ is essentially the seven-dimensional exceptional Malcev algebra. We introduce a triple product in V by

$$\begin{aligned} xyz &= -\tfrac{3}{4}(x,y,z) + \tfrac{1}{4}[[x,y],z] + \langle x|y\rangle z \\ &= -\tfrac{1}{4}(x,y,z) - \langle y|z\rangle x + \langle x|z\rangle y + \langle x|y\rangle z, \end{aligned} \tag{3.10 b}$$

where $(x,y,z) = (x\cdot y)\cdot z - x\cdot(y\cdot z)$ is the associator in A. Note that $\langle e|xyz\rangle = 0$ for $x, y, z \in V$.

From Equations (3.8)–(3.10) we see that the parts of xyz which are independent of $[x,y,z]$ and (x,y,z) are precisely the $U(-1,-1)$-BFKTSs given by Equation (3.2).

Remark 3.2. The reason why we can have arbitrary $\sigma \in F$ for the case of $N = 4$ is due to the validity of the identities

$$\begin{aligned} &\langle w|x\rangle\langle y|[z,u,v]\rangle + \langle w|y\rangle\langle z|[u,v,x]\rangle \\ &\qquad + \langle w|z\rangle\langle u|[v,x,y]\rangle + \langle w|u\rangle\langle v|[x,y,z]\rangle + \langle w|v\rangle\langle x|[y,z,u]\rangle = 0 \end{aligned} \tag{3.11 a}$$

and

$$\langle u|[v,x,y]\rangle w = \langle w|x\rangle[y,u,v] + \langle w|u\rangle[x,y,v] + \langle w|v\rangle[y,x,u] - \langle w|y\rangle[u,v,x] \tag{3.11 b}$$

for $N = 4$, as has already been remarked on in [**13**]. For example, the left-hand side of Equation (3.11 a) is totally antisymmetric in five variables (x, y, z, u and v), so it must be identically zero for $N = 4$.

Remark 3.3. If the underlying field F is of characteristic 3 for the case of $N = 8$, then $[x,y,z]$ represents a Lie triple system.

We will now show that the three cases in Proposition 3.1 will lead to Lie superalgebras $F(4)$, $D(2,1;\alpha)$ ($\alpha \neq 0, \infty$) and $G(3)$, respectively, by first showing that the Lie algebras g_0 in Equation (2.17) are $so(7)$, $su(2) \oplus su(2)$ and G_2, respectively. We discuss these cases separately below.

Case 1. $N = 4$ and $\beta = 0$.

Although we can express $[x,y,z]$ in terms of the quaternion algebra by Equation (3.7), it is more instructive to proceed as follows. We assume for simplicity that the underlying field F is algebraically closed. Let e_1, e_2, e_3, e_4 be a basis of V with

$$\langle e_\mu|e_\nu\rangle = \delta_{\mu\nu} \quad (\mu,\nu = 1,2,3,4). \tag{3.12}$$

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We can then construct the triple product by

$$[e_\mu, e_\nu, e_\lambda] = \sum_{\alpha=1}^{4} \epsilon_{\mu\nu\lambda\alpha} e_\alpha, \tag{3.13}$$

where $\epsilon_{\mu\nu\lambda\alpha}$ is the four-dimensional totally antisymmetric Levi-Civita symbol with $\epsilon_{1234} = 1$. We can then readily verify the validity of Equations (3.3), (3.4) with $\beta = 0$ (see, for example, [**14**]).

Setting

$$\tilde{L}_1 = \ell(e_2, e_3), \qquad \tilde{L}_2 = \ell(e_3, e_1), \qquad \tilde{L}_3 = \ell(e_1, e_2), \tag{3.14 a}$$

$$\tilde{M}_1 = \ell(e_1, e_4), \qquad \tilde{M}_2 = \ell(e_2, e_4), \qquad \tilde{M}_3 = \ell(e_3, e_4), \tag{3.14 b}$$

Equation (2.5 b) then leads to

$$[\tilde{L}_i, \tilde{L}_j] = \sum_{k=1}^{3} \epsilon_{ijk}(\tilde{L}_k + \sigma \tilde{M}_k), \tag{3.15 a}$$

$$[\tilde{L}_i, \tilde{M}_j] = \sum_{k=1}^{3} \epsilon_{ijk}(\tilde{M}_k + \sigma \tilde{L}_k), \tag{3.15 b}$$

$$[\tilde{M}_i, \tilde{M}_j] = \sum_{k=1}^{3} \epsilon_{ijk}(\tilde{L}_k + \sigma \tilde{M}_k) \tag{3.15 c}$$

for $i, j, k = 1, 2, 3$, where ϵ_{ijk} is now the totally antisymmetric Levi-Civita symbol in three dimensions. Suppose $\sigma \neq \pm 1$. Then, if we set

$$L_i^{(\pm)} = \frac{1}{2(1 \pm \sigma)}(\tilde{L}_i \pm \tilde{M}_i) \quad (i = 1, 2, 3), \tag{3.16}$$

Equations (3.15) can be rewritten as the $su(2) \oplus su(2)$ relations

$$[L_i^{(\pm)}, L_j^{(\pm)}] = \sum_{k=1}^{3} \epsilon_{ijk} L_k^{(\pm)}, \tag{3.17 a}$$

$$[L_i^{(+)}, L_j^{(-)}] = 0. \tag{3.17 b}$$

In particular, $L_{\bar{0}}$ is isomorphic to $su(2) \oplus su(2) \oplus su(2)$. Studying the action of $L_{\bar{0}}$ on $L_{\bar{1}}$, we see that the Lie superalgebra L is $\Gamma(1+\sigma, 1-\sigma, -2)$ in the notation of Scheunert [**18**]. In the Kac notation this corresponds to $D(2, 1; \alpha)$ with

$$\alpha = \frac{1 - \sigma}{1 + \sigma}. \tag{3.18}$$

We must have $\alpha \neq 0$, and $\alpha \neq \infty$ since $\sigma \neq \pm 1$. Our construction also effectively reproduces the explicit realization given in [**6**]. Note also that for $\sigma = 0$ we have $\Gamma(1, 1, -2) \simeq osp(4, 2)$ (see [**18**]) in agreement with the result stated in the beginning of this section for $N = 4$, since Equation (3.9) for $\sigma = 0$ is nothing but Equation (3.2).

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However, the case of $\sigma = \pm 1$ leads to an entirely different situation. Suppose $\sigma = +1$. We can still define $L_i^{(+)}$ by Equation (3.10) but *not* $L_i^{(-)}$, and $L_i^{(+)}$ satisfies the same $su(2)$ relation of Equation (3.17 *a*). But we can easily see that

$$[\tilde{L}_i - \tilde{M}_i, \tilde{L}_j - \tilde{M}_j] = 0 = [\tilde{L}_i - \tilde{M}_i, L_j^{(+)}]. \tag{3.19}$$

Therefore, the Lie algebra $L_{\bar{0}}$ is now given as

$$L_{\bar{0}} = su(2) \oplus su(2) \oplus u(1) \oplus u(1) \oplus u(1), \tag{3.20}$$

with the abelian part $u(1) \oplus u(1) \oplus u(1)$ being an ideal of the Lie superalgebra L. Hence, L is no longer simple. However, the quotient Lie superalgebra $L/u(1) \oplus u(1) \oplus u(1)$ is isomorphic to the Lie superalgebra $sp\ell(2,2)/FI_4$ in the notation of [**18**].

Case 2. $N = 7$.

Although the Lie algebra g_0 must be $G(2)$ by its construction given in [**13**], we will also demonstrate it below.

Let L_x be the left multiplication operator in the octonion algebra A by $L_x y \doteq xy$ for $x \in V$ but $y \in A$. Then, $\ell(x,y)$ given in Equation (2.4) is essentially rewritten as

$$\ell(x,y) = -\tfrac{3}{4}\{L_{xy} - L_x L_y\} + \tfrac{1}{4} ad[x,y] \tag{3.21}$$

for $x, y \in V$ with $ad_x y = [x,y]$ since $\ell(x,y)e = 0$. Using alternative properties of A (see, for example, [**14**]), we can rewrite Equation (3.21) as

$$\ell(x,y) = \tfrac{1}{8}\{ad_{[x,y]} + [ad_x, ad_y]\} = \tfrac{1}{4}\{ad_{[x,y]} - 3[L_x, R_y]\},$$

which is the standard derivation operator of the octonion algebra [**17**]. Therefore, the Lie algebra g_0 is G_2 which acts on seven-dimensional V. The resulting Lie superalgebra L is then $G(3)$ in the Kac notation.

Case 3. $N = 8$ and $\beta = \pm 1$.

Without loss of generality, we can set $\beta = -1$ by changing the sign of $[x,y,z]$ if necessary, and we can prove g_0 to be $so(7)$ as follows. We will again assume for simplicity that F is algebraically closed, and consider the Clifford algebra $C(7,0)$ in seven-dimensional carrier space with the defining relation

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu} \quad (\mu, \nu = 1, 2, \dots, 7). \tag{3.22}$$

This admits an eight-dimensional irreducible representation space V. Let $x, y, z \in V$ be eight-dimensional spinors on which 8×8 matrices γ_μ act. We then note that

$$J_{\mu\nu} = \tfrac{1}{4}[\gamma_\mu, \gamma_\nu] \quad (\mu, \nu = 1, 2, \dots, 7) \tag{3.23}$$

defines an $so(7)$ Lie algebra of

$$J_{\mu\nu} = -J_{\nu\mu}, \tag{3.24 a}$$

$$[J_{\mu\nu}, J_{\alpha\beta}] = \delta_{\nu\alpha} J_{\mu\beta} - \delta_{\mu\alpha} J_{\nu\beta} - \delta_{\nu\beta} J_{\mu\alpha} + \delta_{\mu\beta} J_{\nu\alpha}. \tag{3.24 b}$$

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Also, as has been shown in [**15**], there exists the charge-conjugation matrix C satisfying

$$C\gamma_\mu C^{-1} = -\gamma_\mu^{\mathrm{T}} \quad (\mu = 1, 2, \ldots, 7), \tag{3.25 a}$$

$$C^{\mathrm{T}} = C, \tag{3.25 b}$$

where the superscript 'T' denotes transpose. We now introduce a bilinear form in V by

$$\langle x|y\rangle = \sum_{i,j=1}^{8} x_i C_{ij} y_j, \tag{3.26}$$

which is symmetric by Equation (3.25 *b*). Also, it is non-degenerate, as we can easily show. Setting

$$[x, y, z] \doteq \frac{1}{3}\sum_{\mu=1}^{7}\{\langle y|\gamma_\mu z\rangle\gamma_\mu x + \langle z|\gamma_\mu x\rangle\gamma_\mu y + \langle x|\gamma_\mu y\rangle\gamma_\mu z\}, \tag{3.27}$$

we have shown in [**15**] that it satisfies Equations (3.3) and (3.4) with $\beta = -1$. Moreover, we can rewrite Equation (3.27) as

$$\begin{aligned}[x, y, z] &= \sum_{\mu=1}^{7}\langle y|\gamma_\mu z\rangle\gamma_\mu x + \langle x|z\rangle y - \langle x|y\rangle z \\ &= \sum_{\mu=1}^{7}\langle z|\gamma_\mu x\rangle\gamma_\mu y + \langle x|y\rangle z - \langle y|z\rangle x \\ &= \sum_{\mu=1}^{7}\langle x|\gamma_\mu y\rangle\gamma_\mu z + \langle y|z\rangle x - \langle z|x\rangle y \end{aligned} \tag{3.28}$$

when we use the so-called Fierz identities (see [**15**]). We also note the following. Although we have assumed in [**15**] that the underlying field F is real or complex, the results stated here hold true for any algebraically closed field F of characteristic not 2 and not 3. Actually, we can relax the condition of algebraic closure by using a Clifford algebra $C(4, 3)$ or $C(0, 7)$ instead of the Clifford algebra $C(7, 0)$ of Equation (3.22) (see [**15**] for details).

In order to show that the Lie algebra g_0 is $so(7)$, we note further that the triple product xyz for the corresponding $U(-1, -1)$-BFKTS can be written as

$$xyz = -\frac{1}{24}\sum_{\mu,\nu=1}^{7}\langle x|[\gamma_\mu, \gamma_\nu]y\rangle[\gamma_\mu, \gamma_\nu]z + \langle x|y\rangle z \tag{3.29}$$

by using various identities given in the appendix of [**15**]. This implies the validity of

$$\ell(x, y) = -\frac{2}{3}\sum_{\mu,\nu=1}^{7}\langle x|J_{\mu\nu}y\rangle J_{\mu\nu}. \tag{3.30}$$

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Conversely, let $e_1, e_2, \dots, e_8$ be a basis of V with

$$\langle e_j | e_k \rangle = \delta_{jk} \quad (j,k = 1,2,\dots,8). \tag{3.31}$$

We can then express $J_{\mu\nu}$ as

$$J_{\mu\nu} = \frac{3}{8}\sum_{j=1}^{8} \ell(e_j, J_{\mu\nu} e_j) \tag{3.32}$$

in terms of the $\ell(x,y)$. These show that g_0 is indeed $so(7)$. Moreover, these relations reproduce essentially the formula for $F(4)$ given in [**6**]. This completes the demonstration that the case of $N = 8$ will lead to the Lie superalgebra $F(4)$. Also, Equation (3.27) or Equation (3.7 a) will offer new realizations of $F(4)$ in terms of either eight-dimensional $so(7)$ spinors or octonions.

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