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CHARACTERIZING JORDAN MAPS ON C^*-ALGEBRAS THROUGH ZERO PRODUCTS

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Abstract Let A and B be C^*-algebras, let X be an essential Banach A-bimodule and let $T\colon A \to B$ and $S\colon A \to X$ be continuous linear maps with T surjective. Suppose that $T(a)T(b)+T(b)T(a) = 0$ and $S(a)b + bS(a) + aS(b) + S(b)a = 0$ whenever $a, b \in A$ are such that $ab = ba = 0$. We prove that then $T = w\Phi$ and $S = D + \Psi$, where w lies in the centre of the multiplier algebra of B, $\Phi\colon A \to B$ is a Jordan epimorphism, $D\colon A \to X$ is a derivation and $\Psi\colon A \to X$ is a bimodule homomorphism.

Keywords: C^*-algebra; homomorphism; Jordan homomorphism; derivation; Jordan derivation; zero-product-preserving map

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1. Introduction

The question of characterizing homomorphisms on Banach algebras through the action on zero products has attracted the attention of many authors over the last few years. We refer the reader to [**2**] for a full account of the topic and a list of references. The pattern consists in considering the following condition on a linear map T from a Banach algebra A into a Banach algebra B:

$$a, b \in A, \quad ab = 0 \quad \Longrightarrow \quad T(a)T(b) = 0. \tag{H}$$

Such maps are treated in various contexts under different names (Lamperti operators, disjointness-preserving maps, separating maps, zero-product-preserving maps). It is obvious that every homomorphism from A into B satisfies (H) and the standard problem is to show that any map satisfying (H) is 'close' to a homomorphism. In fact, one usually wants to describe a map T satisfying (H) as being a weighted homomorphism, which means that $T = W\Phi$, where $\Phi\colon A \to B$ is a homomorphism and $W\colon B \to B$ is a B-bimodule homomorphism.

A similar problem, of characterizing Jordan homomorphisms through the action on zero products, has recently also attracted some interest [**9**, **10**, **14**, **15**]. Our paper is primarily devoted to this topic. Our purpose is to investigate whether the condition

$$a, b \in A, \quad ab = ba = 0 \quad \Longrightarrow \quad T(a) \circ T(b) = 0 \tag{JH}$$

characterizes Jordan homomorphisms. Here and subsequently, '$\circ$' denotes the Jordan product

$$a \circ b = \tfrac{1}{2}(ab + ba)$$

on a Banach algebra. In this regard our main result is that, in the case when A and B are C^*-algebras, every continuous surjective linear map $T\colon A \to B$ satisfying (JH) is of the form $T = w\Phi$, where $\Phi\colon A \to B$ is a Jordan epimorphism and w lies in the centre of the multiplier algebra of B. Let us recall that the multiplier algebra of B can be thought of as the idealizer of B in its bidual B^{**}, i.e. $\{b \in B^{**} : bB + Bb \subset B\}$. For other versions of linear preservers in the Jordan context we refer the reader to [**8**].

A similar question is concerned with derivations. In this context one is usually involved with the following conditions on a linear map T from a Banach algebra A into a Banach A-bimodule X:

$$a, b, c \in A, \quad ab = bc = 0 \quad \Longrightarrow \quad a \cdot T(b) \cdot c = 0; \tag{D1}$$

$$a, b \in A, \quad ab = 0 \quad \Longrightarrow \quad T(a) \cdot b + a \cdot T(b) = 0. \tag{D2}$$

The preceding conditions have been considered in [**2**, §§ 4.2 and 4.3] and the references therein. It should be pointed out that condition (D1) has proved to be useful for studying local derivations [**1**, **12**]. When dealing with conditions (D1) and (D2) one is intended to describe T as $D + \Psi$, where $D\colon A \to X$ is a derivation and $\Psi\colon A \to X$ is a bimodule homomorphism. The natural way to translate condition (JH) to the context of derivations is to consider the following condition on a linear map $T\colon A \to X$:

$$a, b \in A, \quad ab = ba = 0 \quad \Longrightarrow \quad T(a) \bullet b + a \bullet T(b) = 0. \tag{JD}$$

Here and subsequently, '$\bullet$' denotes the Jordan product on X:

$$a \bullet x = x \bullet a = \tfrac{1}{2}(a \cdot x + x \cdot a), \quad a \in A,\ x \in X.$$

We prove that, in the case when A is a C^*-algebra and X is an essential Banach A-bimodule, condition (JD) implies that T is of the form $T = D + \Psi$, where $D\colon A \to X$ is a derivation and $\Psi\colon A \to X$ is a bimodule homomorphism.

2. Bilinear maps vanishing on zero product

Let A be a Banach algebra and let $\phi : A \times A \to X$ be a continuous bilinear map into a Banach space X. In [**2**] we were concerned with the question of whether the condition

$$a, b \in A, \quad ab = 0 \quad \Longrightarrow \quad \phi(a, b) = 0$$

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implies

$$\phi(ab, c) = \phi(a, bc), \quad a, b, c \in A.$$

It turned out in [**2**] that this is indeed the case for a large class of Banach algebras which includes both C^*-algebras and group algebras, and this provided a powerful tool for characterizing homomorphisms and derivations on that class of algebras. Nevertheless, in order to avoid technicalities, in this paper we will restrict our attention to C^*-algebras.

Theorem 2.1 (Alaminos *et al.* [2]). *Let A be a C^*-algebra, let X be a Banach space and let $\phi\colon A \times A \to X$ be a continuous bilinear map with the property that*

$$a, b \in A, \quad ab = 0 \quad \Longrightarrow \quad \phi(a, b) = 0. \tag{B}$$

Then

$$\phi(ab, c) = \phi(a, bc), \quad a, b, c \in A,$$

and there exists a continuous linear map $\Phi\colon A \to X$ such that

$$\phi(a, b) = \Phi(ab), \quad a, b \in A.$$

Throughout this paper we will be involved with a condition closely related to (B). Now our method consists in considering continuous bilinear maps $\phi\colon A \times A \to X$ satisfying

$$a, b \in A, \quad ab = ba = 0 \quad \Longrightarrow \quad \phi(a, b) = 0. \tag{JB}$$

Theorem 2.2. *Let A be a C^*-algebra, let X be a Banach space and let $\phi\colon A \times A \to X$ be a continuous bilinear map satisfying (JB). Then*

$$\phi(ab, cd) - \phi(a, bcd) + \phi(da, bc) - \phi(dab, c) = 0, \quad a, b, c, d \in A,$$

and there exist continuous linear maps $\Phi, \Psi\colon A \to X$ such that

$$\phi(ab, c) - \phi(b, ca) + \phi(bc, a) = \Phi(abc), \quad a, b, c \in A,$$

and

$$\phi(a, b) + \phi(b, a) = \Psi(a \circ b), \quad a, b \in A.$$

Proof. Pick $a_1, b_1 \in A$ with $a_1 b_1 = 0$ and define a continuous bilinear map $\phi_1\colon A \times A \to X$ by

$$\phi_1(a, b) = \phi(b_1 a, b a_1), \quad a, b \in A.$$

It is straightforward to check that ϕ_1 satisfies (B). From Theorem 2.1 it follows that $\phi_1(ab, c) = \phi_1(a, bc)$ and so

$$\phi(b_1 ab, c a_1) - \phi(b_1 a, b c a_1) = 0 \tag{2.1}$$

for all $a, b, c \in A$. We now fix $a_2, b_2, c_2 \in A$ and consider the continuous bilinear map $\phi_2\colon A \times A \to X$ defined by

$$\phi_2(a_1, b_1) = \phi(b_1 a_2 b_2, c_2 a_1) - \phi(b_1 a_2, b_2 c_2 a_1), \quad a_1, b_1 \in A.$$

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According to (2.1), ϕ_2 satisfies (B), and so Theorem 2.1 now yields $\phi_2(a_1b_1, c_1) - \phi_2(a_1, b_1c_1) = 0$: that is,

$$\phi(c_1a_2b_2, c_2a_1b_1) - \phi(c_1a_2, b_2c_2a_1b_1) + \phi(b_1c_1a_2, b_2c_2a_1) - \phi(b_1c_1a_2b_2, c_2a_1) = 0 \quad (2.2)$$

for all $a_1, b_1, c_1, a_2, b_2, c_2 \in A$.

By taking into account that all the terms in (2.2) involve c_1a_2 and c_2a_1 and that $A^2 = A$ it may be concluded that

$$\phi(ab, cd) - \phi(a, bcd) + \phi(da, bc) - \phi(dab, c) = 0 \quad (2.3)$$

for all $a, b, c, d \in A$, as claimed in the theorem.

We now take a bounded approximate identity $(\rho_i)_{i\in I}$ for A. By applying (2.3) with the element c replaced by ρ_i, $i \in I$, we see that the net $(\phi(dab, \rho_i))_{i\in I}$ is convergent; by taking limits we arrive at

$$\begin{aligned}\phi(ab, d) - \phi(a, bd) + \phi(da, b) - \lim_{i\in I} \phi(dab, \rho_i) \\ &= \lim_{i\in I}(\phi(ab, \rho_i d) - \phi(a, b\rho_i d) + \phi(da, b\rho_i) - \phi(dab, \rho_i)) \\ &= 0\end{aligned}$$

for all $a, b, d \in A$. We can thus define a linear operator $\Phi\colon A^3 \to X$ by $\Phi(a) = \lim_{i\in I} \phi(a, \rho_i)$ for each $a \in A^3$. Since $A^3 = A$, the operator Φ is defined on A and it satisfies

$$\phi(ab, c) - \phi(b, ca) + \phi(bc, a) = \Phi(abc), \quad a, b, c \in A. \quad (2.4)$$

On the other hand, for each $a \in A$ the net $(\phi(a, \rho_i))_{i\in I}$ is bounded and therefore the operator Φ is continuous.

We now apply (2.4) with b replaced by ρ_i to see that the net $(\phi(\rho_i, ca))_{i\in I}$ is convergent and that

$$\begin{aligned}\phi(a, c) - \lim_{i\in I} \phi(\rho_i, ca) + \phi(c, a) &= \lim_{i\in I}(\phi(a\rho_i, c) - \phi(\rho_i, ca) + \phi(\rho_i c, a)) \\ &= \lim_{i\in I} \Phi(a\rho_i c) \\ &= \Phi(ac) \quad (2.5)\end{aligned}$$

for all $a, c \in A$. Since $A^2 = A$, it follows that the net $(\phi(\rho_i, a))_{i\in I}$ is convergent for each $a \in A$ and so we can define a continuous linear operator $\Phi'\colon A \to X$ by

$$\Phi'(a) = \lim_{i\in I} \phi(\rho_i, a), \quad a \in A.$$

Consequently, the identity (2.5) now becomes

$$\phi(a, c) + \phi(c, a) = \Phi(ac) + \Phi'(ca) \quad (2.6)$$

for all $a, c \in A$. By swapping a and c in (2.6) we arrive at

$$\phi(c, a) + \phi(a, c) = \Phi(ca) + \Phi'(ac) \quad (2.7)$$

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for all $a, c \in A$. By adding (2.6) and (2.7) we get

$$2(\phi(a,c) + \phi(c,a)) = \Phi(ac + ca) + \Phi'(ac + ca)$$

and so

$$\phi(a,c) + \phi(c,a) = \Psi(ac + ca)$$

for all $a, c \in A$, where $\Psi\colon A \to X$ is defined by $\Psi = \frac{1}{2}(\Phi + \Phi')$. □

If ϕ is symmetric, i.e. if $\phi(a,b) = \phi(b,a)$ holds for all $a, b \in A$, then the last statement of the theorem shows that ϕ is of the form $\phi(a,b) = \frac{1}{2}\Psi(a \circ b)$. This can be considered as a definitive result on the structure of ϕ. In the general case, when ϕ is not necessarily symmetric, the analogous definitive conclusion would be that $\phi(a,b) = \Phi_1(ab) + \Phi_2(ba)$ for some linear operators Φ_1 and Φ_2. Unfortunately, this does not seem to follow from Theorem 2.2.

Note that every bilinear map $\phi\colon A \times A \to X$ can be written as $\phi = \phi_1 + \phi_2$, where ϕ_1 is symmetric and ϕ_2 is skew-symmetric (i.e. $\phi_2(a,b) = -\phi_2(b,a)$ for all $a, b \in A$); indeed, just define ϕ_1 and ϕ_2 by

$$\phi_1(a,b) = \tfrac{1}{2}(\phi(a,b) + \phi(b,a)) \quad \text{and} \quad \phi_2(a,b) = \tfrac{1}{2}(\phi(a,b) - \phi(b,a)), \quad a, b \in A.$$

It is clear that ϕ satisfies (JB) if and only if both ϕ_1 and ϕ_2 satisfy (JB). Since the structure of ϕ_1 is known by Theorem 2.2, one should only treat ϕ_2. That is to say, in order to describe bilinear maps ϕ satisfying (JB) it suffices to consider the case when ϕ is skew-symmetric. The ultimate goal in this case is to show that there is a linear operator $\Psi : [A, A] \to X$ such that $\phi(a,b) = \Psi([a,b])$ for all $a, b \in A$. By $[A, A]$ we have of course denoted the linear span of all $[a,b]$ with $a, b \in A$.

Remark 2.3. Let A and ϕ be as in Theorem 2.2 and additionally assume that ϕ is skew-symmetric. We claim that

$$\phi(ab, c) + \phi(ca, b) + \phi(bc, a) = 0 \tag{2.8}$$

holds for all $a, b, c \in A$, i.e. the map Φ from the theorem is 0. Pick a self-adjoint element $a \in A$, let B be the C^*-subalgebra of A generated by a and let $b \in B$ with $b^3 = a$. Since B is commutative, the restriction of ϕ to $B \times B$ satisfies the condition (B). Theorem 2.1 then shows that $\phi(a_1 a_2, a_3) = \phi(a_1, a_2 a_3)$ for all $a_1, a_2, a_3 \in B$. In particular, we have $\phi(b^2, b) = \phi(b, b^2)$. However, since ϕ is skew-symmetric, it follows that $\phi(b^2, b) = -\phi(b, b^2)$ and so $\phi(b, b^2) = 0$. On the other hand, according to (2.4) we have $\Phi(a) = \Phi(b^3) = 3\phi(b^2, b) = 0$.

At least for the matrix algebra $A = M_n(\mathbb{C})$ we know that skew-symmetric bilinear maps satisfying (2.8) are indeed of the form $\phi(a,b) = \Psi([a,b])$ [**6**, Theorem 2.1].

3. Characterizing Jordan homomorphisms through zero products

Let A and B be Banach algebras. A *Jordan homomorphism* from A into B is a linear map $\Phi\colon A\to B$ such that

$$\Phi(a\circ b)=\Phi(a)\circ\Phi(b),\quad a,b\in A.$$

It is obvious that each Jordan homomorphism $\Phi\colon A\to B$ satisfies (JH) and we now address the question of whether the converse holds.

Lemma 3.1. *Let A be a C^*-algebra and let $T\colon A\to B$ be a continuous linear map into a Banach algebra B satisfying (JH). Then*

$$T(ab)\circ T(cd)-T(a)\circ T(bcd)+T(da)\circ T(bc)-T(dab)\circ T(c)=0 \tag{3.1}$$

for all $a,b,c,d\in A$. Accordingly, if A and B are unital and $T(\mathbf{1})=\mathbf{1}$, then T is a Jordan homomorphism.

Proof. It suffices to apply Theorem 2.2 to the continuous bilinear map $\phi\colon A\times A\to B$ given by $\phi(a,b)=T(a)\circ T(b)$ for all $a,b\in A$. If A and B are unital and $T(\mathbf{1})=\mathbf{1}$, then by setting $a=c=\mathbf{1}$ in (3.1) we get that T is a Jordan homomorphism. □

Lemma 3.2. *Let A be a C^*-algebra and let $T\colon A\to B$ be a continuous linear map into a Banach algebra B satisfying (JH). Then there exists a continuous linear operator $W\colon\overline{T(A)}\to B$ such that $\|W\|\leqslant\|T\|$ and*

$$W(T(a\circ b))=T(a)\circ T(b),\quad a,b\in A. \tag{3.2}$$

Proof. Let $(\rho_i)_{i\in I}$ be an approximate identity for A of bound 1.

From (3.1) we deduce that

$$\begin{aligned}T(\rho_i)\circ T(abc)&=T(\rho_i a)\circ T(bc)+T(c\rho_i)\circ T(ab)-T(c\rho_i a)\circ T(b)\\&\to T(a)\circ T(bc)+T(c)\circ T(ab)-T(ca)\circ T(b)\end{aligned} \tag{3.3}$$

for all $a,b,c\in A$. Since $A^3=A$, it follows that the net $(T(\rho_i)\circ u)_{i\in I}$ converges for each $u\in T(A)$. On the other hand, from the boundedness of $(T(\rho_i))_{i\in I}$ we deduce that, in fact, the net $(T(\rho_i)\circ u)_{i\in I}$ converges for each $u\in\overline{T(A)}$ so that we can define a linear operator $W\colon\overline{T(A)}\to B$ by

$$W(u)=\lim_{i\in I}T(\rho_i)\circ u,\quad u\in\overline{T(A)}.$$

Of course, the operator W is continuous with $\|W\|\leqslant\|T\|$, and (3.3) now becomes

$$W(T(abc))=T(a)\circ T(bc)+T(c)\circ T(ab)-T(b)\circ T(ca),\quad a,b,c\in A.$$

Hence it follows that

$$\begin{aligned}W(T(ab))&=\lim_{i\in I}W(T(a\rho_i b))\\&=\lim_{i\in I}(T(a)\circ T(\rho_i b)+T(b)\circ T(a\rho_i)-T(\rho_i)\circ T(ba))\\&=T(a)\circ T(b)+T(b)\circ T(a)-W(T(ba))\end{aligned}$$

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for all $a, b \in A$. Therefore, $W(T(a \circ b)) = T(a) \circ T(b)$, $a, b \in A$, as claimed in the lemma. □

Theorem 3.3. *Let A and B be C^*-algebras and let $T\colon A \to B$ be a continuous surjective linear map satisfying the condition*

$$a, b \in A, \quad ab = ba = 0 \implies T(a) \circ T(b) = 0.$$

Then there exist an invertible element w in the centre of the multiplier algebra of B and a Jordan epimorphism $\Phi\colon A \to B$ such that $T = w\Phi$.

Proof. Let W be the map given in Lemma 3.2.

We now claim that W is surjective. Let $b \in B$ positive and let $a \in A$ such that $T(a) = b^{1/2}$. Then (3.2) shows that $W(T(a^2)) = b$. This clearly implies that W is surjective.

By bidualizing (3.2) we obtain

$$W^{**}(T^{**}(x \circ y)) = T^{**}(x) \circ T^{**}(y), \quad x, y \in A^{**}. \tag{3.4}$$

Indeed, to prove the above identity one just observes how the algebraic identity (3.2) carries over by the $\sigma(A^{**}, A^*)$–$\sigma(B^{**}, B^*)$-continuity of T^{**}, the $\sigma(B^{**}, B^*)$–$\sigma(B^{**}, B^*)$-continuity of W^{**} and the separate weak continuity of the products of both A^{**} and B^{**}, one variable at a time.

Let us recall that A^{**} is unital. Write $w = T^{**}(\mathbf{1})$. From (3.4) we deduce that

$$W^{**}(T^{**}(x)) = W^{**}(T^{**}(\mathbf{1} \circ x)) = w \circ T^{**}(x), \quad x \in A^{**}. \tag{3.5}$$

Since $T^{**}(A^{**}) = B^{**}$, the identity (3.5) implies that

$$W(u) = w \circ u, \quad u \in B. \tag{3.6}$$

Let $e \in A^{**}$ be a projection. According to (3.4) and (3.5), we have

$$w \circ T^{**}(e) = W^{**}(T^{**}(e)) = W^{**}(T^{**}(e^2)) = T^{**}(e)^2. \tag{3.7}$$

By multiplying (3.7) by $T^{**}(e)$ on the left we obtain

$$T^{**}(e)wT^{**}(e) + T^{**}(e)^2 w = 2T^{**}(e)^3 \tag{3.8}$$

and multiplying by $T^{**}(e)$ on the right we get

$$wT^{**}(e)^2 + T^{**}(e)wT^{**}(e) = 2T^{**}(e)^3. \tag{3.9}$$

From (3.8) and (3.9) we arrive at $wT^{**}(e)^2 = T^{**}(e)^2 w$, which, on account of (3.7), yields $wW^{**}(T^{**}(e)) = W^{**}(T^{**}(e))w$.

Therefore, $wW^{**}(T^{**}(x)) = W^{**}(T^{**}(x))w$ for each $x \in A^{**}$. Since $T^{**}(A^{**}) = B^{**}$, it follows that $wW^{**}(y) = W^{**}(y)w$ for each $y \in B^{**}$. Since W is surjective, it may be concluded that W^{**} is also surjective (hence, w lies in the centre of B^{**}) and, finally, that w is invertible. According to (3.6), we have

$$W(u) = w \circ u = wu = uw, \quad u \in B.$$

The preceding identity clearly implies that $wB, Bw \subset B$ and hence w lies in the multiplier algebra of B.

We now define $\Phi = W^{-1}T = w^{-1}T$. Of course, Φ is surjective. We proceed to show that Φ is a Jordan homomorphism. Let $a, b \in A$. On account of (3.2), we have

$$\begin{aligned}\Phi(a \circ b) &= W^{-1}(T(a \circ b)) \\ &= W^{-1}(W^{-1}(T(a) \circ T(b))) \\ &= w^{-2}(T(a) \circ T(b)) \\ &= (w^{-1}T(a)) \circ (w^{-1}T(b)) \\ &= \Phi(a) \circ \Phi(b).\end{aligned}$$

□

Corollary 3.4. *Let A and B be C^*-algebras with B prime, and let $T\colon A \to B$ be a continuous surjective linear map satisfying the condition*

$$a, b \in A, \quad ab = ba = 0 \quad \Longrightarrow \quad T(a) \circ T(b) = 0.$$

Then there exist a non-zero complex number λ and either an epimorphism or an anti-epimorphism $\Phi\colon A \to B$ such that $T = \lambda\Phi$.

Proof. Since B is a prime C^*-algebra, it follows that the centre of the multiplier algebra of B is a commutative prime C^*-algebra and so it is isomorphic to $\mathbb{C}$. On account of Theorem 3.3, there exist a non-zero complex number λ and a Jordan epimorphism $\Phi\colon A \to B$ such that $T = \lambda\Phi$. On the other hand, a well-known result by Herstein [**13**] states that every Jordan homomorphism onto a (2-torsion free) prime ring is either a homomorphism or an anti-homomorphism. □

Remark 3.5. A rather natural weakening of condition (H) is the following:

$$a, b \in A, \quad ab = ba = 0 \quad \Longrightarrow \quad T(a)T(b) = 0. \tag{JH1}$$

On the other hand, a natural translation of (H) to Jordan context, which has been considered by a number of authors, is the following:

$$a, b \in A, \quad a \circ b = 0 \quad \Longrightarrow \quad T(a) \circ T(b) = 0. \tag{JH2}$$

A way to unify and generalize both of the preceding conditions consists in considering our condition (JH). Of course, both Theorem 3.3 and Corollary 3.4 remain valid with condition (JH) replaced by any of the above conditions. It is clear that every Jordan

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homomorphism $\Phi\colon A \to B$ satisfies (JH2). However, it is not clear at all that Φ satisfies (JH1), although it does under rather mild assumptions. Indeed, assume that B is semi-prime and $\Phi(A) = B$. From [**4**, Corollary 2.2] we have

$$(\Phi(ab) - \Phi(a)\Phi(b))B(\Phi(cd) - \Phi(d)\Phi(c)) = \{0\}, \quad a, b, c, d \in A. \tag{3.10}$$

If $a, b \in A$ are such that $ab = ba = 0$, then (3.10) with $c = b$ and $d = a$ yields $\Phi(a)\Phi(b)B\Phi(a)\Phi(b) = 0$ and therefore $\Phi(a)\Phi(b) = 0$.

Let us mention the following interesting consequence of identity (2.8) in Remark 2.3 for analysing the Lie-type version of condition (JH). As usual, we write $[\cdot\,,\cdot]$ for the Lie product

$$[a, b] = ab - ba$$

on a Banach algebra.

Corollary 3.6. *Let A and B be unital prime C^*-algebras such that neither A nor B is isomorphic to $M_2(\mathbb{C})$. Let $T\colon A \to B$ be a continuous bijective linear map such that*

$$a, b \in A, \quad ab = ba = 0 \quad \Longrightarrow \quad [T(a), T(b)] = 0. \tag{LH}$$

Then there exist a non-zero complex number λ, either an isomorphism or an anti-isomorphism $\Phi\colon A \to B$ and a linear functional f on A such that $T(a) = \lambda\Phi(a) + f(a)\mathbf{1}$ for all $a \in A$.

Proof. Obviously, the result is true in the case when both A and B are isomorphic to $\mathbb{C}$. Accordingly, from now on we assume that neither A nor B is isomorphic to either $\mathbb{C}$ or $M_2(\mathbb{C})$.

Define $\phi\colon A \times A \to B$ by $\phi(a, b) = [T(a), T(b)]$ for all $a, b \in A$. According to Remark 2.3 we have $\phi(ab, c) + \phi(ca, b) + \phi(bc, a) = 0$ for all $a, b, c \in A$. Setting $a = b = c$, it follows that $\phi(a^2, a) = 0$. That is, $[T(a^2), T(a)] = 0$ for each $a \in A$. Our objective is to apply [**5**, Theorem 2] and then the desired conclusion follows. Nevertheless, to this end we are required to check that neither A nor B satisfies the standard polynomial identity S_4 and that both A and B are centrally closed.

We first claim that neither A nor B satisfies S_4. Indeed, if a C^*-algebra $\mathfrak{A}$ satisfies S_4, then it can be embedded into $M_2(C)$ for a commutative C^*-algebra C [**3**, Theorem 6.1.7]. If $\mathfrak{A}$ is in addition prime, then C is easily seen to be prime, which implies that $C \cong \mathbb{C}$ and therefore that $\mathfrak{A}$ embeds into $M_2(\mathbb{C})$. According to the Wedderburn Structure Theorem [**11**, Theorem 1.5.9], $\mathfrak{A}$ is isomorphic to a full matrix algebra, so that either $\mathfrak{A} \cong \mathbb{C}$ or $\mathfrak{A} \cong M_2(\mathbb{C})$.

Our final observation is that all prime C^*-algebras are centrally closed [**3**, Proposition 2.2.10]. □

It should be mentioned that the assumption that both A and B are different from $M_2(\mathbb{C})$ is certainly necessary in Corollary 3.6. By [**17**, Theorem 1.1], every linear map $T\colon M_2(\mathbb{C}) \to M_2(\mathbb{C})$ that sends the identity matrix $\mathbf{1}$ into a scalar multiple of $\mathbf{1}$ automatically preserves commutativity, and hence also satisfies (LH). However, not every such map has the form described in the corollary.

Remark 3.7. It should be pointed out that the Lie-type version of the associative condition (H) and the Jordan condition (JH2) is merely the classical condition of preserving commutativity:

$$a, b \in A, \quad [a, b] = 0 \quad \Longrightarrow \quad [T(a), T(b)] = 0. \tag{C}$$

The standard goal is to express such maps through (anti)homomorphisms and maps having their range in the centre. The literature on this subject is really vast. Let us just refer to [**7**, Chapter 7] for references and history. The condition (LH) simultaneously generalizes two seemingly unrelated conditions: the condition (C) that T preserves commutativity and the condition that T preserves zero products (more precisely, the condition (JH1), which of course is more general than (H)).

4. Characterizing derivations through zero products

Let A be a Banach algebra, let X be a Banach A-bimodule and let $D\colon A \to X$ be a linear map. Then D is a *Jordan derivation* if

$$D(a \circ b) = D(a) \bullet b + a \bullet D(b), \quad a, b \in A.$$

Since we shall be concerned with Jordan derivations on a C^*-algebra A, it should be pointed out that Johnson showed in [**16**] that every continuous Jordan derivation $D\colon A \to X$ into any Banach A-bimodule is a derivation.

Theorem 4.1. *Let A be a C^*-algebra, let X be an essential Banach A-bimodule and let $T\colon A \to X$ be a continuous linear map satisfying*

$$a, b \in A, \quad ab = ba = 0 \quad \Longrightarrow \quad T(a) \bullet b + a \bullet T(b) = 0.$$

Then there exist a derivation $D\colon A \to X$ and a bimodule homomorphism $\Phi\colon A \to X$ such that $T = D + \Phi$.

Proof. Throughout this proof we will use the fact that, for every Banach algebra A and every Banach A-bimodule X, X^{**} turns into a Banach A^{**}-bimodule with respect to the operations defined by

$$u \cdot \xi = \lim_{i \in I} \lim_{j \in J} a_i \cdot x_j, \quad \xi \cdot u = \lim_{j \in J} \lim_{i \in I} x_j \cdot a_i, \quad u \in A^{**},\ \xi \in X^{**},$$

where $(a_i)_{i \in I}$ is any net in A with $\sigma(A^{**}, A^*) - \lim a_i = u$ and $(x_j)_{j \in J}$ is any net in X with $\sigma(X^{**}, X^*) - \lim x_j = \xi$, and A^{**} is endowed with the first Arens product [**11**, Theorem 2.6.15]. We shall use the following basic facts about the weak continuity of the above-defined products which the reader can find in [**11**, Proposition A.3.52].

(i) For all $u \in A^{**}$ and $a \in A$, the maps $\xi \mapsto \xi \cdot u$ and $\xi \mapsto a \cdot \xi$ from X^{**} into itself are $\sigma(X^{**}, X^*)$-continuous.

(ii) For all $\xi \in X^{**}$ and $x \in X$, the maps $u \mapsto u \cdot \xi$ and $u \mapsto x \cdot u$ from A^{**} into X^{**} are $\sigma(A^{**}, A^*)$–$\sigma(X^{**}, X^*)$-continuous.

It is well known that every C^*-algebra A has the property that every continuous linear map from A into its dual A^* is weakly compact [**11**, Corollary 3.2.43]. This property implies that every continuous bilinear map $\phi\colon A \times A \to X$ into some Banach space X is Arens regular, which means that the two ways of extending to the second dual [**11**, Identities (A.3.8) and (A.3.9), p. 824] give the same result: that is,

$$\lim_{i\in I}\lim_{j\in J}\phi(a_i, b_j) = \lim_{j\in I}\lim_{i\in I}\phi(a_i, b_j) \tag{4.1}$$

for all $\sigma(A^{**}, A^*)$-convergent nets $(a_i)_{i\in I}$ and $(b_j)_{j\in J}$ in A; the limits in (4.1) are taken with respect to the topology $\sigma(X^{**}, X^*)$. Of course, this property implies that A is Arens regular.

By applying Theorem 2.2 to the bilinear map

$$(a, b) \to T(a) \bullet b + a \bullet T(b)$$

we obtain

$$\begin{aligned} T(ab) \bullet cd + ab \bullet T(cd) - T(a) \bullet bcd - a \bullet T(bcd) + T(da) \bullet bc \\ + da \bullet T(bc) - T(dab) \bullet c - dab \bullet T(c) = 0 \end{aligned} \tag{4.2}$$

for all $a, b, c, d \in A$. We bidualize (4.2), taking into account the regularity of A, the $\sigma(A^{**}, A^*)$–$\sigma(X^{**}, X^*)$-continuity of T^{**}, the separate weak continuity properties of the module operations on X^{**}, and the identity (4.1). We thus get

$$\begin{aligned} T^{**}(ab) \bullet cd + ab \bullet T^{**}(cd) - T^{**}(a) \bullet bcd - a \bullet T^{**}(bcd) \\ + T^{**}(da) \bullet bc + da \bullet T^{**}(bc) - T^{**}(dab) \bullet c - dab \bullet T^{**}(c) = 0 \end{aligned} \tag{4.3}$$

for all $a, b, c, d \in A^{**}$.

Let $\mathbf{1}$ be the identity of A^{**} and write $\xi = T^{**}(\mathbf{1}) \in X^{**}$. By applying (4.3) with $a = c = \mathbf{1}$ and arbitrary $b, d \in A^{**}$ we get

$$\begin{aligned} T^{**}(b) \bullet d + b \bullet T^{**}(d) - \xi \bullet bd - \mathbf{1} \bullet T^{**}(bd) \\ + T^{**}(d) \bullet b + d \bullet T^{**}(b) - T^{**}(db) \bullet \mathbf{1} - db \bullet \xi = 0. \end{aligned} \tag{4.4}$$

On the other hand, since X is essential it follows that $\mathbf{1} \cdot x = x \cdot \mathbf{1} = x$ for each $x \in X$. On account of the $\sigma(X^{**}, X^*)$-denseness of X in X^{**} and the $\sigma(X^{**}, X^*)$-continuity of the map $x \mapsto x \cdot \mathbf{1}$, we have

$$\zeta \cdot \mathbf{1} = \zeta, \quad \zeta \in X^{**}. \tag{4.5}$$

Unfortunately, we cannot be sure about the identity $\mathbf{1} \cdot \zeta = \zeta$ for an arbitrary $\zeta \in X^{**}$. Nevertheless, we claim that

$$\mathbf{1} \cdot T^{**}(a) = T^{**}(a), \quad a \in A^{**}. \tag{4.6}$$

Indeed, let $a \in A^{**}$ and let $(\rho_i)_{i\in I}$ and $(a_j)_{j\in J}$ be nets in A converging to $\mathbf{1}$ and a, respectively, with respect to the topology $\sigma(A^{**}, A^*)$. Then

$$\mathbf{1} \cdot T^{**}(a) = \lim_{i\in I}\lim_{j\in J}\rho_i \cdot T(a_j) = \lim_{j\in J}\lim_{i\in I}\rho_i \cdot T(a_j) = \lim_{j\in J} T(a_j) = T^{**}(a),$$

where the limits above are taken with respect to the topology $\sigma(X^{**}, X^*)$.

According to (4.5) and (4.6), (4.4) reads as follows:

$$T^{**}(b \circ d) = T^{**}(b) \bullet d + b \bullet T^{**}(d) - \xi \bullet (b \circ d), \quad b, d \in A^{**}. \tag{4.7}$$

In particular, we have

$$T(a \circ b) = T(a) \bullet b + a \bullet T(b) - \xi \bullet (a \circ b), \quad a, b \in A. \tag{4.8}$$

Our next objective is to show that $\xi \cdot a = a \cdot \xi$ for each $a \in A^{**}$. Of course, it suffices to prove the identity for each projection in A^{**}. Let $e \in A^{**}$ be a projection. Then we take $b = d = e$ in (4.7) and we thus get

$$T(e) = T(e) \cdot e + e \cdot T(e) - \tfrac{1}{2}\xi \cdot e - \tfrac{1}{2}e \cdot \xi. \tag{4.9}$$

We multiply (4.9) on the right by e to obtain

$$T(e) \cdot e = T(e) \cdot e + e \cdot T(e) \cdot e - \tfrac{1}{2}\xi \cdot e - \tfrac{1}{2}e \cdot \xi \cdot e$$

and so

$$0 = e \cdot T(e) \cdot e - \tfrac{1}{2}\xi \cdot e - \tfrac{1}{2}e \cdot \xi \cdot e. \tag{4.10}$$

Similarly, by multiplying (4.9) on the left by e, we arrive at

$$0 = e \cdot T(e) \cdot e - \tfrac{1}{2}e \cdot \xi \cdot e - \tfrac{1}{2}e \cdot \xi. \tag{4.11}$$

From (4.10) and (4.11) we deduce that $\xi \cdot e = e \cdot \xi$, as required.

We now claim that $\xi \cdot A \subset X$. It suffices to prove that $\xi \cdot a \in X$ for each positive element $a \in A$. Let a be a positive element in A and let $b \in A$ with $b^2 = a$. According to (4.8) and the commutativity property of ξ, we have

$$\xi \cdot a = \xi \bullet b^2 = 2T(b) \bullet b - T(a) \in X.$$

Observe that the map $\Phi\colon A \to X$ defined by $\Phi(a) = \xi \cdot a$, $a \in A$, is a continuous A-bimodule homomorphism. We then define $D\colon A \to X$ by $D = T - \Phi$. From (4.8) it is easily checked that D is a Jordan derivation. On account of [**16**], D is a derivation. □

Remark 4.2. The natural way to translate conditions (JH1) and (JH2) to the context of derivations consists in considering the following conditions on a linear map $T\colon A \to X$:

$$a, b \in A, \quad ab = ba = 0 \implies T(a) \cdot b + a \cdot T(b) = 0; \tag{JD1}$$

$$a, b \in A, \quad a \circ b = 0 \implies T(a) \bullet b + a \bullet T(b) = 0. \tag{JD2}$$

It should be pointed out that each of the conditions (JD1) and (JD2) implies (JD) and therefore that Theorem 4.1 still works with condition (JD2) replaced by any of the above conditions.

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