

INFINITE PRODUCTS OF CYCLOTOMIC POLYNOMIALS

WILLIAM DUKE[✉] and HA NAM NGUYEN

(Received 10 November 2014; accepted 21 December 2014; first published online 26 February 2015)

Abstract

We study analytic properties of certain infinite products of cyclotomic polynomials that generalise some products introduced by Mahler. We characterise those that have the unit circle as a natural boundary and use associated Dirichlet series to obtain their asymptotic behaviour near roots of unity.

2010 *Mathematics subject classification*: primary 11M41; secondary 11B85, 11R18, 30B30.

Keywords and phrases: infinite products, cyclotomic polynomials, natural boundary, Mahler functions, Stern diatomic sequence.

1. Introduction

In this paper we will investigate analytic properties of certain infinite products of cyclotomic polynomials. The power series expansions of these products have interesting integer coefficients. We will determine those that have the unit circle as a natural boundary and then, at least in some cases, find their asymptotic behaviour near roots of unity. This behaviour is subtle and is controlled by the size of certain cyclotomic integers and by the residues and special values of associated Dirichlet series.

First we will recall the definition of the cyclotomic polynomials. For $\ell \in \mathbb{Z}^+$, let $\varphi(\ell)$ be Euler's function giving the number of positive integers less than or equal to ℓ that are relatively prime to ℓ . Let $\Phi_\ell(x) \in \mathbb{Z}[x]$ be the integral polynomial of degree $\varphi(\ell)$ with $\Phi_\ell(0) = 1$ whose zeros are the primitive ℓ th roots of unity. The first few are

$$\Phi_1(x) = 1 - x, \quad \Phi_2(x) = 1 + x, \quad \Phi_3(x) = 1 + x + x^2, \quad \Phi_4(x) = 1 + x^2, \dots$$

Generally, for $\ell \geq 2$, we have

$$\Phi_\ell(x) = \prod_{\substack{a \pmod{\ell} \\ \gcd(a, \ell) = 1}} (x - e(a/\ell))$$

where we set $e(z) = e^{2\pi iz}$. Thus $\Phi_\ell(x)$ is the ℓ th cyclotomic polynomial if $\ell \geq 2$, and is minus the usual cyclotomic polynomial if $\ell = 1$.¹

Supported by NSF grant DMS-10-01527.

© 2015 Australian Mathematical Publishing Association Inc. 0004-9727/2015 \$16.00

¹This nonstandard convention serves to make some identities more uniform.

For a fixed prime p consider the infinite product

$$F(z) = F_{p,\ell}(z) = \prod_{k \geq 0} \Phi_\ell(z^{p^k}). \tag{1.1}$$

This product defines an analytic function in the unit disc $\mathbb{D} = \{z : |z| < 1\}$. It is given by a power series with integer coefficients $a(n) \in \mathbb{Z}$:

$$F(z) = \sum_{n \geq 0} a(n)z^n.$$

To see that these coefficients can be interesting let us consider some examples. When $p = 2$ and $\ell = 1$,

$$F(z) = \prod_{m \geq 0} (1 - z^{2^m}) = \sum_{n \geq 0} (-1)^{t_n} z^n = 1 - z - z^2 + z^3 - z^4 + z^5 + z^6 - z^7 - z^8 + \dots \tag{1.2}$$

Here t_n is the Thue–Morse sequence defined by $t_n = 0$ or $t_n = 1$ according as the sum of the binary digits of n is even or odd. See [1] for a survey about this important sequence. Another interesting example occurs when $p = 2$ and $\ell = 3$:

$$F(z) = \prod_{k \geq 0} (1 + z^{2^k} + z^{2^{k+1}}) = \sum_{n \geq 0} b(n)z^n = 1 + z + 2z^2 + z^3 + 3z^4 + 2z^5 + \dots \tag{1.3}$$

The coefficients $b(n)$ define the Stern diatomic sequence [24]:

$$1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, 5, 4, 7, 3, 8, \dots$$

Here $b(n)$ is the number of partitions of the integer n into powers of 2, in which no power of 2 is used more than twice. See [23] for an account of some of the amazing properties of this sequence. Among them are that $b(n + 1)$ and $b(n)$ are relatively prime and that the sequence of quotients defined by $r(n) = (b(n + 1)/b(n))$ enumerates all the positive rational numbers:

$$1, 2, \frac{1}{2}, 3, \frac{2}{3}, \frac{3}{2}, \frac{1}{3}, 4, \frac{3}{4}, \frac{5}{3}, \frac{2}{5}, \frac{5}{2}, \dots$$

Another well-known class of examples comes when ℓ is prime and $p > \ell$:

$$F(z) = \prod_{k \geq 0} (1 + z^{p^k} + z^{2p^k} + \dots + z^{(\ell-1)p^k}) = \sum_{n \geq 0} a(n)z^n. \tag{1.4}$$

Now $a(n) = 1$ if the base p expansion of n has only digits less than ℓ and $a(n) = 0$ otherwise. When $\ell = 2$ the sequence $a(n)$ was studied by Lehmer *et al.* [12]. When $p = \ell$ we have the trivial example

$$F(z) = \sum_{n \geq 0} z^n = \frac{1}{1 - z}, \tag{1.5}$$

as follows by the uniqueness of the base p expansion of an integer.

A famous result of Carlson and Pólya (see [22]) says that a power series with integral coefficients that converges in \mathbb{D} either is a rational function or has the unit circle as a natural boundary. As (1.5) illustrates, for our $F(z)$ this dichotomy is settled by whether or not $p \mid \ell$.

THEOREM 1.1. *The function $F(z)$ is rational if and only if $p \mid \ell$, in which case*

$$F(z) = \frac{1}{\Phi_m(z^{p^{r-1}})}, \tag{1.6}$$

where $\ell = p^r m$ with $r \geq 1$ and $p \nmid m$. Otherwise $F(z)$ has the unit circle as a natural boundary.

We will prove this result, parts of which are already known, in the next section. The main originator of this line of research was Mahler. The fact that (1.2) and (1.4) have the unit circle as a natural boundary follows from Mahler’s early papers [13, 14], respectively. For (1.4) one can apply the well-known result that a power series with coefficients from a finite set is rational if and only if the coefficients are eventually periodic (see [21, page 138, #158]). That the generating function of the Stern sequence (1.3) has the unit circle as a natural boundary is also known (see [6]). More generally, it follows from [10] that $F(z)$ has the unit circle as a natural boundary when $p \nmid \ell$ and ℓ is square-free. The reader may consult [5, 20] and references therein for background on the relation of such results to transcendence theory.

Our proof that $F(z)$ with $p \nmid \ell$ has the unit circle as a natural boundary is short and uses an approach given by Mahler in one of his last papers [16]. Actually, Mahler’s result includes the case $\ell = 1$ but not $\ell > 1$. By suitably modifying his method, we will see that $F(z) \rightarrow 0$ as z approaches a primitive $(p^n \ell)$ th root of unity along a radius of the unit circle, for any nonnegative integer n . This implies the result since the set of such roots of unity is dense in the unit circle. In fact, this approach can be developed much further with interesting applications. Dumas and Flajolet [10] give a very precise asymptotic formula for the m th coefficient of the reciprocal function

$$F(z)^{-1} = \prod_{k \geq 0} \Phi_\ell(z^{p^k})^{-1},$$

when $p \nmid \ell$ and ℓ is square-free.² They apply Cauchy’s formula in the manner of the circle method and utilise asymptotic expansions of $\log F(z)$ near $(p^n \ell)$ th roots of unity. Their method is a refinement of that of de Bruijn [4], who considered the case where $\ell = 1$ and $p = 2$, which had been studied earlier by Mahler [15].

Our main object is to determine the asymptotic behaviour of $F(z)$ as z approaches a q th root of unity $e(a/q)$ along a radius of the unit circle when $p \nmid q$ and $q \nmid \ell$. We will see that this behaviour depends on the value of the cyclotomic integer

$$S = \prod_{k=1}^{\text{ord}_q(p)} \Phi_\ell\left(e\left(\frac{p^k a}{q}\right)\right).$$

Here $\text{ord}_q(b)$ for any integer b with $\text{gcd}(b, q) = 1$ is the usual multiplicative order of b modulo q . Clearly S is a nonzero real number. Set

$$\alpha_q = (\text{ord}_q(p) \log p)^{-1}.$$

²Actually they allow p to be any integer greater than 1 that is prime to ℓ .

THEOREM 1.2. *Suppose that $p \nmid q$ and $q \nmid \ell$. If $|S| < 1$ then for some constant c depending on a/q ,*

$$F\left(e\left(\frac{a}{q} + iy\right)\right) = c + o(1)$$

as $y \rightarrow 0^+$. If $|S| \geq 1$ there exists a continuous 1-periodic function $g(x)$ depending on a/q so that as $y \rightarrow 0^+$,

$$F\left(e\left(\frac{a}{q} + iy\right)\right) \sim g(\alpha_q \log y) y^{-\alpha_q \log |S|}.$$

Now S is an element of $\mathbb{Z}[e(1/r)]$ where $r = \varphi(\ell)/\text{ord}_q(p)$. In fact, it follows from a well-known result about resultants of cyclotomic polynomials (see [3]) that

$$\prod_{\substack{a \pmod{q} \\ \gcd(a,q)=1}} \Phi_\ell\left(e\left(\frac{a}{q}\right)\right) = \begin{cases} p_1^{\varphi(\ell)} & \text{if } q/\ell = p_1^m \text{ for some prime } p_1, \\ 1 & \text{otherwise.} \end{cases}$$

In particular, if q/ℓ is not a power of a prime then S is a cyclotomic unit. In the special case where p is also a primitive root modulo q we see that $S = 1$. Some other cases where S can be evaluated follow from the results of [11].

For example, when $p = 2$ is a primitive root modulo q with q an odd prime (for example, $q = 3, 5, 11, 13, \dots$) we have for the Thue–Morse function (1.2) that

$$F\left(e\left(\frac{1}{q} + iy\right)\right) \sim g\left(\frac{\log y}{(q-1)\log 2}\right) y^{-(\log q/(q-1)\log 2)}, \quad (1.7)$$

while if $q > 3$ we have for the Stern function (1.3) that

$$F\left(e\left(\frac{1}{q} + iy\right)\right) \sim g\left(\frac{\log y}{(q-1)\log 2}\right).$$

Example (1.7) should be compared with results about the behaviour of partial sums of the Thue–Morse sequence in progressions first obtained by Newman [18] (see also [8, 9] and the references cited therein).

We remark that Theorem 1.2 holds as well when $p \mid \ell$. Of course, after Theorem 1.1 it is not very interesting in this case.

We will see that the value of c and the Fourier expansion of g are determined by special values and residues of Dirichlet series formed from the coefficients of $F(z)$ twisted by exponentials. Dirichlet series associated to more general infinite products are studied in Section 3. The main result of that section, Theorem 3.3, is then applied to prove Theorem 1.2 in Section 4. Some concluding remarks are made in Section 5.

2. Proof of Theorem 1.1

To prove that (1.6) holds when $p \mid \ell$ we need some basic results about cyclotomic polynomials.

LEMMA 2.1. *If m is odd and $r \geq 1$ then*

$$\Phi_{p^r m}(z) = \frac{\Phi_m(z^{p^r})}{\Phi_m(z^{p^{r-1}})}.$$

PROOF. We have the well-known identities (see [17, page 160]) for $n \in \mathbb{Z}^+$ and p a prime:

$$\Phi_{pn}(z) = \Phi_n(z^p) \tag{2.1}$$

when $p \mid n$, and

$$\Phi_{pn}(z) = \frac{\Phi_n(z^p)}{\Phi_n(z)} \tag{2.2}$$

when $p \nmid n$. Taking $n = p^{r-1}m$ and applying (2.1) $r - 1$ times and then applying (2.2) once, we get the result. \square

If $\ell = p^r m$ with $r \geq 1$ and $p \nmid m$ then by Lemma 2.1 we have the telescoping product

$$\prod_{0 \leq k \leq n} \Phi_{p^r m}(z^{p^k}) = \frac{\Phi_m(z^{p^r})}{\Phi_m(z^{p^{r-1}})} \cdots \frac{\Phi_m(z^{p^{r+n}})}{\Phi_m(z^{p^{r+n-1}})} = \frac{\Phi_m(z^{p^{r+n}})}{\Phi_m(z^{p^{r-1}})}.$$

Thus by (1.1) we get (1.6) since

$$F_{p,p^r m}(z) = \lim_{n \rightarrow \infty} \prod_{0 \leq k \leq n} \Phi_{p^r m}(z^{p^k}) = \frac{1}{\Phi_m(z^{p^{r-1}})}.$$

Turning to the case where $p \nmid \ell$, we have the following estimate.

LEMMA 2.2. *Suppose that $p \nmid \ell$. There is an absolute constant $C > 0$ so that for $\gcd(a, \ell) = 1$ and $0 < y < 1$,*

$$\left| F\left(e\left(\frac{a}{\ell} + iy\right)\right) \right| \ll_{p,\ell} e^{-C(\log y)^2}.$$

PROOF. Recall that

$$|\Phi_\ell(z)| = \prod_{\epsilon} |z - \epsilon|,$$

where the product runs over all primitive ℓ th roots of unity ϵ . Thus there is a constant A depending only on ℓ so that for any such ϵ we have the estimate

$$|\Phi_\ell(\epsilon e^{-2\pi y})| \leq A(1 - e^{-2\pi y}).$$

Since $p \nmid \ell$ is odd we know that $e(p^k a / \ell)$ is a primitive ℓ th root of unity for all k , so that have

$$\begin{aligned} \left| F\left(e\left(\frac{a}{\ell} + iy\right)\right) \right| &\leq \prod_{0 \leq k \leq \log(y^{-1})} A(1 - e^{-2\pi(p^k y)}) \prod_{k > \log(y^{-1})} \left| \Phi_\ell\left(e\left(p^k\left(\frac{a}{\ell} + iy\right)\right)\right) \right| \\ &\ll_{\ell} \prod_{0 \leq k \leq \log(y^{-1})} (2\pi A)p^k y \ll e^{-C(\log y)^2}. \end{aligned}$$

\square

Now it follows from (1.1) that for any $n \geq 1$,

$$F(z) = \prod_{k=0}^{n-1} \Phi_\ell(z^{p^k})F(z^{p^n}),$$

so by Lemma 2.2 we have that $F(z) \rightarrow 0$ as z approaches any primitive ($p^n \ell$)th root of unity along a radius of the unit circle. Since the set of all these points is dense in the unit circle, F must have the unit circle as a natural boundary.

3. Associated Dirichlet series

Our proof of Theorem 1.2 makes use of analytic properties of a Dirichlet series associated to $F(z)$. In this section we will proceed a little more generally. Let

$$P(\tau) = \sum_{m=0}^d c_m e(m\tau)$$

be a trigonometric polynomial of degree $d \geq 1$ with $c_m \in \mathbb{C}$ and $c_0 = 1$. Let $b \geq 2$ be an integer and consider the Fourier series defined for $\tau \in \mathcal{H}$, the upper half-plane, by

$$f(\tau) = \prod_{k \geq 0} P(b^k \tau). \tag{3.1}$$

This is easily seen to define an analytic function in \mathcal{H} by comparison with $\sum_{n \geq 0} e(b^n \tau)$. Thus f has a Fourier expansion

$$f(\tau) = 1 + \sum_{n \geq 1} a(n) e(n\tau), \tag{3.2}$$

which converges uniformly on compact subsets of \mathcal{H} . Note that $a(n) = a_\ell(n)$ from the introduction in the case where $P(\tau) = \Phi_\ell(e(\tau))$ and $b = 2$.

We are concerned with the associated Dirichlet series

$$\psi(s) = \sum_{n \geq 1} a(n) n^{-s}$$

and, more generally, its twist defined for $a \in \mathbb{Z}$ and $q \in \mathbb{Z}^+$ by

$$\psi(s, a/q) = \sum_{n \geq 1} a(n) e(na/q) n^{-s}.$$

We will show that these series converge absolutely for $\text{Re}(s)$ sufficiently large. Also, $\psi(s, a/q)$ may be meromorphically continued to the entire s -plane, provided $\text{gcd}(q, b) = 1$, and its possible poles lie on or to the left of

$$\text{Re}(s) = \frac{\log |S(a/q)|}{\text{ord}_q(b) \log b},$$

where

$$S(a/q) = \prod_{k=1}^{\text{ord}_q(b)} \sum_{m=0}^d c_m e\left(m \frac{ab^k}{q}\right) \tag{3.3}$$

and $\text{ord}_q(b)$ is, as before, the order of b modulo q .

In particular, if $S(a/q) = 0$ then $\psi(s, a/q)$ is entire. For $q = 1$, this last fact was proven by Mahler [16].

LEMMA 3.1. *Let $M = \max_m |c_m|$ and $A = \log M(d + 1)/\log b$. Then:*

- (a) $a(n) = O(n^A)$; and
- (b) $f(\tau) = O((\text{Im } \tau)^{-A-1})$ for $\text{Im}(\tau) \leq 1$.

PROOF. For statement (a), note that the number of factors in (3.1) that contribute a term greater than one to $a(n)e(n\tau)$ in (3.2) is less than or equal to

$$N = \frac{\log n}{\log b} + 1 = \frac{\log nb}{\log b}.$$

The number of possible products from these factors is $(d + 1)^N$ and the coefficient of each product is bounded in absolute value by M^N , so

$$|a(n)| \leq ((d + 1)M)^N = (d + 1)Mn^A.$$

For statement (b), we use statement (a) to obtain

$$\sum_{n \geq 1} |a(n)||e(n\tau)| \leq \sum_{n \geq 1} n^A e^{-2\pi ny} = O(y^{-A-1})$$

for $y \leq 1$. □

REMARK 3.2. If $P(0) = 0$, Mahler showed in [16] that for all $c > 0$ we have $f(\tau) = O((\text{Im } \tau)^c)$, from which the fact that $\psi(s)$ is entire follows easily from the Mellin transform.

By Lemma 3.1(a) we see that the Dirichlet series $\psi(s, a/q)$ converges absolutely for $\text{Re}(s) > A + 1$ and that we have the Mellin representation

$$\Psi(s, a/q) \stackrel{\text{(def)}}{=} (2\pi)^{-s} \Gamma(s) \psi(s, a/q) = \int_0^\infty \left(f\left(\frac{a}{q} + iy\right) - 1 \right) y^s \frac{dy}{y} \tag{3.4}$$

also for $\text{Re}(s) > A + 1$.

It follows immediately from definition (3.1) that $f(\tau)$ satisfies the functional equation

$$f(\tau) = f(b^n \tau) f_n(\tau) \tag{3.5}$$

for all positive integers n , where

$$f_n(\tau) = \prod_{k=0}^{n-1} P(b^k \tau).$$

Note that $f_1(\tau) = P(\tau)$ and that we may write

$$f_n(\tau) = \sum_{m=0}^{d_n} c_{n,m} e(m\tau), \tag{3.6}$$

where

$$d_n = d(b^n - 1)(b - 1)^{-1}.$$

It is also convenient to define the Dirichlet polynomial

$$\psi_n(s, a/q) = \sum_{m=1}^{d_n} c_{n,m} e\left(m \frac{a}{q}\right) m^{-s}. \tag{3.7}$$

Observe that $c_{n,m} = a(m)$ for $n > (\log m / \log b)$.

THEOREM 3.3. *Suppose that $\gcd(b, q) = 1$. If $S(a/q) = 0$ then $\psi(s, a/q)$ has an analytic continuation to an entire function. Otherwise, $\psi(s, a/q)$ has a meromorphic continuation to the entire s -plane with at most simple poles at*

$$s_{j,k} = \frac{1}{\text{ord}_q(b) \log b} (\log |S(a/q)| + i \arg S(a/q) \pm 2\pi i j - k)$$

for $j, k = 0, 1, 2, \dots$. Moreover, there is a constant $C > 0$ so that for

$$\text{Re}(s) \geq \text{Re}(s_{0,0}) - \frac{1}{2}$$

we have

$$(b^{s \text{ ord}_q(b)} - S(a/q))\psi(s, a/q) = O(|\text{Im } s| + 1)^C. \tag{3.8}$$

PROOF. We establish a kind of recursion formula for $\psi(s, a/q)$ that allows us to continue $\psi(s, a/q)$ meromorphically in vertical strips of width one, inductively.

Set $n = \text{ord}_q(b)$, which is defined since $\gcd(b, q) = 1$. By the functional equations (3.5) and (3.4) we have for $\text{Re}(s) > A + 1$ that

$$\begin{aligned} \Psi(s, a/q) &= \int_0^\infty f_n\left(\frac{a}{q} + iy\right) \left[f\left(\frac{a}{q} + iyb^n\right) - 1 \right] y^s \frac{dy}{y} + \int_0^\infty \left(f_n\left(\frac{a}{q} + iy\right) - 1 \right) y^s \frac{dy}{y} \\ &= \sum_{m=0}^{d_n} c_{n,m} e\left(m \frac{a}{q}\right) \int_0^\infty e(imy) \left[f\left(\frac{a}{q} + iyb^n\right) - 1 \right] y^s \frac{dy}{y} \\ &\quad + \sum_{m=1}^{d_n} c_{n,m} e\left(m \frac{a}{q}\right) \int_0^\infty e(imy) y^s \frac{dy}{y} \\ &= b^{-ns} \sum_{m=0}^{d_n} c_{n,m} e\left(m \frac{a}{q}\right) \int_0^\infty e(imb^{-n}y) \left[f\left(\frac{a}{q} + iy\right) - 1 \right] y^s \frac{dy}{y} \\ &\quad + (2\pi)^{-s} \Gamma(s) \psi_n(s, a/q) \quad \text{by (3.7)} \\ &= b^{-ns} \sum_{m=0}^{d_n} c_{n,m} e\left(m \frac{a}{q}\right) \int_0^\infty [e(imb^{-n}y) - 1] \left[f\left(\frac{a}{q} + iy\right) - 1 \right] y^s \frac{dy}{y} \\ &\quad + b^{-ns} \sum_{m=0}^{d_n} c_{n,m} e\left(m \frac{a}{q}\right) \int_0^\infty \left[f\left(\frac{a}{q} + iy\right) - 1 \right] y^s \frac{dy}{y} + (2\pi)^{-s} \Gamma(s) \psi_n(s, a/q) \end{aligned}$$

$$\begin{aligned}
 &= b^{-ns} \sum_{m=1}^{d_n} c_{n,m} e\left(m \frac{a}{q}\right) \int_0^\infty [e(imb^{-n}y) - 1] \left[f\left(\frac{a}{q} + iy\right) - 1 \right] y^s \frac{dy}{y} \\
 &\quad + b^{-ns} f_n(a/q) \Psi(s, a/q) + (2\pi)^{-s} \Gamma(s) \psi_n(s, a/q) \quad \text{by (3.6)}.
 \end{aligned}$$

Thus for $\text{Re}(s) > A + 1$ we have, upon using $n = \text{ord}_q(b)$ and (3.3), that

$$\begin{aligned}
 &\Psi(s, a/q)(b^{ns} - S(a/q)) \\
 &= b^{ns} (2\pi)^{-s} \Gamma(s) \psi_n(s, a/q) \\
 &\quad + \sum_{m=1}^{d_n} c_{n,m} e\left(m \frac{a}{q}\right) \int_0^\infty [e(imb^{-n}y) - 1] \left[f\left(\frac{a}{q} + iy\right) - 1 \right] y^s \frac{dy}{y}.
 \end{aligned}$$

Now write for K a nonnegative integer

$$e(imb^{-n}y) - 1 = \sum_{k=1}^K \frac{(-2\pi mb^{-n}y)^k}{k!} + E_K(my)$$

where $E_K(y) = O(y^K)$ for $y \geq 1$, while for $y \leq 1$,

$$E_K(y) = O(y^{K+1}). \tag{3.9}$$

Thus for $\text{Re}(s) > A + 1$,

$$\begin{aligned}
 &\Psi(s, a/q)(b^{ns} - S(a/q)) \\
 &= b^{ns} (2\pi)^{-s} \Gamma(s) \psi_n(s, a/q) \\
 &\quad + \sum_{k=1}^K \frac{b^{-nk} (-2\pi)^k}{k!} \psi_n(-k, a/q) \Psi(s + k, a/q) + G_K(s), \tag{3.10}
 \end{aligned}$$

where by Lemma 3.1(b) and (3.9),

$$G_K(s) = \sum_{m=1}^{d_n} c_{n,m} e\left(m \frac{a}{q}\right) \int_0^\infty E_K(my) \left[f\left(\frac{a}{q} + iy\right) - 1 \right] y^s \frac{dy}{y}$$

is holomorphic for $\text{Re}(s) > A - K$.

Now $b^{ns} - S(a/q) = 0$ if and only if

$$s = \frac{1}{\text{ord}_q(b) \log b} (\log |S(a/q)| + i \arg S(a/q) + 2\pi i j)$$

for $j \in \mathbb{Z}$.

We use (3.10) for $K = 0$ to define $\psi(s, a/q)$ for $\text{Re}(s) > A$ with possible simple poles at these points. We then continue this process with $K = 1, 2, \dots$

Finally, (3.8) follows from (3.10) and the fact that

$$\frac{\Gamma(s + k)}{\Gamma(s)} = O(|s|^k). \quad \square$$

4. Proof of Theorem 1.2

In Theorem 3.3 take $P(\tau) = \Phi_\ell(e(\tau))$ and $b = p$. To prove Theorem 1.2 we use the inverse Mellin transform from (3.4) to represent $F(e(a/q + iy))$ in terms of $\Psi(s, a/q)$:

$$\begin{aligned} F(e(a/q + iy)) - 1 &= \frac{1}{2\pi i} \int_{\text{Re}(s)=c} \Psi(s, a/q)y^{-s} ds \\ &= \frac{1}{2\pi i} \int_{\text{Re}(s)=c} (2\pi)^{-s}\Gamma(s)\psi(s, a/q)y^{-s} ds \end{aligned}$$

where c is sufficiently large. Now we push the contour to the line $\text{Re}(s) = \text{Re}(s_{0,0}) - 1/2$ and pick up residues at the (possible) simple poles $s_{j,0}$. Using (3.8) and the exponential decay of the gamma function on vertical lines, we now easily derive the asymptotic formula of Theorem 1.2. Explicitly, if $|S| < 1$ we get the main contribution from the pole of $\Gamma(s)$ at $s = 0$:

$$F\left(e\left(\frac{a}{q} + iy\right)\right) = c + O(y^{-\alpha_q \log |S|}),$$

where $c = 1 + \psi(0, a/q)$. If $|S| \geq 1$ we get the Fourier expansion

$$g(x) = \sum_{n \in \mathbb{Z}} \rho(n)e(nz)$$

where

$$\rho(n) = \text{res}_{s=s_n} \Psi(s, a/q)$$

and

$$s_n = \alpha_q(\log |S| - 2\pi in).$$

The absolute convergence of this Fourier series follows from (3.8) and the exponential decay of $\Gamma(s)$ on vertical lines.

5. Concluding remarks

As should be clear, many of the results of this paper can be generalised in various ways. In particular, a generalisation of Theorem 1.2 may be given for more general products of the form (3.1). Our restriction to cyclotomic polynomials and $b = p$ was mainly to give easily stated and perhaps more elegant results that apply to the Thue–Morse and Stern diatomic sequences (see [2] and [7] for related results). Also, there could be some interest in further understanding the nature of the residues and special values of the associated Dirichlet series $\psi(s, a/q)$ in this case. Is it possible to express them in terms of invariants of cyclotomic fields? The Dirichlet series used by de Bruijn [4] and Dumas and Flajolet [10] arise from the Mellin transform of $\log F(z)$, and are directly related to classical zeta functions. It might be interesting to try to connect them with $\psi(s, a/q)$.

Acknowledgements

We thank the editor and the referee for pointing out to us the papers [4, 10] and for their helpful comments. Parts of this paper, especially in Section 3, appear in the master's thesis [19] of the second named author Ha Nam Nguyen. She thanks Werner Horn, Bharath Sethuraman and Daniel Katz for their comments and suggestions.

References

- [1] J.-P. Allouche and J. Shallit, 'The ubiquitous Prouhet–Thue–Morse sequence', in: *Sequences and their Applications (Singapore, 1998)*, Springer Series in Discrete Mathematics and Theoretical Computer Science (Springer, London, 1999), 1–16.
- [2] J.-P. Allouche, M. Mendès France and J. Peyrière, 'Automatic Dirichlet series', *J. Number Theory* **81** (2000), 359–373.
- [3] T. Apostol, 'Resultants of cyclotomic polynomials', *Proc. Amer. Math. Soc.* **24** (1970), 457–462.
- [4] N. G. de Bruijn, 'On Mahler's partition problem', *Ned. Akad. Wet. Proc.* **51** (1948), 659–669; *Indagationes Math.* **10** (1948), 210–220.
- [5] P. Bundschuh, 'Algebraic independence of infinite products and their derivatives', in: *Number Theory and Related Fields*, Springer Proceedings in Mathematics and Statistics, 43 (Springer, New York, 2013), 143–156.
- [6] M. Coons, 'The transcendence of series related to Stern's diatomic sequence', *Int. J. Number Theory* **6**(1) (2010), 211–217.
- [7] M. Coons, '(Non)automaticity of number theoretic functions', *J. Théor. Nombres Bordeaux* **22** (2010), 339–352.
- [8] J. Coquet, 'A summation formula related to the binary digits', *Invent. Math.* **73**(1) (1983), 107–115.
- [9] M. Drmota and Th. Stoll, 'Newman's phenomenon for generalized Thue–Morse sequences', *Discrete Math.* **308**(7) (2008), 1191–1208.
- [10] P. Dumas and P. Flajolet, 'Asymptotique des récurrences mahlériennes: le cas cyclotomique', *J. Théor. Nombres Bordeaux* **8**(1) (1996), 1–30.
- [11] R. P. Kurshan and A. M. Odlyzko, 'Values of cyclotomic polynomials at roots of unity', *Math. Scand.* **49**(1) (1981), 15–35.
- [12] D. H. Lehmer, K. Mahler and A. J. van der Poorten, 'Integers with digits 0 or 1', *Math. Comp.* **46**(174) (1986), 683–689.
- [13] K. Mahler, 'Arithmetische Eigenschaften der Lösungen einer Klasse von Funktionalgleichungen', *Math. Ann.* **101** (1929), 342–366.
- [14] K. Mahler, 'Über die Taylorkoeffizienten rationaler Funktionen', in: *Akad. Amsterdam*, 38 (1935), 51–60.
- [15] K. Mahler, 'On a special functional equation', *J. Lond. Math. Soc.* (2) **15** (1940), 115–123.
- [16] K. Mahler, 'On two analytic functions', *Acta Arith.* **49**(1) (1987), 15–20.
- [17] T. Nagell, *Introduction to Number Theory* (Wiley, New York, 1951).
- [18] D. J. Newman, 'On the number of binary digits in a multiple of three', *Proc. Amer. Math. Soc.* **21** (1969), 719–721.
- [19] H. N. Nguyen, *Dirichlet series associated to B-multiplicative arithmetic functions*, Master's Thesis, California State University, 2014.
- [20] F. Pellarin, 'An introduction to Mahler's method for transcendence and algebraic independence', Preprint, 2005, arXiv:1005.1216v2.
- [21] G. Pólya and G. Szegő, *Problems and Theorems in Analysis. II. Theory of Functions, Zeros, Polynomials, Determinants, Number Theory, Geometry* (Springer, Berlin, 1998).
- [22] R. Remmert, *Classical Topics in Complex Function Theory*, Graduate Texts in Mathematics, 172 (Springer, New York, 1998), Translated from the German by Leslie Kay.

- [23] R. P. Stanley and H. S. Wilf, Refining the Stern Diatomic sequence, Preprint available at <http://www-math.mit.edu/~rstan/papers/stern.pdf>.
- [24] M. A. Stern, 'Über eine zahlentheoretische Funktion', *J. reine angew. Math.* **55** (1858), 193–220.

WILLIAM DUKE, Department of Mathematics, University of California Los Angeles,
Box 951555, Los Angeles, CA 90095-1555, USA
e-mail: wdduke@ucla.edu

HA NAM NGUYEN, Department of Mathematics, California State University,
18111 Nordhoff Street, Northridge, CA 91330, USA
e-mail: ha.nguyen@csun.edu