J. Austral. Math. Soc. (*Series A*) **58** (1995), 167–182

REGULAR SKEW GROUP RINGS

RICARDO ALFARO, PERE ARA and ANGEL DEL RÍO

(Received 1 February 1992; revised 6 May 1992)

Communicated by P. Schultz

Dedicated to the memory of Pere Menal

Abstract

Let G be a group acting on a ring R. We study the problem of determining necessary and sufficient conditions in order that the skew group ring RG be von Neumann regular. Complete characterizations are given in some particular situations, including the case where all idempotents of R are central. For a regular ring R admitting a G-invariant pseudo-rank function N, with G finite, we obtain a necessary condition for RG being regular in terms of the induced action of G on the N-completion of R.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 16S35; secondary 23B45.

Introduction

Associated to an action of a group G on a ring R, there are two rings: the fixed ring R^G and the skew group ring RG (see Section 1 for definitions). The relationship between the structure of these three rings, R, R^G and RG has been intensively studied by many authors, see [11, 17, 19] for instance.

In this paper we study when the skew group ring is regular in the sense of von Neumann. When the action is trivial the skew group ring is just the classical group ring $R[G]$, and it is well known that $R[G]$ is regular if and only if R is regular, G is locally finite and the order of any finite subgroup of G is invertible in R; this result is due to contributions of Auslander [2], Connell [5] and Villamayor [21].

The first author was partially supported by a grant from the Faculty Development Fund of the University of Michigan-Flint and a fellowship from the C. R. M. Bellaterra, Spain. The second author was partially supported by CICYT grant PB89-0296 and the third author was partially supported by CICYT grant PB87-0703.

In the history of the study of skew group rings (or more generally crossed products) there is Maschke's standard argument which has been extensively used when G is finite and the order of G is invertible in R. This argument also works when the trace of the center of R contains the unit of R, and allows us to obtain a sufficient condition for RG to be regular. This extension was inspired by some results on separable functors; see [13, 20]. In the first section we prove this extension and give general notation and definitions.

In the second section we obtain some necessary conditions for RG to be regular. Namely we prove that if RG is regular, then R is regular, G is locally finite and for every finite subgroup H of G, $1 \in \mathrm{tr}_H(R)$ and R^H is regular. As a consequence, the general problem can be reduced to the particular case of finite groups. In some particular cases (R commutative or G-Galois actions) these necessary conditions are also sufficient.

In the third section we study the particular case when R is abelian regular (that is, regular with all idempotents central). In this case the necessary conditions turn out to be sufficient. It is worth noting that R^G is always abelian regular if R is abelian regular (because every element of an abelian regular ring has a (unique) group inverse).

In the fourth section we start by giving necessary and sufficient conditions for RG to be regular when R is regular right self-injective and G is finite. We show that regular, semiprimitive and semiprime are equivalent conditions for the skew group ring RG. In this particular case, Passman's results on semiprime crossed products are useful. Finally we deal with pseudo-rank functions, obtaining that if N is a G-invariant pseudo-rank function then there is a natural pseudo-rank function $\tilde{N}$ on RG and the action of G on R can be extended to its completion $\bar{R}$ under the topology associated to N. Moreover, the completion of RG under $\tilde{N}$ is homeomorphic under a ring isomorphism to $\bar{R}G$. This result allows us to give an example of a simple regular ring R with a finite group G of outer automorphisms such that RG and R^G are not regular.

1. Sufficient Conditions

Throughout R will denote an associative ring with unity, $Z(R)$ will denote the center of the ring R, and G will stand for a group action on R, that is, there is a group homomorphism $\sigma : G \to \mathrm{Aut}(R)$. If $r \in R$ and $g \in G$, then ${}^g r$ will stand for $\sigma_g(r)$.

The skew group ring RG is a free left R-module with basis G (that is, the elements of RG are finite linear combinations $\sum_{g\in G} r_g g$, with $r_g \in R$ and $g \in G$) and product given by $(rg)(sh) = r({}^g s)gh$. The fixed subring of R under G is $R^G = \{r \in R \mid {}^g r = r, \text{ for all } g \in G\}$. If G is finite, the trace is the map $\mathrm{tr}_G : R \to R^G$, with $\mathrm{tr}_G(r) = \sum_{g\in G} {}^g r$. Note that if H is a subgroup of G and X is a G-invariant subset of R, then $\mathrm{tr}_G(X) \subseteq \mathrm{tr}_H(X)$. Indeed, if $x \in X$ and A is a right transversal of H, then $\mathrm{tr}_G(x) = \mathrm{tr}_H(\sum_{a \in A} {}^a x)$.

The main interest of this paper is to obtain necessary and sufficient conditions for RG to be regular. The case when G acts trivially on R was solved by Auslander [2], Connell [5] and Villamayor [21]: the group ring $R[G]$ is regular if and only if R is regular, G is locally finite and the order of every finite subgroup of G is invertible in R. In the general case it is well known that the sufficient conditions remain true.

PROPOSITION 1.1 (see [19, Prop. 17.2]). *If R is regular, G is locally finite and the order of any finite subgroup is invertible in R then RG is regular. The same is true when we consider a crossed product $R * G$.*

The converse does not hold as the following example shows:

EXAMPLE 1.2. Let A be a regular ring, $n \geq 1$, and set $R = A^n$. Let σ be an automorphism of R given by a cyclic permutation of order n on the factors of R. Put $G = \langle \sigma \rangle$. Then RG is regular (see Theorem 1.3 below) but $|G| = n$ is invertible in R if and only if it is invertible in A.

Proposition 1.1 can be improved using Maschke's standard argument as follows:

THEOREM 1.3. *Let G be a locally finite group acting on a regular ring R. Assume that for every finite subgroup H of G, $1 \in \mathrm{tr}_H(Z(R))$. Then RG is regular.*

PROOF. It is clear that it is enough to prove the theorem for G finite with $1 \in \mathrm{tr}_G(Z(R))$. Let $r \in Z(R)$ be such that $\mathrm{tr}_G(r) = 1$. Let I be a principal left ideal of RG. Then I is a direct summand of RG as left R-modules. Let $\rho \in \mathrm{Hom}_R(RG, I)$ be a projection. Then $f(m) = \sum_{g \in G} gr\rho(g^{-1}m)$ is a projection and $f \in \mathrm{Hom}_{RG}(RG, I)$. It follows that RG is a regular ring.

In Section 3 we will give an example of a regular skew group ring RG with G finite such that 1 does not belong to $\mathrm{tr}(Z(R))$.

2. Necessary conditions

In this section we give some necessary conditions for RG to be regular. We then prove that in some particular cases these conditions are also sufficient, and also give some examples which show that these necessary conditions are not sufficient in general.

Let G be a group acting on a ring R. We can consider R as a right R^G-module by restriction of scalars and as a left RG-module by the rule $(rg) \cdot s = r({}^g s)$ for $r,\ s \in R$,

$g \in G$; so R is an (RG, R^G)-bimodule. Let $A : RG \to R$ be the canonical bimodule homomorphism given by $A(\sum_{g\in G} a_g g) = \sum_{g\in G} a_g$. When G is finite π will denote the element $\sum_{g\in G} g$ in RG.

The following lemmas are well known:

LEMMA 2.1. *Assume G is finite. For any $\alpha \in RG$, $\alpha\pi = A(\alpha)\pi$.*

LEMMA 2.2. *Let X be a set of generators of G. Then $\{x - 1 | x \in X\}$ is a set of generators of* Ker(A) *as a left RG-module.*

PROOF. If $\sum_{g\in G} a_g g \in \operatorname{Ker}(A)$, then $\sum_{g\in G} a_g g = \sum_{g\in G} a_g g - \sum_{g\in G} a_g = \sum_{g\in G} a_g(g-1)$. Thus $\{g - 1 | g \in G\}$ generates Ker(A). Consider now $(g - 1)$ as an element of $R^G G = R^G[G]$. Then, by [18, Lemma 3.1.1], $\{x - 1 | x \in X\}$ generates the augmentation ideal of $R^G[G]$ and hence $(g - 1) \in \sum_{x\in X} R^G(x - 1)$ for all $g \in G$.

LEMMA 2.3 ([11, Prop. 0.3]). *R^G is canonically isomorphic to* End$({}_{RG}R)$.

Now we can give the necessary conditions:

THEOREM 2.4. *Let G be a group acting on a ring R. Assume that RG is regular. Then the following conditions hold:*

(a) *For every subgroup H of G, RH is regular. In particular R is regular.*
(b) *G is locally finite.*
(c) *For every finite subgroup H of G, $1 \in \operatorname{tr}_H(R)$.*
(d) *For every finite subgroup H of G, R^H is regular.*

PROOF. (a) is trivial. By (a) we can assume that G is finitely generated. By Lemma 2.2, Ker(A) is finitely generated; therefore ${}_{RG}R \cong RG/\operatorname{Ker}(A)$ is finitely presented and hence projective. Let $A' : R \to RG$ be the split homomorphism of A. Set $A'(1) = \sum_{h\in G} a_h h$. Then for any $g \in G$,

$$\sum_{h\in G} a_h h = A'(1) = A'(g.1) = gA'(1) = g\sum_{h\in G} a_h h = \sum_{h\in G} {}^g(a_h)gh.$$

Thus $a_{gh} = {}^g(a_h)$ for all g, $h \in G$ and hence $a_g \neq 0$ for any $g \in G$. We conclude that G is finite and this proves (b). Moreover, $1 = AA'(1) = \operatorname{tr}(a_e)$, where e is the unit of G, so (c) follows. Finally, since ${}_{RG}R$ is finitely generated projective and RG is regular, $R^G \cong \operatorname{End}({}_{RG}R)$ is regular.

By Theorem 2.4 the problem of studying when RG is regular can be reduced to the case when G is finite. Moreover the case when R is commutative is completely solved by Theorems 1.3 and 2.4. Explicitly:

https://doi.org/10.1017/S1446788700038210 Published online by Cambridge University Press

COROLLARY 2.5. *Assume R is commutative. RG is regular if and only if R is regular, G is locally finite and $1 \in \operatorname{tr}_H(R)$ for every finite subgroup H of G.*

REMARK. When R is not commutative, a similar result is not possible. Indeed, let G be a finite group and let A be a gr-*regular G-graded ring* (that is, every graded left A-module is flat, see [14]); also assume A is not regular. (For example a group ring $S[G]$with S regular, G finite and $|G|$ not invertible in S.) Then G acts on the smash product $R = A\#G$, and $RG \cong M_{|G|}(A)$ is not regular although R is regular and $1 = \operatorname{tr}(p_1)$ ([4, 3]).

Theorem 2.4 also has some consequences for the regularity of crossed products. A *crossed product $R * G$* is a *G-graded ring A* (that is, a ring with a decomposition into additive subgroups $A = \bigoplus_{g \in G} A_g$ such that $A_g A_h \subseteq A_{gh}$, for every $g,\ h \in G$), in which $A_e = R$ and A_g contains an invertible element for every $g \in G$.

COROLLARY 2.6. *If $R * G$ is a regular crossed product, then G is torsion.*

PROOF. For any subgroup H of G, $R * H$ is also regular, therefore it is enough to prove that if G is cyclic, then it is finite. For that we use the fact that any crossed product on a cyclic group can be expressed as a skew group ring using a diagonal change of basis. Now, by Theorem 2.4 (b), G is cyclic and locally finite, and hence finite.

REMARK. It would be interesting to know if in regular crossed products the group must be locally finite.

From Theorem 2.4 we have that if RG is regular and G is finite, then R^G is regular. But if G is not finite, the regularity of RG does not imply the regularity of R^G, as the following example shows:

EXAMPLE 2.7. Let K be a field of characteristic 0, $S = M_2(K)$ and $P = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. Consider α, the inner automorphism of S defined by P. Let $\mathbb{N}$ denote the set of natural numbers; for any element $x \in S^{\mathbb{N}}$ and any $n \in \mathbb{N}$, x_n will denote the n-th entry of x. Let R be the following subring of the product ring $S^{\mathbb{N}}$:

$$R = \{x \in S^{\mathbb{N}} \mid \text{there exists } n \in \mathbb{N} \text{ such that } x_n = x_{n+k} \text{ for all } k \in \mathbb{N}\}.$$

For every $n \in \mathbb{N}$ consider the automorphism of R given by:

$$(\sigma_n(x))_m = \begin{cases} \alpha^{-1}(x_{m-1}) & \text{if } 1 < m \le 2n \text{ and } m \text{ is even,} \\ \alpha(x_{m-1}) & \text{if } 1 < m < 2n \text{ and } m \text{ is odd,} \\ \alpha(x_{2n}) & \text{if } m = 1, \\ x_m & \text{if } 2n < m. \end{cases}$$

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If any element of $S^{\mathbb{N}}$ is represented by a column vector with matrix entries, then σ_n can be represented by the $\mathbb{N} \times \mathbb{N}$ matrix:

$$A_n = \left(\begin{array}{cccccc|c} 0 & 0 & 0 & \dots & 0 & \alpha & \\ \alpha^{-1} & 0 & 0 & \dots & 0 & 0 & \\ 0 & \alpha & 0 & \dots & 0 & 0 & \mathrm{O} \\ 0 & 0 & \alpha^{-1} & \dots & 0 & 0 & \\ . & . & . & \dots & . & . & \\ . & . & . & \dots & . & . & \\ 0 & 0 & 0 & \dots & \alpha^{-1} & 0 & \\ \hline & & \mathrm{O} & & & & \mathrm{I} \end{array}\right).$$

Indeed $\sigma_n(x) = A_n x$ for any $x \in S^{\mathbb{N}}$. Let G be the subgroup of Aut (R) generated by $\{\sigma_n | n \in \mathbb{N}\}$.

G is locally finite. To prove this it is enough to show that for every $n \in \mathbb{N}$, $\sigma_1, \sigma_2, \dots, \sigma_n$ generates a finite group. Let $k_1, k_2, \dots, k_p \in \{1, 2, \dots, n\}$. Then $(\sigma_{k_p} \cdots \sigma_{k_2}\sigma_{k_1}(x))_m = \beta_p(\sigma_{k_{p-1}} \cdots \sigma_{k_2}\sigma_{k_1}(x))_{r_p}$ where

$$\begin{cases} r_p = m & \text{and } \beta_p = 1 & \text{if } 2k_p < m, \\ r_p = m-1 & \text{and } \beta_p = \alpha^{(-1)^{m+1}} & \text{if } 1 \neq m \leq 2k_p, \\ r_p = 2k_p & \text{and } \beta_p = \alpha & \text{if } m = 1. \end{cases}$$

Thus $(\sigma_{k_p} \cdots \sigma_{k_2}\sigma_{k_1}(x))_m = \beta_p \cdots \beta_2\beta_1(x_{r_1})$, where

$$\begin{cases} r_i = r_{i+1} & \text{and } \beta_i = 1 & \text{if } 2k_i < r_{i+1}, \\ r_i = r_{i+1} - 1 & \text{and } \beta_i = \alpha^{(-1)^{r_{i+1}+1}} & \text{if } 1 \neq r_{i+1} \leq 2k_i, \\ r_i = 2k_i & \text{and } \beta_i = \alpha & \text{if } r_{i+1} = 1. \end{cases}$$

If β_i and β_{i+1} are not 1, then they are inverse to each other because in this case r_i and r_{i+1} have different parity and therefore $\beta_p \cdots \beta_2\beta_1$ is equal to either 1, α or α^{-1}. It is now clear that $\sigma_{k_p} \cdots \sigma_{k_2}\sigma_{k_1}$ can be represented by a matrix $\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$, where $A \in M_{2n}(S)$ has only one non-zero entry in each row and column and this non-zero entry is either 1, α, or α^{-1}. Since there are only a finite number of such matrices, $\langle \sigma_1, \cdots, \sigma_n \rangle$ is finite.

RG is regular. This follows from Proposition 1.1.

R^G is not regular. Indeed, if $x \in R^G$, then there exists $n \in \mathbb{N}$ such that $x_n = x_{n+h}$ for every $h \in \mathbb{N}$. let $k \in \mathbb{N}$ be such that $n + 1 \leq 2k$; then $x_n = x_{n+1} = \alpha(x_n)$ or $x_n = x_{n+1} = \alpha^{-1}(x_n)$ and hence $x_n = \begin{pmatrix} a & 0 \\ b & a \end{pmatrix}$ for some $a, b \in K$. But in this case $x_i = \begin{pmatrix} a & 0 \\ b & a \end{pmatrix}$ for every $i \in \mathbb{N}$. Thus R^G is isomorphic to $\left\{\begin{pmatrix} a & 0 \\ b & a \end{pmatrix} | a, b \in K\right\}$ which is not regular.

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The necessary conditions of Theorem 2.4 turn out to be sufficient when the action is G-Galois. The action is said to be *G-Galois* if R_{R^G} is finitely generated projective and the canonical homomorphism $RG \to \mathrm{End}\,(R_{R^G})$ is a ring isomorphism. This is equivalent to G being finite and $R\pi R = RG$. (See [6, 16].) If R is simple, G is finite, and the action is outer, then RG is simple and consequently the action is G-Galois.

PROPOSITION 2.8. *Let R be a ring with G-Galois action. RG is regular if and only if R^G is regular. Furthermore, in this case RG and R^G are Morita equivalent.*

PROOF. We already know that if RG is regular, then R^G is regular. Assume R^G is regular; then $RG \cong \mathrm{End}\,(R_{R^G})$ is also regular [8, Theorem 1.7]. In this case $1 \in \mathrm{tr}(R)$, hence R_{R^G} is a progenerator and therefore RG and R^G are Morita equivalent.

REMARK. Unfortunately Proposition 2.8 cannot be generalized to an arbitrary group action. Indeed, let K be a field of characteristic 2 and let $R = M_2(K)$. Consider G to be the group of invertible matrices in $M_2(\mathbb{Z}_2)$ acting on R by conjugation. It is easy to verify that $R^G = \left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} | a \in K\right\}$ and $\mathrm{tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. If RG is regular then RH is also regular, where H is the subgroup generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Hence R^H is also regular; but this is not the case because $R^H = \left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} | a,\, b \in K\right\}$. Therefore RG is not regular.

We finish this section with another special case where we have a complete characterization of regular skew group rings.

PROPOSITION 2.9. *Assume that any non-trivial element of G has order 2 and that R is an algebra over a field K. Then RG is regular if and only if R^H is regular and $1 \in \mathrm{tr}_H(R)$ for every finite subgroup H of G.*

PROOF. We already know that the necessary conditions hold. In the other direction, from the assumption on G, we know that G is abelian and locally finite. Therefore, in order to prove that RG is regular, it is enough to assume that G is finite and isomorphic to a finite direct product of copies of the group with two elements. Moreover, if K has characteristic different from 2, then the claim follows from Proposition 1.1.

Assume K has characteristic 2. We will finish the proof by induction on the order of G. Assume $|G| = 2$ and let $r \in R$ with $\mathrm{tr}(R) = 1$; then $r\pi + \pi({}^g r) = 1$ and hence the action is G-Galois. The regularity of RG follows from Proposition 2.8. Now assume that $2 < |G|$. There exist two subgroups H and L with $|H| = 2$ and $G = H \times L$. Then H acts on RL by ${}^h\alpha = h\alpha h^{-1}$. By the induction hypothesis RL is regular and obviously $1 \in \mathrm{tr}_H(RL)$. Moreover, $RG = (RL)H$. Therefore it only remains to show that $(RL)^H$ is regular. Note that R^H is L-invariant and $(RL)^H = (R^H)L$, furthermore for every subgroup L' of L, $(R^H)^{L'} = R^{H\times L'}$ is regular, and if $\mathrm{tr}_{H\times L'}(r) = 1$, then $\mathrm{tr}_{L'}(\mathrm{tr}_H(r)) = \mathrm{tr}_{H\times L'}(r) = 1$ and $\mathrm{tr}_H(r) \in R^H$. Therefore $(R^H)L$ is regular by the induction hypothesis.

3. Abelian regular rings

In this section we present another case in which the necessary conditions are also sufficient.

Let R be an *abelian regular ring* as in [8] (that is, R is a regular ring and all idempotents are central). Let G be a finite group acting on R and let $B = B(R)$ be the boolean algebra of idempotents of R, with operations $e \vee f = e + f - ef$ and $e \wedge f = ef$. Clearly $\sigma_g(e \vee f) = \sigma_g(e) \vee \sigma_g(f)$ and $\sigma_g(e \wedge f) = \sigma_g(e) \wedge \sigma_g(f)$; therefore G acts on B and $B^G = B(R^G)$. Since $B^G \subseteq Z(RG)$, for every prime ideal m of B^G we can consider the central localizations $(RG)_m = (RG)_{B^G\setminus m}$ and R_m. Obviously R_m is contained in $(RG)_m$, R_m is G-invariant and $(RG)_m \cong R_m G$.

PROPOSITION 3.1. *With the above conditions, R_m is regular and* 0 *and* 1 *are the unique idempotents of $(R_m)^G$.*

PROOF. Let $\phi_m : R \to R_m$ be the canonical map. If $e \in B^G\setminus m$ then $e(e-1) = 0$ and hence $\phi_m(e) = \phi_m(1) = 1$. Therefore $r/e = \phi_m(r)\phi_m(e)^{-1} = \phi_m(r)$ and hence ϕ_m is epic. We now show that $\mathrm{Ker}(\phi_m) = mR$. If $x \in m$, then $1 - x \in B^G\setminus m$ and $(1-x)x = 0$; thus $\phi_m(x) = 0$. Now let $x \in \mathrm{Ker}(\phi_m)$. Then $ex = 0$ for some $e \in B^G\setminus m$; and thus $x = (1-e)x$ and $1 - e \in m$. Therefore $R_m \cong R/Rm$. If p is an idempotent of R/Rm which is fixed by the action of G, then there exists $e \in B$ such that $\bar{e} = p$. Since p is fixed, e can be chosen in B^G (otherwise we can change it by $f = \bigvee_{g\in G}\sigma_g(e)$) and thus $e \in m$ or $1 - e \in m$. But $p = \bar{e}$, so $p = 0$ or $p = 1$.

LEMMA 3.2. *Let G be a finite group acting on an abelian regular ring R such that the only fixed idempotents are* 0 *and* 1. *Then R is semisimple artinian. In fact, $R \cong D^n$, where D is a division ring and the action of G on R permutes transitively the factors of D^n.*

PROOF. First we will see that R has minimal idempotents. If not, for every non-zero idempotent $f \in R$ there are orthogonal non-zero idempotents f_1 and f_2 such that $f = f_1 + f_2$. Now $\bigvee_{g\in G}\sigma_g(f_1) = 1$. So there exists $x \in G$ such that $f_2\sigma_{x^{-1}}(f_1) \neq 0$, and then $g_1 = f_2\sigma_{x^{-1}}(f_1)$ and $g_2 = f_1$ are non-zero orthogonal idempotents such that

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$\sigma_x(g_1) \leq g_2$. We can construct in this way, by induction, two sequences of non-zero idempotents $\{f_{1,k}\}$ and $\{f_{2,k}\}$ such that for all $k \in \mathbb{N}$, $f_{1,k}$ and $f_{2,k}$ are orthogonal, $f_{1,k+1} < f_{1,k}$ and $f_{2,k+1} < f_{1,k}$ and $\sigma_{x_k}(f_{1,k}) \leq f_{2,k}$ for some $x_k \in G$. Observe that all x_k must be different; thus since G is finite we obtain a contradiction and there exists a minimal idempotent.

Let e be a minimal idempotent of R. If $D = eR$, then D is a division ring. Therefore, if $e_1, e_2, \ldots, e_n$ are the different elements of the form $\sigma_g(e)$ for some $g \in G$, we can express $R = e_1R \oplus e_2R \oplus \cdots \oplus e_nR \cong D^n$ and the action of G permutes transitively the factors.

THEOREM 3.3. *Let R be an abelian regular ring and let G be a finite group acting on R. If $1 \in \mathrm{tr}(R)$, then RG is regular.*

PROOF. Firstly we note that $(RG)_m$ is regular for any $m \in \operatorname{Spec}(B^G)$. This is because $(RG)_m = R_mG$, R_m has no fixed idempotents other than 0 and 1 and then, by Lemma 3.2, R_m and hence R_mG is artinian. By [19, Theorem 27.7] R_mG is also semiprime and hence it is regular.

Let $x \in RG$ and set $I = \{f \in B^G \mid \text{there exists } y \in RG \text{ such that } fx = xyx\}$. Clearly if $f \in I$, the corresponding element y can be chosen in $f(RG)$. If $e \in B^G$ and $f \in I$, put $y_1 = ey \in RG$; thus $efx = xy_1x$. On the other hand, if $f_1, f_2 \in I$, $f_1x = xy_1x$ and $f_2x = xy_2x$ by taking $y = y_1 + (1 - f_1)y_2$ one has:

$$(f_1 \vee f_2)x = (f_1 + f_2 - f_1f_2)x = xy_1x + x(1 - f_1)y_2x = xyx$$

So I is an ideal of B^G. It only remains to prove that $I = B^G$. If not, let m be a maximal ideal of B^G containing I. Since $(RG)_m$ is regular, there exists $y \in RG$ such that $\phi_m(x) = \phi_m(x)\phi_m(y)\phi_m(x)$. Thus there exists $e \in B^G \setminus m$ such that $e(x - xyx) = 0$. So $e \in I \subseteq m$ which yields a contradiction.

We finish with an example showing that RG regular does not imply that $1 \in \mathrm{tr}(Z(R))$ in general.

EXAMPLE 3.4. Let $K = \mathbb{Z}_2(t)$ be the field of fractions of $\mathbb{Z}_2[t]$ and consider the automorphism $\sigma : K \to K$ for which $\sigma(t) = t + 1$. Let D be the classical ring of quotients of the skew polynomial ring $K[x, \sigma]$. The automorphism σ extends to an automorphism σ of D by setting $\sigma(x) = x$. Consider the group $G = \langle \sigma \rangle$; since $\mathrm{tr}(t) = 1$ and D is abelian regular, DG is regular but $\mathrm{tr}(Z(D))$ does not contain 1.

4. G-invariant pseudo-rank functions

In this section we first obtain a characterization of the regular skew group rings RG, where G is a finite group acting on a regular right self-injective ring R. Then we give a new necessary condition for certain skew group rings to be regular.

If R is a regular ring and N is a G-invariant pseudo-rank function on R, we can extend N to RG following techniques of [10, 1]. We also can extend G to an action on the N-completion $\bar{R}$ of R. Generalizing a result of Kado [10, Theorem 7], we show that $\overline{RG} \cong \bar{R}G$ where $\overline{RG}$ is the completion of RG with respect to the extension of N. This implies that if RG is regular, then so is $\bar{R}G$. Since $\bar{R}$ is regular and self-injective, this gives us a useful tool to prove that certain skew group rings are not regular. As an application we construct a certain skew group ring RG which is not regular, where R is a simple regular ring and G is an outer group of automorphisms which has order 2. An example of such a skew group ring was given in [9, Prop. 8.2], but it turns out that their example gives a regular skew group ring. If R is a ring, $J(R)$ will denote the Jacobson radical of R.

PROPOSITION 4.1. *Let G be a finite group acting on a regular right self-injective ring R. The following conditions are equivalent:*

(a) *RG is regular.*
(b) *RG is semiprimitive.*
(c) *RG is semiprime.*

PROOF. The proofs that (a) implies (b) implies (c) are well known. Assume RG is semiprime. By [19, Theorem 4.2] $J(RG)^{|G|} \subseteq J(R)G = 0$. Moreover, RG is regular right self-injective (see [12]) and consequently $RG = RG/J(RG)$ is regular and right self-injective.

One can see, by using the techniques in [19, Section 18], that, for a regular right self-injective ring R and a finite group G acting on R, RG is semiprime if and only if the centralizer $C_{RG}(R)$ of R in RG is semiprime; and the latter is a finite extension of $C = Z(R)$. If, in addition, R is prime, then $C_{RG}(R)$ is isomorphic to $C'[G_{\text{inn}}]$ where G_{inn} is the normal subgroup of G consisting of elements that act by inner automorphisms, and $C'[G_{\text{inn}}]$ is some twisted group ring of G_{inn} over C. See [19, Lemma 12.4].

Let R be any ring. A *pseudo-rank function* on R is a map $N : R \to [0, 1]$ such that:

(a) $N(x + y) \leq N(x) + N(y)$;
(b) $N(xy) \leq \min(N(x), N(y))$;
(c) $N(e + f) = N(e) + N(f)$ for any two orthogonal idempotents $e, f \in R$;
(d) $N(1) = 1$.

A *rank function* is a pseudo-rank function with the additional property:

(e) $N(x) = 0$ if and only if $x = 0$.

When R is a regular ring, Condition (a) follows from (b), (c) and (d). If R is a ring with pseudo-rank function N the rule $\delta(x, y) = N(x - y)$ defines a pseudo-metric on R, which is a metric if and only if N is a rank function. Since the operations on R are continuous in the pseudo-metric, the (Hausdorff) completion $\bar{R}$ of R becomes a ring called the N-completion of R. If R is regular then $\bar{R}$ is a unit-regular, left and right self-injective ring: see [8, Theorem 19.7]. Let G be a group acting on a ring R. A pseudo-rank function N on R is called *G-invariant* if $N({}^g r) = N(r)$ for all $r \in R$, $g \in G$. It is easy to prove that if the action of G is *inner* or *locally inner* (in the sense that for every $x \in R$, $g \in G$ there exists a unit $u \in R$ such that ${}^g x = u^{-1} x u$), then every pseudo-rank function on R is G-invariant. If N is a G-invariant pseudo-rank function on R, the action of G can be extended to the completion $\bar{R}$ in the obvious way.

Denote by $P(R)$ the set of all pseudo-rank functions on a regular ring R. Then $P(R)$ is a compact convex subset of $\mathbb{R}^R$; moreover, there exists an affine homeomorphism $P(R) \cong S(K_o(R), [R])$, where $S(K_o(R), [R])$ is the state space of the partially pre-ordered abelian group $K_o(R)$ with order-unit $[R]$; see [8, Propositions 16.17 and 17.12].

If G is a group acting on a ring R, the rule $N^g(x) = N({}^g x)$ defines an action of G on $P(R)$ by affine homeomorphisms. Similarly we can define an action of G on $S(K_o(R), [R])$ such that the canonical affine homeomorphism from $P(R)$ onto $S(K_o(R), [R])$ is G-equivariant. If G is finite we can see, as in [1, Theorem 3], that there exists an affine embedding from $S(K_o(R), [R])^G$ into $S(K_o(RG), [RG])$.

It is obvious that, if R admits a unique pseudo-rank function N, then N is G-invariant for any group G acting on R. Note also that if N is a pseudo-rank function on R, then $N_o = |G|^{-1} \sum_{g \in G} N^g$ is a G-invariant pseudo-rank function on R.

The following result was obtained by Kado [10] in the special case that $|G|^{-1} \in R$ and N is an extreme point in $P(R)$. For a Cauchy sequence $\{x^{(k)}\}$ in X, $[x^{(k)}]$ will denote its equivalence class in the completion $\bar{X}$.

THEOREM 4.2. *Let G be a finite group acting on a regular ring R and let N be a G-invariant pseudo-rank function on R. Let $\bar{N}$ be the rank function obtained by extending N to $\bar{R}$. Then:*

(a) *N induces a pseudo-rank function $\tilde{N}$ on RG and the $\tilde{N}$-completion of RG is isomorphic to $\bar{R}G$.*
(b) *The rank function induced by $\bar{N}$ on $\bar{R}G$ corresponds with the natural extension of $\tilde{N}$ to $\overline{RG}$ under the isomorphism in* (a).

PROOF. As in [10, 1] we can define $\tilde{N}$ in the following way:

$$\tilde{N}(x) = |G|^{-1} s([xRG_R]), \qquad x \in RG$$

https://doi.org/10.1017/S1446788700038210 Published online by Cambridge University Press

where s is the state corresponding to the pseudo-rank function N in the affine homeomorphism $P(R) \cong S(K_o(R), [R])$. Set $n = |G|$ and note that $RG \cong R^n$ as right R-modules. We will see that the $\tilde{N}$-topology on RG is the same as the product topology on R^n. In order to show this, we need the following:

Claim 1. $s([(\sum_{g\in G} a_g g)RG_R]) \leq \sum_{g\in G} s([a_g RG_R])$ where $a_g \in R$ for all $g \in G$.

Claim 2. For all $a \in R$, $s([aRG_R]) = |G|N(a)$.

Proof of claim 1: We have $(\sum_{g\in G} a_g g)RG \leq \sum_{g\in G}(a_g g)RG = \sum_{g\in G} a_g RG$ and $\sum_{g\in G}(a_g RG)_R$ is isomorphic to a direct summand of $\bigoplus_{g\in G}(a_g RG)_R$ since $\sum_{g\in G}(a_g RG)_R$ is projective. We conclude that $s([(\sum_{g\in G} a_g g)RG_R]) \leq \sum_{g\in G} s([a_g RG_R])$.

Proof of claim 2: Observe that $RG = \bigoplus_{g\in G} gR$ and obviously $gR_R \cong R_R$ by left multiplication by g. Now $aRG_R = \bigoplus_{g\in G} agR = \bigoplus_{g\in G} g({}^{g^{-1}}a)R \cong \bigoplus_{g\in G}({}^{g^{-1}}a)R$. Consequently $s([aRG_R]) = \sum_{g\in G} s([({}^{g^{-1}}a)R]) = \sum_{g\in G} N({}^{g^{-1}}a) = |G|N(a)$ since N is G-invariant.

To see that the isomorphism $RG \cong R^n$ is a homeomorphism, it suffices to show that for sequences $\{a_g^{(k)}\}_{k=1}^{\infty}$ for each $g \in G$, $\lim_{k\to\infty} \sum_{g\in G} a_g^{(k)} g = 0$ in the $\tilde{N}$-metric if and only if $\lim_{k\to\infty} a_g^{(k)} = 0$ in the N-metric for all $g \in G$.

Assume first that $\lim_{k\to\infty} a_g^{(k)} = 0$ in the N-metric. By using Claims 1 and 2, we compute:

$$\begin{aligned}\tilde{N}\left(\sum_{g\in G} a_g^{(k)} g\right) &= |G|^{-1} s\left(\left[\left(\sum_{g\in G} a_g^{(k)} g\right) RG_R\right]\right)\\ &\leq |G|^{-1} \sum_{g\in G} s\left(\left[a_g^{(k)} RG_R\right]\right) = \sum_{g\in G} N\left(a_g^{(k)}\right)\end{aligned}$$

and it follows that $\tilde{N}(\sum_{g\in G} a_g^{(k)} g) \to 0$ as $k \to \infty$.

Now assume that $\lim_{k\to\infty} \sum_{g\in G} a_g^{(k)} g = 0$ in the $\tilde{N}$-metric. Fix $h \in G$ and observe that $(\sum_{g\in G} a_g^{(k)} g)R \leq \bigoplus_{g\in G} a_g^{(k)} gR$. Consider the canonical projection $\bigoplus_{g\in G} a_g^{(k)} gR \to a_h^{(k)} hR$. The restriction of this map to $(\sum_{g\in G} a_g^{(k)} g)R$ gives us an onto homomorphism of R-modules $(\sum_{g\in G} a_g^{(k)} g)R \to a_h^{(k)} hR$. Since $a_h^{(k)} hR$ is a projective R-module, $a_h^{(k)} hR$ is isomorphic to a direct summand of $(\sum_{g\in G} a_g^{(k)} g)R$. Consequently:

$$\begin{aligned}\tilde{N}\left(\sum_{g\in G} a_g^{(k)} g\right) &= |G|^{-1} s\left(\left[\left(\sum_{g\in G} a_g^{(k)} g\right) RG_R\right]\right) \geq |G|^{-1} s\left(\left[\left(\sum_{g\in G} a_g^{(k)} g\right) R\right]\right)\\ &\geq |G|^{-1} s\left(\left[a_h^{(k)} hR\right]\right) = |G|^{-1} s\left(\left[{}^{h^{-1}}\left(a_h^{(k)}\right) R\right]\right) = |G|^{-1} N\left({}^{h^{-1}}\left(a_h^{(k)}\right)\right)\\ &= |G|^{-1} N\left(a_h^{(k)}\right).\end{aligned}$$

It follows that $\lim_{k\to\infty} N(a_h^{(k)}) = 0$. Since this holds for every $h \in G$, we have proved our statement about the topologies.

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In particular we see that a sequence $\{\sum_{g\in G} a_g^{(k)} g\}$ is a Cauchy sequence in the $\tilde{N}$-topology if and only if $\{a_g^{(k)}\}$ is a Cauchy sequence in the N-topology for all $g \in G$. Consequently the map $\overline{RG} \to \bar{R}G$ which sends $[\sum_{g\in G} a_g^{(k)} g]$ to $\sum_{g\in G}[a_g^{(k)}]g$, is well defined and a ring isomorphism. It is now straightforward to see that the natural extension of $\tilde{N}$ to $\overline{RG}$ corresponds to the extension of the (G-invariant) rank function $\bar{N}$ to $\bar{R}G$ under the isomorphism defined above.

COROLLARY 4.3. *Let G be a finite group acting on a regular ring R and N be a G-invariant pseudo-rank function on R.*

(a) *If RG is regular, then $\bar{R}G$ is regular.*
(b) *If $\bar{R}G$ is regular, then $J(RG) \subseteq \operatorname{Ker}(\tilde{N})$. If, in addition, N is a rank function then RG is semiprimitive.*

PROOF. (a) If RG is regular, then so is its completion with respect to a pseudo-rank function. By Theorem 4.2 we have $\overline{RG} \cong \bar{R}G$, so we deduce that $\bar{R}G$ is regular. (b) By [19, Theorem 4.2] we have $J(RG)^n = 0$ where $n = |G|$. Let $x \in J(RG)$. There exists a sequence $\{y_k\}$ in RG such that the limit of $\{\tilde{N}(x - xy_kx)\}$ is 0. Now we have $\tilde{N}(x) = \tilde{N}(x - (xy_k)^n x) \le n\tilde{N}(x - xy_kx)$. It follows that $x \in \operatorname{Ker}(\tilde{N})$ and consequently $J(RG) \subseteq \operatorname{Ker}(\tilde{N})$.

We shall use Corollary 4.3 to obtain an example of a simple regular ring with an outer action such that the skew group ring is not regular. Such an example was offered in [9, Prop. 8.2], but in fact it gives a regular ring as we now show:

The correct form of the fixed ring should be

$$A^{\sigma} \cap M_{2^{n+1}}(F) = \left\{ \begin{pmatrix} P & Q \\ \operatorname{tr}(P) & P + \operatorname{tr}(P+Q) \end{pmatrix} \mid P,\ Q \in M_{2^n}(F) \right\}$$

and if $[\alpha] \in A^{\sigma}$ with $P \in A_n = M_{2^n}(F)$ such that $[\alpha][P][\alpha] = [\alpha]$, then it is easy to see that $[\alpha]\left[\begin{pmatrix} P & 0 \\ \operatorname{tr}(P) & P + \operatorname{tr}(P) \end{pmatrix}\right][\alpha] = [\alpha]$ and the fixed ring is regular. Furthermore since the action is G-Galois, the skew group ring is also regular.

For a non-regular skew group ring we give the following example, which is a variation of the one given in [9]:

EXAMPLE 4.4. A simple regular ring with an outer action such that the skew group ring is not regular:

Let F be the field with two elements. For any $n \ge 1$ let $R_n = M_{t(n)}(F)$ where $t(1) = 1$ and $t(n+1) = 2^n t(n)$. For any $n \ge 1$ consider the diagonal map $\psi_n : R_n \to R_{n+1}$ and the canonical map $\phi_n : R_n \to R = \lim_{n\ge 1} M_{t(n)}(F)$. Clearly R is a simple regular ring.

For any $n \geq 1$ define $\lambda_n \in R_n$ by $\lambda_1 = 1$ and $\lambda_n = \begin{pmatrix} I & 0 & \dots & 0 & I \\ 0 & I & \dots & 0 & 0 \\ . & . & \dots & . & . \\ 0 & 0 & \dots & 0 & I \end{pmatrix}$ where the matrix is decomposed into $t(n-1)$-blocks in R_n and I denotes the identity matrix. Define x_n by induction as follows: $x_1 = 1 \in R_1$ and

$$x_{n+1} = \begin{pmatrix} x_n & 0 & \dots & 0 & \lambda_n x_n \\ 0 & x_n & \dots & 0 & 0 \\ . & . & \dots & . & . \\ 0 & 0 & \dots & 0 & x_n \end{pmatrix} \in R_{n+1}.$$

It is easy to see that λ_n commutes with x_n and that $x_n^2 = 1$ for all $n \geq 1$. Now define an automorphism σ of R as follows:

$$\sigma(\phi_n(A)) = \phi_{n+1}(x_{n+1} \cdot \psi_n(A) \cdot x_{n+1}) \qquad (A \in R_n).$$

Since $\lambda_{n+1} \in C_{R_{n+1}}(\psi_n(R_n))$, it is easy to see that σ is a well defined automorphism of order two on R, and clearly σ is outer; furthermore let $G = \langle \sigma \rangle$.

Let N be the unique rank function on R, given by $N(\phi_n(A)) = \operatorname{rank}(A)/t(n)$ for $A \in R_n$. The rank function N is G-invariant and σ extends to an automorphism $\bar{\sigma}$ of $\bar{R}$. Since $N(\phi_{n+1}(x_{n+1}) - \phi_n(x_n)) = 1/2^n$, we see that $\{\phi_n(x_n)\}$ is a Cauchy sequence in R. Now put $x = \lim_{n\to\infty} \phi_n(x_n) \in \bar{R}$. It is clear that $\bar{\sigma}$ is an inner automorphism of $\bar{R}$, given by conjugation by x. Since the center of $\overline{R}$ is isomorphic to F (see [7, Theorem 2.8(c)]), and $G_{\text{inn}} = G$ on $\bar{R}$, we obtain that $C^t[G_{\text{inn}}] = F[G]$ is not semiprime. It follows from [19, Lemma 18.8(ii)] that $\bar{R}G$ is not semiprime and consequently not regular. Finally, Corollary 4.3(a) implies that RG is not regular.

REMARK. Note that in the previous example R^G is not regular and $1 \in \operatorname{tr}(R)$. Indeed, the first is a consequence of Proposition 2.8, and an element of trace 1 is $\phi_3(\omega)$ where ω, decomposed in 2-blocks, is given by

$$\omega = \begin{pmatrix} x & 0 & 0 & y \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{pmatrix} \qquad \text{where} \quad x = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

It follows that RG and R^G are Morita equivalent. In particular R^G is a simple ring and its completion with respect to the rank function N restricted to R^G is $\bar{R}^G$ which is a self-injective non-regular ring.

Finally we obtain a result which implies that in Example 4.4 the ring RG is neither left nor right P. P. (A ring R is said to be *left P. P.* if every cyclic left ideal is projective.)

PROPOSITION 4.5. *Let G be a finite group acting on a regular ring R. If RG is a left P.P. ring, then R^G is regular.*

PROOF. Let $r \in R^G$. Since $P = (RG)(\pi r)$ is a projective RG-module, the map $RG \to P$ given by right multiplication by πr splits. Let $\varphi : P \to RG$ be a splitting for this map. Write $\varphi(\pi r) = \sum_{g\in G} a_g g$. As in the proof of Theorem 2.4, $a_g = {}^g a_1$ for all $g \in G$. Set $a = a_1$. Thus $a \in r_R(l_R(r)) = rR$ and consequently $a = rb$ for some $b \in R$. Note that $\pi r = \varphi(\pi r)\pi r = \pi a \cdot \pi r = \pi \mathrm{tr}(a)r$. So $r = \mathrm{tr}(a)r = \mathrm{tr}(rb)r = r\mathrm{tr}(b)r$. It follows that R^G is regular.

It was claimed in [15, Corollary 4] that, for a simple ring S and a finite group G of outer automorphisms of S, S^G is a simple ring if and only if the homological dimension of any left SG-module A coincides with the homological dimension of A viewed as an S-module. However, the ring R of Example 4.4 gives a counterexample to this claim. Indeed, every cyclic left ideal of RG is projective as a left R-module because R is regular, but it follows from Proposition 4.5 that at least one cyclic left ideal of RG is not projective as an RG-module.

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Department of Mathematics
University of Michigan-Flint
Flint, MI 48502
U.S.A.

Departament de Matemàtiques
Universitat Autònoma de Barcelona
(08193) Bellaterra (Barcelona)
Spain

Departamento de Matemáticas
Universidad de Murcia
(30001) Murcia
Spain