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EXTENSIONS OF UNIFORMLY SMOOTH NORMS ON BANACH SPACES

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We give a characterisation for the extension of uniformly smooth norms from subspaces Y of superreflexive spaces X to uniformly smooth norms on all of X. This characterisation is applied to obtain results in various contexts.

1. INTRODUCTION

Consider the following problem. Let $\mathcal{P}$ be some rotundity or smoothness property of a norm on a Banach space X. Then given a subspace $Y \subset X$, and an equivalent norm $\|\cdot\|_Y$ on Y with property $\mathcal{P}$, is it possible to extend $\|\cdot\|_Y$ to an equivalent norm on X with property $\mathcal{P}$? Equivalently, can $\|\cdot\|_Y$ with property $\mathcal{P}$ be seen as the restriction of an equivalent norm on X with property $\mathcal{P}$? For separable spaces, and $\mathcal{P}$ the property of being rotund or locally uniformly rotund, this problem has a positive solution ([**9**]). For general X, if $\mathcal{P}$ represents the property of rotundity, local uniform rotundity, or uniform rotundity, then the recent result of [**5**] gives a positive solution provided $Y \subset X$ is reflexive.

For the case in which $\mathcal{P}$ is a smoothness property, the situation appears to be more delicate, and in certain situations is related to the complementability of the subspace Y. There is an example from [**2**], which exhibits a Gâteaux smooth norm $|\cdot|$ on c_0 and a $y \in c_0\backslash\{0\}$, such that $|\cdot|$ cannot be extended to a norm on l_∞ which is Gâteaux smooth at y.

We also have the following "negative" result of [**14**]. There exists a separable Banach space X, a non-complemented subspace $Y \subset X$, and a Gâteaux differentiable norm on Y such that this norm cannot be extended to a Gâteaux differentiable norm on X. This result is proven via contradiction by using the supposed existence of such an extension to show that Y is then complemented in X. In the same paper, additional connections between smooth extensions and the complementability of subspaces are established by showing that if X^* is separable, and Y is a Hilbertian subspace of X with unit sphere S_Y, then the Hilbertian norm on Y extends to a map $\varphi : X \to \mathbb{R}$ which as a function on X is

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Fréchet smooth on S_Y, with φ' locally Lipschitz on S_Y, if and only if Y is complemented ([**14**, Theorem 1]). It is also shown in [**14**] that if Y is (linearly) complemented, the smooth extension problem is easily solved.

Concerning positive results for the case in which $\mathcal{P}$ is a smoothness property, to the author's knowledge, essentially no progress has been made from the time of [**14**]. In fact, for non-complemented subspaces, we know of no positive result concerning the smooth extension of norms in the infinite dimensional setting. We address this issue in Proposition 1 where we give a characterisation for the extension of uniformly smooth norms from subspaces of superreflexive spaces to uniformly smooth norms on the whole space, somewhat in the spirit of [**14**, Theorem 1] described above. The techniques of our main proposition are used to obtain a result concerning the approximation of norms on subspaces of superreflexive spaces by the restrictions of uniformly smooth norms defined on the whole space. This approximate solution to the uniformly smooth extension problem also follows from a result mentioned in a Remark in [**11**], which uses a different approach.

We also discuss the relationship between uniformly smooth extensions of norms and subspaces $Y \subset X$ which are nonlinearly complemented. Here the situation is subtle. Indeed, from classical results any closed subspace Y of a Banach space X is nonlinearly complemented by a continuous projection (see for example, [**13**]), however, Y may not be linearly complemented and the projection may possess no smoothness properties. On the other hand, by a result of Lindenstrauss (see for example, [**1**]), if Y is reflexive and nonlinearly complemented by a projection uniformly continuous on all of X, then in fact Y is linearly complemented. Then again, there is a result of Holmes [**8**] which states in part that for X superreflexive and $Y \subset X$, the metric projection onto Y is uniformly continuous on bounded sets, although not Fréchet smooth in general. From these results one can see that if $\nu : X \to Y$ is a continuous, nonlinear projection with X superreflexive and Y is not linearly complemented, then the continuity or smoothness conditions on ν must be balanced with some care. In this direction we show, using our main proposition, that the uniformly smooth extension problem has a positive solution if the continuous nonlinear projection ν is uniformly smooth and bounded on a neighbourhood of S_X (the unit sphere of X).

2. Notation and Definitions

All Banach spaces are assumed real and are denoted by X, Y, et cetera. The closed unit ball and sphere of X are written B_X and S_X respectively. A closed ball of radius $r > 0$ and centre $p \in X$ is denoted $B_r(p)$. If $G \subset X$, then the *distance function to G*, $\operatorname{dist}(\cdot, G) : X \to \mathbb{R}$, is given by $\operatorname{dist}(x, G) = \inf\{\|x - y\| : y \in G\}$. The norm on a Banach space is said to be *uniformly Fréchet smooth* (or simply uniformly smooth) if the limit,

$$\lim_{t \to 0} t^{-1}\big(\|x + th\| - \|x\|\big),$$

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exists, is continuous and linear in h, and is uniform in $(x,h) \in S_X \times S_X$. Let $U \subset X$ be an open subset of a Banach space, and Y a Banach space. A map, $f : U \to Y$, is similarly said to be Fréchet differentiable or Fréchet smooth at $x \in U$ if the limit,

$$df(x)(h) \equiv \lim_{t\to 0} t^{-1}\big(f(x+th) - f(x)\big), \tag{2.1}$$

exists, is continuous and linear in h, and is uniform in $h \in S_X$.

If $f : U \to Y$ is Fréchet differentiable at all $x \in U$ with $U \subset X$ open, $A \subset U$ is a subset, and the limit (2.1) is uniform for $(x,h) \in A \times S_X$, then we shall say that f is *uniformly Fréchet smooth* on A, or simply *uniformly smooth* on A for short. The collection of all such functions is written $UF(A,Y)$. It is worth noting that if $f(x) = \|x\|$, and we define $\phi : S_X \to S_{X^*}$ by $\phi(x) = df(x)$, then the condition that $\phi : S_X \to S_{X^*}$ be uniformly continuous is equivalent to the limit (2.1) being uniform in $(x,h) \in S_X \times S_X$ (see for example, [**10**, Lemma 5.5.9]). In this note, smoothness is meant in the Fréchet sense. $(X, \|\cdot\|)$ is said to be *superreflexive* if it admits a uniformly smooth norm equivalent to $\|\cdot\|$. For further information on superreflexive spaces, we refer the reader to [**4**, **10**]. All subspaces are assumed closed.

3. A Characterisation of Uniformly Smooth Extensions

For the purposes of this paper, let X be a superreflexive Banach space with uniformly smooth norm $\|\cdot\|$, and Y a subspace with a given equivalent uniformly smooth norm $\|\cdot\|_Y$. We suppose without loss of generality, that $\|\cdot\| \leqslant \|\cdot\|_Y$ on Y.

Our first result gives a characterisation of those subspaces Y of superreflexive spaces X for which $\|\cdot\|_Y$ can be extended to a uniformly smooth norm on all of X. The techniques of the following proof shall then be adapted to obtain results concerning such extensions in other contexts.

Proposition 1. *Let X be a superreflexive Banach space, and Y a subspace with an equivalent uniformly smooth norm $\|\cdot\|_Y$. Then there exists an extension of $\|\cdot\|_Y$ to a map uniformly smooth and bounded on a neighbourhood of S_X if and only if there exists an equivalent uniformly smooth norm on X extending the norm $\|\cdot\|_Y$.*

Proof: Fix a subspace $Y \subset X$, a uniformly smooth norm $\|\cdot\|$ on X, and let $\|\cdot\|_Y$ be an equivalent uniformly smooth norm on Y, which we can assume satisfies $A\,\|\cdot\|_{|_Y} \geqslant \|\cdot\|_Y \geqslant \|\cdot\|_{|_Y}$, for some $A > 0$. Unless mentioned otherwise, all closed balls are taken with respect to $\|\cdot\|$. Sufficiency is clear. For necessity, let $f : X \to \mathbb{R}$ be an extension of $\|\cdot\|_Y$ which is uniformly smooth on a neighbourhood of S_X with $\sup\{f(x) : x \in S_X\} \equiv \sqrt{M} < \infty$.

Now, for $y \in Y$, $f(y) = \|y\|_Y \geqslant \|y\|$, and hence for all $y \in Y\backslash\{0\}$, $f(y/\|y\|) \geqslant 1$. Since f is uniformly continuous on S_X, there is a $\delta > 0$ such that for any $y_0 \in S_Y$ and $y \in B_{3\delta}(y_0) \cap S_X$, we have $f(y) > 1/2$. We define the sets,

$$S_1 = \{x \in X : \operatorname{dist}(x,Y) \leqslant \delta\}, \text{ and } S_2 = \{x \in X : \operatorname{dist}(x,Y) \geqslant 2\delta\}.$$

Let $\zeta \in C^\infty(\mathbb{R}, [0,1])$ be such that $\zeta(t) = 0$ if $t \leqslant 3\delta/2$, and $\zeta(t) = 1$ if $t \geqslant 2\delta$, and put $h(x) = \zeta(\operatorname{dist}(x, Y))$. Since X is superreflexive, we have that $h \in UF(X, [0,1])$ (see for example, [**12**, Proposition 4.2.5]), and we have $h = 0$ on S_1, and $h = 1$ on S_2. For $x \in X$, set

$$g(x) = \sqrt{f^2(x) + h(x)}.$$

Note that we have $g_{|Y} = \|\cdot\|_Y$, and $1/2 \leqslant g(x/\|x\|) \leqslant 1 + M$, for all $x \in X \backslash \{0\}$. Also, both $g(x/\|x\|)$ and $\left(g(x/\|x\|)\right)'$ are uniformly continuous on the sets $\{x \in X : \|x\| > r\}$, $r > 0$.

By composing $\|\cdot\|$ with appropriate smooth bump functions on $\mathbb{R}$, we construct maps $\xi_n \in UF(X, [0,1])$ such that ξ_n vanishes in a neighbourhood of the origin, and $\xi_n(x) \equiv 1$ for $\|x\| \geqslant 1/3n$. Define $\psi_n : X \to \mathbb{R}$ by,

$$\psi_n(x) = \begin{cases} \|x\| \, g(x/\|x\|)\xi_n(x) & \text{for} \quad x \neq 0 \\ 0 & \text{for} \quad x = 0 \end{cases}$$

It follows from the definition of ξ_n, that ψ_n is uniformly smooth on bounded subsets of X. Note that, $\psi_n(x) \geqslant \max\{0, 1/2\|x\| - 1/3n\}$ for all $x \in X$, and also that for $y \in Y$ with $\|y\| \geqslant 1/3n$, we have $\psi_n(y) = \|y\|_Y$.

Following the proof of [**4**, Theorem V.3.2] or [**7**, Theorem 10.7], define a convex map $\Psi_n : \operatorname{int}(4B_X) \to \mathbb{R}$ by,

$$\Psi_n(x) = \inf\left\{\sum_{j=1}^{n} \lambda_j \psi_n(x_j) : x = \sum_{j=1}^{n} \lambda_j x_j,\ \sum_{j=1}^{n} \lambda_j = 1,\ \lambda_j \geqslant 0,\ n \in \mathbb{N}\right\}.$$

Using the method of [**7**], since ψ_n is uniformly smooth on $4B_X$, we have that Ψ_n is uniformly smooth on $\operatorname{int}(3B_X)$. We write the derivative of Ψ_n at x as $\Psi_n'(x)$. Because $\psi_n(x) \geqslant \max\{0, 1/2\|x\| - 1/3n\}$, it follows that $\Psi_n(x) \geqslant \max\{0, 1/2\|x\| - 1/3n\}$, and hence that $\Psi_n(x) \leqslant 1$ implies $\|x\| < 3$. Set $\widetilde{\Psi}_n(x) = \left(\Psi_n(x) + \Psi_n(-x)\right)/2$, and μ_n equal to the Minkowski functional of $B_n = \{x \in X : \widetilde{\Psi}_n(x) \leqslant 1\}$. Since $\widetilde{\Psi}_n(0) = 0$ and $B_n \subset \operatorname{int}(4B_X)$ for all n, we have that μ_n is an equivalent norm on X for each n. Further, since

$$\psi_n(x) \geqslant \max\left\{0, \frac{1}{2}\|x\| - \frac{1}{3n}\right\} \geqslant \max\left\{0, \frac{1}{2}\|x\| - \frac{1}{3}\right\},$$

and

$$\psi_n(x) = \|x\| \, g(x/\|x\|)\xi_n(x) \leqslant (1 + M)\|x\|,$$

the same inequalities hold for $\widetilde{\Psi}_n$, and so there are constants $A_1, A_2 > 0$, independent of n, such that for all $x \in X$ and $n \geqslant 1$,

$$A_1 \|x\| \leqslant \mu_n(x) \leqslant A_2 \|x\| \tag{3.1}$$

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Now, as in the proof of [4, Theorem V.1.3], we use the Implicit Function Theorem on the equation $\widetilde{\Psi}_n\Big(x/(\mu_n(x))\Big) = 1$ to obtain,

$$\mu_n'(x) = -\big(\widetilde{\Psi}_n'(x)(x)\big)^{-1}\widetilde{\Psi}_n'(x) \quad \text{for } x \text{ such that } \mu_n(x) = 1.$$

Note that since $\widetilde{\Psi}_n$ is convex, we have $\widetilde{\Psi}_n'(x)(x) \geqslant \widetilde{\Psi}_n(x) - \widetilde{\Psi}_n(0) = \widetilde{\Psi}_n(x)$, and hence for x such that $\mu_n(x) = 1$, we have $\widetilde{\Psi}_n'(x)(x) \geqslant 1$. It follows that $\mu_n(x)$ is Fréchet smooth. Further, since $\widetilde{\Psi}_n'$ is uniformly continuous on the set $S_n = \{x \in X : \mu_n(x) = 1\}$, we have that μ_n is uniformly smooth on S_n, and therefore μ_n is an equivalent uniformly smooth norm on X.

Next, fix any $x_0 \in X\backslash\{0\}$, pick n_0 with $\|x_0\| > 1/3n_0$, and choose $\delta > 0$ so that $x \in B_\delta(x_0)$ implies $\|x\| > 1/3n_0$. Then for all $m, n > n_0$, and $x \in B_\delta(x_0)$, we have $\mu_n(x) = \mu_m(x)$, and so $|\mu_n(x) - \mu_m(x)| \to 0$ uniformly on $B_\delta(x_0)$. Since $\mu_n(x) \leqslant A_2\|x\|$ for all n, μ_n also converges uniformly about the origin. It follows that there exists a continuous map μ with $\mu_n \to \mu$. A similar argument shows that μ_n' converges uniformly in a neighbourhood about any $x \neq 0$, and hence μ is continuously Fréchet differentiable on $X\backslash\{0\}$. Now, for $x \in S \equiv \{x \in X : \mu(x) = 1\}$, we have that $\mu_n(x) = \mu_m(x)$ for all $n, m > (1/3)(1 + M)$, and hence $\mu_n' \to \mu'$ uniformly on S. Since the μ_n' are uniformly continuous on S, it follows that μ' is uniformly continuous on S. This, together with (3.1), show that μ is an equivalent uniformly smooth norm on X.

Next, let $\varepsilon \in (0,1)$ and choose n_0 so that $(1 + A + M)/3n_0 < \varepsilon/4$. Now, for $y \in Y$ and any n, $\psi_n(y) = \xi_n(y)\|y\|_Y$, and hence for $y \in Y$ with $\|y\| \geqslant 1/3n_0$, we have $\psi_{n_0}(y) = \|y\|_Y$. Therefore, for all $n \geqslant n_0$ and $y \in Y$, $|\psi_n(y) - \|y\|_Y| < \varepsilon/2$, or $\|y\|_Y - \varepsilon/2 < \psi_n(y) < \|y\|_Y + \varepsilon/2$. A convexity argument now gives that $\|y\|_Y - \varepsilon/2 < \Psi_n(y) < \|y\|_Y + \varepsilon/2$, and so for all $n \geqslant n_0$ and $y \in Y$, $|\widetilde{\Psi}_n(y) - \|y\|_Y| < \varepsilon/2$.

It follows that $|\mu_n(y) - \|y\|_Y| < \varepsilon\|y\|_Y$ for $n \geqslant n_0$ and $y \in Y\backslash\{0\}$, since $\|\cdot\|_Y$ is a norm and μ_n on Y is the Minkowski functional of the set $\{y \in Y : \widetilde{\Psi}_n(y) \leqslant 1\}$. Indeed, let $n \geqslant n_0$, $y \in Y\backslash\{0\}$ and $\lambda > 0$ so that $\widetilde{\Psi}_n(\lambda^{-1}y) = 1$. Then we have, $|1 - \|\lambda^{-1}y\|_Y| < \varepsilon/2$, which implies that $1/1 - \varepsilon/2 > \lambda/\|y\|_Y > 1/(1 + \varepsilon/2)$, and hence $\|y\|_Y\big((\varepsilon/2)/(1 - \varepsilon/2)\big) > \lambda - \|y\|_Y > \|y\|_Y\big((-\varepsilon/2)/(1 + \varepsilon/2)\big)$, from which the desired inequality follows.

Finally, for any fixed $y_0 \in Y\backslash\{0\}$ and $\varepsilon' \in (0,1)$, working in a neighbourhood $B_\delta(y_0) \subset Y$ of y_0 such that $0 \notin B_\delta(y_0)$, and using our above estimate with $\varepsilon < \varepsilon'/(\delta + \|y_0\|)$, we can find an $n_0 = n_0(y_0)$ so that for all $n \geqslant n_0$, $|\mu_n(y) - \|y\|_Y| < \varepsilon'$ on $B_\delta(y_0)$. Since $\mu_n \to \mu$ locally uniformly on Y, this implies $|\mu(y) - \|y\|_Y| < \varepsilon'$ on a neighbourhood of y_0, and so $\mu_{|Y} = \|\cdot\|_Y$, since ε' and y_0 were arbitrary (the case $y_0 = 0$ is clear). ∎

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4. Some Applications

4.1. Uniformly Smooth Extensions and Nonlinear Projections. Let us observe (see for example, [**14**]), as mentioned in the introduction, that if $Y \subset X$ is linearly complemented, with $P : X \to Y$ a continuous, linear projection, then the smooth extension problem can be solved. Indeed, with the notation mentioned above, define a norm on X by

$$\|x\|_E = \sqrt{\|x - Px\|^2 + \|Px\|_Y^2}.$$

Then $\|\cdot\|_E$ is an equivalent uniformly smooth norm on X which extends the norm $\|\cdot\|_Y$ on Y. As noted previously, for Y nonlinearly complemented one must proceed more carefully. The following proposition addresses the case in which Y is complemented by a continuous, nonlinear projection uniformly smooth and bounded on a neighbourhood of S_X.

Proposition 2. *Let X be superreflexive, and Y a subspace. Suppose that there exists a continuous nonlinear projection $\nu : X \to Y$ which is uniformly smooth and bounded on a neighbourhood of S_X. Then any equivalent uniformly smooth norm on Y can be extended to an equivalent uniformly smooth norm on all of X.*

Proof: The proof proceeds almost exactly as the proof for Proposition 1, by putting $f(x) = \|\nu(x)\|_Y^2$ and $g(x) = \sqrt{f(x) + h(x)}$. ∎

4.2. Approximate Uniformly Smooth Extensions. We next use the uniform approximation result from [**3**] and the techniques of the proof of Proposition 1 to obtain the following. This result also follows from a variation of a result mentioned in [**11**] (see Propostion 2.5 there and the Remark following), where the techniques of infimal convolutions are used.

Proposition 3. ([**11**]) *Let X be superreflexive, and $Y \subset X$ a subspace. Then any equivalent norm on Y can be uniformly approximated on bounded subsets of Y by the restrictions of norms uniformly smooth on X.*

Proof: Let $(X, \|\cdot\|)$ and Y be as in the statement of the theorem, $\|\cdot\|_Y$ an equivalent uniformly smooth norm on Y, $B \subset Y$ bounded and $\varepsilon \in (0,1)$. Fix $r > 4$ so that $B \subset B_r \equiv B_r(0) \subset X$. We let $A > 0$ be as in the proof of Proposition 1, and fix n for the remainder of the proof large enough so $1/3n < \varepsilon/\big((3+A)4r\big)$. We first observe that $\|\cdot\|_Y$ can be extended to an equivalent norm $\|\cdot\|_E$ on X (see for example, Lemma II.8.1 [**4**]) which we can suppose satisfies $\|\cdot\|_E \leqslant \|\cdot\|$. Because $\|\cdot\|_E$ is Lipschitz, by [**3**] there exists a uniformly smooth map $\rho_\varepsilon : X \to \mathbb{R}$ such that

$$\big|\|x\|_E - \rho_\varepsilon(x)\big| < \varepsilon/2r^2 \ \text{ for all } \ x \in B_r. \tag{4.1}$$

The proof proceeds by replacing the extension f in Proposition 1 with the uniformly smooth map ρ_ε defined above for which (4.1) holds. The method of proof is essentially the same as for Proposition 1, and so we present only a few details for the readers convenience, using the same notation as above. As mentioned, we use here $f(x) = \rho_\varepsilon(x)$, and again choose $g(x) = \sqrt{f^2(x) + h(x)}$. We have similar to before that $1/4 \leqslant g(x/\|x\|) \leqslant 3$, for all $x \in X\backslash\{0\}$, and also have similar bounds on $\psi_n(x)$. For $y \in B_r \cap Y$ with $\|y\| \geqslant 1/3n$, we have $\psi_n(y) = \|y\| f(y/\|y\|)$, and so, $|\psi_n(y) - \|y\|_Y| = \|y\| \left|f(y/\|y\|) - \|y/\|y\|\|_E\right|$ $\leqslant \|y\|(\varepsilon/2r^2) < \varepsilon/2r$. Hence by choice of n we have that $|\psi_n(y) - \|y\|_Y| < \varepsilon/2r$ for all $y \in B_r \cap Y$. If we let μ_n be the uniformly smooth norm on X associated with ψ_n as given in Proposition 1, then one can check as before that $|\|y\| - \mu_n(y)| < \varepsilon$ on $B_r \cap Y$. Hence, μ_n is the required extension. ∎

We end this note with the simple observation that the previous proposition can be cast in a slightly different form as follows. If $(Y, |\cdot|)$ is a superreflexive Banach space, let Z be the space of all uniformly smooth norms on Y equivalent to $|\cdot|$ (the norm $|\cdot|$ need not be uniformly smooth.) Define a metric on Z by,

$$\rho(n_1, n_2) = \sup\Big\{|n_1(x) - n_2(x)| : x \in (B_Y, |\cdot|)\Big\}.$$

Then in this notation we have,

Corollary 1. *Let X be superreflexive, and $(Y, |\cdot|)$ a subspace. Then the set of equivalent uniformly smooth norms on Y which can be extended to a uniformly smooth norm on X is dense in (Z, ρ).*

Proof: Let $\varepsilon > 0$, and fix any uniformly smooth norm $\|\cdot\|_Y \in Z$. Then from Corollary 1 we have that there exists a uniformly smooth norm μ_ε on X with $|\|y\|_Y - \mu_\varepsilon(y)| < \varepsilon$ for all $y \in (B_Y, |\cdot|)$. Therefore μ_ε is the desired norm. ∎

This corollary should be compared with the result of [**6**] which states that if $(X, |\cdot|)$ admits a locally uniformly rotund norm, then the set of all equivalent locally uniformly rotund norms on X is residual in (Z, ρ), where here Z is the collection of all norms on X equivalent to $|\cdot|$.

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