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Continuity of critical exponent of quasiconvex-cocompact groups under Gromov–Hausdorff convergence

NICOLA CAVALLUCCI

Mathematisches Institut, Universitat Köln, Weyertal 86–90, 50931 Köln, Germany
(e-mail: n.cavallucci23@gmail.com)

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Abstract. We show continuity under equivariant Gromov–Hausdorff convergence of the critical exponent of discrete, non-elementary, torsion-free, quasiconvex-cocompact groups with uniformly bounded codiameter acting on uniformly Gromov-hyperbolic metric spaces.

Key words: Gromov hyperbolicity, Gromov–Hausdorff convergence, entropy, continuity, convex-cocompact
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Contents

1. *Introduction*

The *critical exponent* of a discrete group of isometries Γ of a proper metric space X is defined as

$$h_\Gamma := \limsup_{T\to+\infty} \frac{1}{T} \log \Gamma x \cap \overline{B}(x, T),$$

where x is any point of X. In [Cav21b] the author proved that if X is a Gromov-hyperbolic space then the limit superior above is a true limit (see also Lemma 4.12).

A discrete group Γ of isometries of a proper, δ-hyperbolic metric space is said to be quasiconvex-cocompact if it acts cocompactly on the quasiconvex hull of its limit set $\Lambda(\Gamma)$, namely QC-Hull($\Lambda(\Gamma)$). In this case the codiameter is by definition the diameter of the quotient metric space $\Gamma\backslash$QC-Hull($\Lambda(\Gamma)$).

In the sequel we denote by $\mathcal{M}(\delta, D)$ the class of triples (X, x, Γ) where X is a proper, δ-hyperbolic metric space, Γ is a discrete, torsion-free, non-elementary, quasiconvex-cocompact group of isometries of X with codiameter less than or equal to D and x is a point of QC-Hull($\Lambda(\Gamma)$). We refer to §4 for the details of all these definitions.

We are interested in convergence of sequences of triples in $\mathcal{M}(\delta, D)$ in the equivariant pointed Gromov–Hausdorff sense, as defined by Fukaya in [Fuk86]. This is a version of the classical pointed Gromov–Hausdorff convergence that considers also the groups acting on the spaces. Its precise definition is recalled in §3. Our main result is the following theorem.

THEOREM A. *Let $\delta, D \geq 0$ and let $(X_n, x_n, \Gamma_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}(\delta, D)$. If the sequence (X_n, x_n, Γ_n) converges in the equivariant pointed Gromov–Hausdorff sense to $(X_\infty, x_\infty, \Gamma_\infty)$ then:*

(i) $(X_\infty, x_\infty, \Gamma_\infty) \in \mathcal{M}(\delta, D)$*; and*

(ii) $h_{\Gamma_\infty} = \lim_{n\to+\infty} h_{\Gamma_n}$.

The first difficulty in the proof of (i) is to show that the limit group Γ_∞ is discrete. The proof is based on a result of [BCGS21]: if $(X, x, \Gamma) \in \mathcal{M}(\delta, D)$ satisfies $h_\Gamma \leq H < +\infty$, then the global systole of Γ is bigger than some positive constant depending only on δ, D and H (cf. Proposition 5.3). This is a powerful tool when used together with Corollary 5.9: under the assumptions of Theorem A the critical exponents of the groups Γ_n are uniformly bounded above by some $H < +\infty$. All the assumptions on the class $\mathcal{M}(\delta, D)$ are necessary in order to get the discreteness of the limit group; see §8. The second difficulty is to show that Γ_∞ is quasiconvex-cocompact. In order to do so we will show that the limit of the Gromov boundaries ∂X_n can be seen as a canonical subset of ∂X_∞; see Proposition 5.11. Under this identification the limit of the sets $\Lambda(\Gamma_n)$ coincides with $\Lambda(\Gamma_\infty)$.

The proof of the continuity statement, Theorem A(ii), is based on the following uniform equidistribution of the orbits. It is a quantified version of a result of Coornaert [Coo93].

THEOREM B. *Under the assumptions of Theorem A there exists $K > 0$ such that, for every n and for every $T \geq 0$,*

$$\frac{1}{K} \cdot e^{T \cdot h_{\Gamma_n}} \leq \Gamma_n x_n \cap \overline{B}(x_n, T) \leq K \cdot e^{T \cdot h_{\Gamma_n}}.$$

In the literature the behaviour of the critical exponent under another kind of convergence, algebraic convergence, was previously studied by Bishop and Jones in the case of hyperbolic manifolds [**BJ97**] and in a more general setting by Paulin in [**Pau97**]. The definition of algebraic convergence, as well as the notation below, are recalled in §7. They proved the following result, which we refer to as the BJP theorem.

THEOREM. [**BJ97**, **Pau97**] *Let X be a geodesic, δ-hyperbolic metric space such that for each $x, y \in X$ there is a geodesic ray issuing from x passing at distance no more than δ from y and let G be a finitely generated group. Let $\varphi_n, \varphi_\infty \colon G \to \mathrm{Isom}(X)$ be homomorphisms. If $\varphi_n(G)$ converges algebraically to $\varphi_\infty(G)$, if $\varphi_n(G)$, $\varphi_\infty(G)$ are discrete and if $\varphi_\infty(G)$ has no global fixed point at infinity, then $h_{\varphi_\infty(G)} \leq \liminf_{n\to+\infty} h_{\varphi_n(G)}$.*

We point out here the main differences and analogies between this statement and Theorem A(ii).

- In the BJP theorem the isomorphism type of the group G is fixed. A posteriori this is not restrictive: under the assumptions of Theorem A the isomorphism type of the groups Γ_n is eventually constant (Corollary 7.7). The proof of Theorem A(i) does not use this property.
- The algebraic limit is a priori different from the equivariant pointed Gromov–Hausdorff limit (see Example 7.3). However, if the spaces X_n are all isometric and satisfy the assumptions of Theorem A then the two limits coincide (Theorem 7.4).
- In Theorem A the spaces X_n can be pairwise non-isometric. In that case the notion of algebraic convergence cannot be defined. This is a posteriori the main difference between algebraic convergence and equivariant pointed Gromov–Hausdorff convergence on the class $\mathcal{M}(\delta, D)$.

2. *Preliminaries on metric spaces*

Throughout this paper X will denote a metric space and d will denote the metric on X. The open (respectively, closed) ball of radius r and centre x is denoted by $B(x, r)$ (respectively, $\overline{B}(x, r)$). A geodesic segment is an isometry $\gamma \colon I \to X$ where $I = [a, b]$ is a bounded interval of $\mathbb{R}$. The points $\gamma(a), \gamma(b)$ are called the endpoints of γ. A metric space X is said to be geodesic if for all couple of points $x, y \in X$ there exists a geodesic segment whose endpoints are x and y. We will denote any geodesic segment between two points x and y, in an abuse of notation, by $[x, y]$. A geodesic ray is an isometry $\gamma \colon [0, +\infty) \to X$, while a geodesic line is an isometry $\gamma \colon \mathbb{R} \to X$.

The group of isometries of a proper metric space X (that is, closed balls are compact) is denoted by $\mathrm{Isom}(X)$ and it is endowed with the topology of uniform convergence on compact subsets of X.

If Γ is a subgroup of $\mathrm{Isom}(X)$ we define $\Sigma_R(\Gamma, x) := \{g \in \Gamma \text{ s.t. } d(x, gx) \leq R\}$ and $\Gamma_R(x) := \langle \Sigma_R(\Gamma, x) \rangle$, for every $x \in X$ and $R \geq 0$. When the context is clear we simply write $\Sigma_R(x)$. A subgroup Γ is said to be discrete if equivalently:

(i) it is a discrete subspace of $\mathrm{Isom}(X)$;

(ii) $\#\Sigma_R(x) < +\infty$ for all $x \in X$ and all $R \geq 0$.

The *systole* of Γ at $x \in X$ is the quantity

$$\mathrm{sys}(\Gamma, x) := \inf\{d(x, gx) \text{ s.t. } g \in \Gamma \setminus \{\mathrm{id}\}\},$$

while the *global systole* of Γ is $\mathrm{sys}(\Gamma, X) := \inf_{x\in X} \mathrm{sys}(\Gamma, x)$. If any non-trivial isometry of Γ has no fixed points then the systole at a point is always strictly positive by discreteness.

3. *Convergence of group actions*

First, we recall the definition and the properties of the equivariant pointed Gromov–Hausdorff convergence. Then we compare this notion with ultralimit convergence.

3.1. *Equivariant pointed Gromov–Hausdorff convergence.*

We consider triples (X, x, Γ) where (X, x) is a pointed, proper metric space and $\Gamma < \mathrm{Isom}(X)$. The following definitions are due to Fukaya [Fuk86].

Definition 3.1. Let (X, x, Γ), (Y, y, Λ) be two triples as above and $\varepsilon > 0$. An equivariant ε-approximation between them is a triple (f, φ, ψ) where:

- $f: B(x, 1/\varepsilon) \to B(y, 1/\varepsilon)$ is a map such that
  - $f(x) = y$,
  - $|d(f(x_1), f(x_2)) - d(x_1, x_2)| < \varepsilon$ for every $x_1, x_2 \in B(x, 1/\varepsilon)$,
  - for every $y_1 \in B(y, 1/\varepsilon)$ there exists $x_1 \in B(x, 1/\varepsilon)$ such that $d(f(x_1), y_1) < \varepsilon$;
- $\varphi: \Sigma_{1/\varepsilon}(\Gamma, x) \to \Sigma_{1/\varepsilon}(\Lambda, y)$ is a map satisfying $d(f(gx_1), \varphi(g)f(x_1)) < \varepsilon$ for every $g \in \Sigma_{1/\varepsilon}(\Gamma, x)$ and every $x_1 \in B(x, 1/\varepsilon)$ such that also $gx_1 \in B(x, 1/\varepsilon)$;
- $\psi: \Sigma_{1/\varepsilon}(\Lambda, y) \to \Sigma_{1/\varepsilon}(\Gamma, x)$ is a map satisfying $d(f(\psi(g)x_1), gf(x_1)) < \varepsilon$ for every $g \in \Sigma_{1/\varepsilon}(\Lambda, y)$ and every $x_1 \in B(x, 1/\varepsilon)$ such that $\psi(g)x_1 \in B(x, 1/\varepsilon)$.

Definition 3.2. A sequence of triples (X_n, x_n, Γ_n) is said to converge in the equivariant pointed Gromov–Hausdorff sense to a triple (X, x, Γ) if for every $\varepsilon > 0$ there exists $n_\varepsilon \geq 0$ such that if $n \geq n_\varepsilon$ then there exists an equivariant ε-approximation between (X_n, x_n, Γ_n) and (X, x, Γ). One of these equivariant ε-approximations will be denoted by (f_n, φ_n, ψ_n).

In this case we will write $(X_n, x_n, \Gamma_n) \underset{\text{eq-pGH}}{\longrightarrow} (X, x, \Gamma)$.

Remark 3.3. A few observations are in order.

- If $(X_n, x_n, \Gamma_n) \underset{\text{eq-pGH}}{\longrightarrow} (X, x, \Gamma)$ then (X_n, x_n) converges in the classical pointed Gromov–Hausdorff sense to (X, x). We denote this convergence by $(X_n, x_n) \underset{\text{pGH}}{\longrightarrow} (X, x)$.
- In the definition the limit space X is assumed to be proper. This is not restrictive, as we will see in a moment. If $(X_n, x_n, \Gamma_n) \underset{\text{eq-pGH}}{\longrightarrow} (X, x, \Gamma)$ we denote by $\hat{X}$ the completion of X. Any isometry of X defines uniquely an isometry of $\hat{X}$, so there is a well-defined group of isometries $\hat{\Gamma}$ of $\hat{X}$ associated to Γ. It follows from the definition that $(X_n, x_n, \Gamma_n) \underset{\text{eq-pGH}}{\longrightarrow} (\hat{X}, \hat{x}, \hat{\Gamma})$ too. Moreover, if a sequence of proper metric spaces

converges in the pointed Gromov–Hausdorff sense to a complete metric space then the limit is proper by [**Her16**, Corollary 3.10].

We recall that $\mathrm{Isom}(X)$ is endowed with the topology of uniform convergence on compact subsets of X, when X is a proper space. It is classically known (see also the remark above) that the pointed Gromov–Hausdorff limit of a sequence of metric spaces is unique up to pointed isometry when we restrict to the class of complete (and therefore proper) spaces. In order to obtain uniqueness of the equivariant pointed Gromov–Hausdorff limit we need to restrict to groups that are closed in the isometry group of the limit space.

PROPOSITION 3.4. [**Fuk86**, Proposition 1.5] *Suppose* $(X_n, x_n, \Gamma_n) \underset{\text{eq-pGH}}{\longrightarrow} (X, x, \Gamma)$ *and* $(X_n, x_n, \Gamma_n) \underset{\text{eq-pGH}}{\longrightarrow} (Y, y, \Lambda)$, *where* X, Y *are proper and* Γ, Λ *are closed subgroups of* $\mathrm{Isom}(X)$, $\mathrm{Isom}(Y)$, *respectively. Then there exists an isometry* $F \colon X \to Y$ *such that:*

- $F(x) = y$;
- $F_* \colon \mathrm{Isom}(X) \to \mathrm{Isom}(Y)$ *defined by* $F_*(g) = F \circ g \circ F^{-1}$ *is an isomorphism between* Γ *and* Λ.

Definition 3.5. Two triples (X, x, Γ) and (Y, y, Λ) are said equivariantly isometric if there exists an isometry $F \colon X \to Y$ satisfying the thesis of the previous proposition. In this case F is called an equivariant isometry and we write $(X, x, \Gamma) \cong (Y, y, \Lambda)$.

From now on we will consider the equivariant pointed Gromov–Hausdorff convergence only restricted to triples (X, x, Γ) where (X, x) is a pointed proper metric space and Γ is a closed subgroup of $\mathrm{Isom}(X)$. This condition is not restrictive.

LEMMA 3.6. *If* $(X_n, x_n, \Gamma_n) \underset{\text{eq-pGH}}{\longrightarrow} (X, x, \Gamma)$ *then* $(X_n, x_n, \Gamma_n) \underset{\text{eq-pGH}}{\longrightarrow} (X, x, \bar{\Gamma})$, *where* $\bar{\Gamma}$ *is the closure of* Γ *in* $\mathrm{Isom}(X)$.

Proof. By definition for every $\varepsilon > 0$ there is $n_\varepsilon \geq 0$ such that for every $n \geq n_\varepsilon$ there is an equivariant $\varepsilon/2$-approximation (f_n, φ_n, ψ_n) between (X_n, x_n, Γ_n) and (X, x, Γ). We want to define an equivariant 2ε-approximation $(f_n, \varphi_n, \bar{\psi}_n)$ between (X_n, x_n, Γ_n) and $(X, x, \bar{\Gamma})$.

For every $g \in \Sigma_1/\varepsilon(\bar{\Gamma}, x)$ there is a sequence of isometries $g_k \in \Gamma$ such that $g_k \to g$ uniformly on compact subsets of X. In particular, for every $\delta > 0$ there exists $k_\delta \geq 0$ such that if $k \geq k_\delta$ then $d(g_k(y), g(y)) \leq \delta$ for every $y \in \overline{B}(x, 2/\varepsilon)$. Choosing δ small enough, we have $g_k \in \Sigma_{2/\varepsilon}(\Gamma, x)$ for $k \geq k_\delta$. We define $\bar{\psi}_n(g) := \psi_n(g_{k_\delta})$. Observe that for every $y_n \in B(x_n, 1/\varepsilon)$ we have

$$\begin{aligned} d(f_n(\bar{\psi}(g)y_n), gf_n(y_n)) &= d(f_n(\psi_n(g_{k_\delta})), gf_n(y_n)) \\ &\leq d(f_n(\psi_n(g_{k_\delta})), g_{k_\delta} f_n(y_n)) + d(g_{k_\delta} f_n(y_n), gf_n(y_n)) \\ &\leq \varepsilon + \delta, \end{aligned}$$

where the last inequality follows since $f_n(y_n) \in \overline{B}(x, 2/\varepsilon)$. Taking $\delta \leq \varepsilon$, we conclude that $(f_n, \varphi_n, \bar{\psi}_n)$ is the desired equivariant 2ε-approximation and it is defined for all $n \geq n_\varepsilon$. □

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3.2. *Ultralimit of groups.* For more detailed notions on ultralimits we refer to [**CS21**, **DK18**]. A non-principal ultrafilter ω is a finitely additive measure on $\mathbb{N}$ such that $\omega(A) \in \{0, 1\}$ for every $A \subseteq \mathbb{N}$ and $\omega(A) = 0$ for every finite subset of $\mathbb{N}$. Accordingly we write ω-a.s. and for ω-a.e. (n) in the usual measure-theoretic sense.

Given a bounded sequence (a_n) of real numbers and a non-principal ultrafilter ω, there exists a unique $a \in \mathbb{R}$ such that for every $\varepsilon > 0$ the set $\{n \in \mathbb{N} \text{ s.t. } |a_n - a| < \varepsilon\}$ has ω-measure 1; see, for instance, [**DK18**, Lemma 10.25]. The real number a is called the ultralimit of the sequence a_n and it is denoted by $\omega\text{-lim } a_n$.

Given a sequence of pointed metric spaces (X_n, x_n), we denote by (X_ω, x_ω) the ultralimit pointed metric space. It is the set of sequences (y_n), where $y_n \in X_n$ for every n, for which there exists $M \in \mathbb{R}$ such that $d(x_n, y_n) \leq M$ for ω-a.e. (n), modulo the relation $(y_n) \sim (y'_n)$ if and only if $\omega\text{-lim } d(y_n, y'_n) = 0$. The point of X_ω defined by the class of the sequence (y_n) is denoted by $y_\omega = \omega\text{-lim } y_n$. The formula $d(\omega\text{-lim } y_n, \omega\text{-lim } y'_n) = \omega\text{-lim } d(y_n, y'_n)$ defines a metric on X_ω which is called the ultralimit distance on X_ω.

A sequence of isometries $g_n \in \text{Isom}(X_n)$ is admissible if there exists $M \geq 0$ such that $d(g_n x_n, x_n) \leq M$ ω-a.s. Any such sequence defines an isometry $g_\omega = \omega\text{-lim } g_n$ of X_ω by the formula $g_\omega y_\omega = \omega\text{-lim } g_n y_n$ [**DK18**, Lemma 10.48]. Given a sequence of groups of isometries Γ_n of X_n, we set

$$\Gamma_\omega = \{\omega\text{-lim } g_n \text{ s.t. } g_n \in \Gamma_n \text{ for } \omega\text{-a.e.}(n)\}.$$

In particular, the elements of Γ_ω are ultralimits of admissible sequences.

LEMMA 3.7. *The composition of admissible sequences of isometries is an admissible sequence of isometries and the limit of the composition is the composition of the limits.*

(Indeed, if $g_\omega = \omega\text{-lim } g_n$, $h_\omega = \omega\text{-lim } h_n$ belong to Γ_ω then their composition belongs to Γ_ω, as $\omega\text{-lim } d(g_n h_n \cdot x_n, x_n) \leq \omega\text{-lim } d(g_n h_n \cdot x_n, g_n \cdot x_n) + \omega\text{-lim } d(g_n \cdot x_n, x_n) < +\infty$.)

Analogously one proves that (id_n) belongs to Γ_ω and defines the identity map of X_ω, and that if $g_\omega = \omega\text{-lim } g_n$ belongs to Γ_ω then also the sequence (g_n^{-1}) defines an element of Γ_ω, which is the inverse of g_ω.

So we have a well-defined composition law on Γ_ω, that is, for $g_\omega = \omega\text{-lim } g_n$ and $h_\omega = \omega\text{-lim } h_n$ we set $g_\omega \circ h_\omega = \omega\text{-lim}(g_n \circ h_n)$. With this operation Γ_ω is a group of isometries of X_ω and we call it *the ultralimit group* of the sequence of groups Γ_n.

The ultralimit space X_ω may be not proper in general, even if X_n is proper for every n. When X_ω is proper, Γ_ω is closed with respect to the uniform convergence on compact subsets.

PROPOSITION 3.8. *Let (X_n, x_n, Γ_n) be a sequence of proper metric spaces and ω be a non-principal ultrafilter. If X_ω is proper then Γ_ω is a closed subgroup of* $\text{Isom}(X_\omega)$.

We remark that the proof is analogous to [**DK18**, Corollary 10.64].

Proof. Let $(g_\omega^k)^{k \in \mathbb{N}}$ be a sequence of isometries of Γ_ω converging to an isometry g^∞ of X_ω with respect to the uniform convergence on compact subsets. We want to show that g^∞ coincides with the ultralimit of some sequence of admissible isometries g_n^∞ of X_n.

First of all, we can extract a subsequence, denoted again by (g_ω^k), satisfying $d(g_\omega^k y_\omega, g_\omega^{k+1} y_\omega) \leq 1/2^k$ for all $y_\omega \in \overline{B}(x_\omega, k)$, for all $k \in \mathbb{N}$. Now, for every fixed $k \in \mathbb{N}$, let $S^k = \{y_\omega^1, \ldots, y_\omega^{N_k}\}$ be a $1/2^{k+2}$-dense subset of $B(x_\omega, k)$. It is finite since X_ω is proper. If $y_\omega^i = \omega\text{-lim } y_n^i$ for $i = 1, \ldots, N_k$, it is clear that the set $S_n^k = \{y_n^1, \ldots, y_n^{N_k}\}$ is a $1/2^{k+1}$-dense subset of $B(x_n, k)$ for ω-a.e. (n). For every $k \in \mathbb{N}$ we define the set

$$A_k = \bigg\{ n \in \mathbb{N} \text{ s.t. } d(g_n^j y_n^i, g_n^{j+1} y_n^i) \leq \frac{1}{2^j} \text{ for all } 1 \leq j \leq k,\ i = 1, \ldots, N_j \bigg\}.$$

By definition $\omega(A_k) = 1$ and $A_{k+1} \subseteq A_k$ for every $k \in \mathbb{N}$. We set $B = \bigcap_{k \in \mathbb{N}} A_k$. There are two cases.

Case 1: $\omega(B) = 1$. In this case $d(g_n^k y_n^i, g_n^{k+1} y_n^i) \leq 1/2^k$ for all $i = 1, \ldots, N_k$ and all $k \in \mathbb{N}$, ω-a.s. Then ω-a.s. we have $d(g_n^k y_n, g_n^{k+1} y_n) \leq 1/2^{k-1}$ for every $k \in \mathbb{N}$ and every $y_n \in B(x_n, k)$. We set $g_n^\infty := g_n^n \in \Gamma_n$. This sequence of isometries is admissible and we denote its ultralimit by $g_\omega^\infty \in \Gamma_\omega$. Now we fix $y_\omega = \omega\text{-lim } y_n \in X_\omega$. By definition $d(x_n, y_n) \leq M$ for ω-a.e. (n). We have

$$\begin{aligned} d(g^\infty y_\omega, g_\omega^\infty y_\omega) \leq d(g_\omega^k y_\omega, g_\omega^\infty y_\omega) + \frac{1}{2^k} &= \omega\text{-lim } d(g_n^k y_n, g_n^n y_n) + \frac{1}{2^k} \\ &\leq \sum_{j=0}^{n-k-1} d(g_n^{k+j} y_n, g_n^{k+j+1} y_n) + \frac{1}{2^k} \\ &\leq \sum_{j=k-1}^{\infty} \frac{1}{2^j} + \frac{1}{2^k} \leq \frac{1}{2^{k-2}} + \frac{1}{2^k} \leq \frac{1}{2^{k-3}} \end{aligned}$$

for all fixed $k \geq M$. We conclude that $g^\infty y_\omega = g_\omega^\infty y_\omega$. By the arbitrariness of y_ω we get $g^\infty = g_\omega^\infty$ that belongs to Γ_ω.

Case 2: $\omega(B) = 0$. Since $A_1 = \bigsqcup_{j=1}^\infty (A_j \setminus A_{j+1}) \sqcup B$ and since $\omega(A_1) = 1$, we have that $\omega(\bigsqcup_{j=1}^\infty (A_j \setminus A_{j+1})) = 1$. We set $C = \bigsqcup_{j=1}^\infty (A_j \setminus A_{j+1})$. For all $n \in C$ we set $g_n^\infty := g_n^{j(n)}$, where $j(n)$ is the unique $j \geq 1$ such that $n \in A_j \setminus A_{j+1}$. We claim that the corresponding ultralimit isometry $g_\omega^\infty \in \Gamma_\omega$ equals g^∞. Indeed, let $y_\omega = \omega\text{-lim } y_n \in X_\omega$. For every fixed $k \in \mathbb{N}$, consider the set $C_k = \bigsqcup_{j=k}^\infty (A_j \setminus A_{j+1})$. Each of the sets $A_j \setminus A_{j+1}$ has ω-measure 0, so $\omega(C_k) = 1$ for every fixed k. Let $n \in C_k \subset C$. By definition $j(n) \geq k$ since $n \in C_k$. Since $A_{j(n)} \subseteq A_{j(n)-1} \subseteq \cdots \subseteq A_k$ we have

$$d(g_n^{j(n)} y_n, g_n^k y_n) \leq 2 \cdot \frac{1}{2^{k+2}} + \sum_{m=k}^{j(n)} \frac{1}{2^m} \leq \frac{1}{2^{k+1}} \cdot \frac{1}{2^{k-1}} \leq \frac{1}{2^{k-2}}$$

as soon as $k > d(y_\omega, x_\omega)$ and $n \in C_k$. The set C_k has ω-measure 1, so we conclude that $d(g_\omega^\infty y_\omega, g^\infty y_\omega) \leq 1/2^{k-2}$. By the arbitrariness of k and y_ω we finally get $g^\infty = g_\omega^\infty \in \Gamma_\omega$. □

LEMMA 3.9. *Let (X, x) be a proper metric space and $\Gamma \subseteq \text{Isom}(X)$ be a closed subgroup. Then the ultralimit of the constant sequence (X, x, Γ) is naturally equivariantly isometric to (X, x, Γ) for every non-principal ultrafilter.*

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Proof. Let $(X_n, x_n, \Gamma_n) = (X, x, \Gamma)$ for every n and let ω be a non-principal ultrafilter. By [**CS21**, Proposition A.3] the map $\iota\colon (X, x) \to (X_\omega, x_\omega)$ that sends each point y to the ultralimit point corresponding to the constant sequence $y_n = y$ is an isometry sending x to x_ω. This means that each point y_ω of X_ω can be written as $y_\omega = \omega\text{-lim}\, y$ for some $y \in X$. We need to show that $\iota_*\Gamma = \Gamma_\omega$. We have $\iota_* g(\omega\text{-lim}\, y) = \omega\text{-lim}\, gy$ for every $g \in \Gamma$ and $y_\omega = \omega\text{-lim}\, y \in X_\omega$, that is, $\iota_* g$ coincides with the ultralimit of the constant sequence $g_n = g$. In particular, $\iota_*\Gamma \subseteq \Gamma_\omega$. Now we take $g_\omega = \omega\text{-lim}\, g_n \in \Gamma_\omega$, where $g_n \in \Gamma$ is an admissible sequence, that is, $d(x, g_n x) \leq M$ for some M. The set $\Sigma_M(\Gamma, x)$ is compact by the Arzelà–Ascoli theorem [**Kel17**, Ch. 7 and Theorem 17] since Γ is closed, so by [**DK18**, Lemma 10.25] there exists $g \in \Sigma_M(\Gamma, x)$ such that for every $\varepsilon > 0$ and $R \geq 0$ the set

$$\{n \in \mathbb{N} \text{ s.t. } d(g_n y, gy) < \varepsilon \text{ for all } y \in \overline{B}(x, R)\}$$

belongs to ω. It is clear that the constant sequence g defines the ultralimit isometry g_ω, that is, $\iota_*(g) = g_\omega$. Since $g \in \Gamma$ we conclude that $\iota_*\Gamma = \Gamma_\omega$. □

3.3. *Comparison between the two convergences.* We now compare the ultralimit convergence to the equivariant pointed Gromov–Hausdorff convergence. Analogues of the next results for the classical pointed Gromov–Hausdorff convergence can be found, for instance, in [**Jan17**].

PROPOSITION 3.10. *Suppose* $(X_n, x_n, \Gamma_n) \underset{\text{eq-pGH}}{\longrightarrow} (X, x, \Gamma)$ *and denote by* (f_n, φ_n, ψ_n) *some corresponding equivariant approximations. Let ω be a non-principal ultrafilter and let* $(X_\omega, x_\omega, \Gamma_\omega)$ *be the ultralimit triple. Then the map*

$$F\colon (X_\omega, x_\omega, \Gamma_\omega) \to (X, x, \Gamma)$$

defined by sending $y_\omega = \omega\text{-lim}\, y_n \in X_\omega$ *to* $\iota^{-1}(\omega\text{-lim}\, f_n(y_n))$ *is a well-defined equivariant isometry. Here ι is the natural equivariant isometry of Lemma 3.9 between* (X, x, Γ) *and the ultralimit of its constant sequence.*

Proof. We divide the proof into steps.

Good definition. Given a point $y_\omega = \omega\text{-lim}\, y_n \in X_\omega$, by definition there exists $M \geq 0$ such that $d(x_n, y_n) \leq M$ ω-a.s. For sufficiently large n the map f_n is defined on y_n and it satisfies $d(f_n(y_n), f_n(x_n)) \leq M + 1$ and $f_n(x_n) = x$. Then the sequence $(f_n(y_n))$ is ω-a.s. bounded and $\iota^{-1}(\omega\text{-lim}\, f_n(y_n))$ is a well-defined point of X.

Now suppose (y_n') is another sequence such that $\omega\text{-lim}\, d(y_n, y_n') = 0$. For every $\varepsilon > 0$ we have $d(y_n, y_n') < \varepsilon$ ω-a.s. Moreover, arguing as before, $d(f_n(y_n), f_n(y_n')) \leq d(y_n, y_n') + \varepsilon \leq 2\varepsilon$ ω-a.s. By the arbitrariness of $\varepsilon > 0$ we get $d(\omega\text{-lim}\, f_n(y_n), \omega\text{-lim}\, f_n(y_n')) = 0$. In particular, F is well defined.

Isometric embedding. We fix $y_\omega = \omega\text{-lim}\, y_n$, $z_\omega = \omega\text{-lim}\, z_n \in X_\omega$ and $\varepsilon > 0$. As usual, all the conditions

$$d(\iota^{-1}(\omega\text{-lim}\, f_n(y_n)), f_n(y_n)) < \varepsilon, \quad d(\iota^{-1}(\omega\text{-lim}\, f_n(z_n)), f_n(z_n)) < \varepsilon,$$
$$|d(y_\omega, z_\omega) - d(y_n, z_n)| < \varepsilon, \quad |d(f_n(y_n), f_n(z_n)) - d(y_n, z_n)| < \varepsilon$$

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hold ω-a.s. Therefore

$$|d(F(y_\omega), F(z_\omega)) - d(y_\omega, z_\omega)| < 4\varepsilon.$$

By the arbitrariness of ε we conclude that F is an isometric embedding.

Surjectivity. We fix $y \in X$, $\varepsilon > 0$ and we set $L := d(x, y)$. By definition there exists $y_n \in X_n$ such that $d(f_n(y_n), y) < \varepsilon\omega$-a.s. The sequence y_n is clearly admissible and defines a point $y_\omega = \omega\text{-lim } y_n$ of X_ω. Since $d(f_n(y_n), y) < \varepsilon\omega$-a.s., we have that $F(y_\omega) = \omega\text{-lim } f_n(y_n)$ satisfies $d(F(y_\omega), y) < 2\varepsilon$. This shows that y belongs to the closure of $F(X_\omega)$. Every ultralimit space is a complete metric space [**DK18**, Corollary 10.64], so F is a closed map. Indeed, if $F(y_\omega^k)$ is a convergent sequence, then it is Cauchy. Since F is an isometric embedding, the sequence (y_ω^k) is Cauchy and therefore converges. By continuity of F we conclude that the limit point of the sequence $F(y_\omega^k)$ belongs to the image of F. Hence $F(X_\omega)$ is closed and $y \in F(X_\omega)$, showing that F is surjective.

F is equivariant. It is clear that $F(x_\omega) = \iota^{-1}(\omega\text{-lim } f_n(x_n)) = x$. It remains to show that $F_*(\Gamma_\omega) = \Gamma$. We take $g_\omega = \omega\text{-lim } g_n \in \Gamma_\omega$. Then $F_*(g_\omega)$ acts on the point y of X as $F_*g_\omega(y) = F \circ g_\omega \circ F^{-1}(y)$. Clearly $F^{-1}(y) = \omega\text{-lim } y_n$, where y_n is any sequence such that $\omega\text{-lim } f_n(y_n) = y$. So $g_\omega \circ F^{-1}y = \omega\text{-lim } g_n y_n$ by definition of g_ω. Finally, $F \circ g_\omega \circ F^{-1} = \iota^{-1}(\omega\text{-lim } f_n(g_n y_n))$. Now we consider the isometries $\varphi_n(g_n) \in \Gamma$: they are defined for sufficiently large n since g_ω displaces x_ω of some finite quantity. We define the isometry $\iota_*^{-1}(\omega\text{-lim } \varphi_n(g_n))$: it is an isometry of X, which is proper, and it belongs to Γ by Lemma 3.9. We have

$$d(\iota_*^{-1}\omega\text{-lim } \varphi_n(g_n)(y), F \circ g_\omega \circ F^{-1}(y)) = \omega\text{-lim } d(\varphi_n g_n(y), f_n(g_n y_n))$$

for every $y \in X$. Moreover,

$$\begin{aligned} d(\varphi_n g_n(y), f_n(g_n y_n)) &\leq d(\varphi_n g_n(y), \varphi_n g_n(f_n(y_n))) + d(\varphi_n g_n(f_n(y_n)), f_n(g_n y_n)) \\ &\leq 2\varepsilon \end{aligned}$$

if n is sufficiently large. This means that $F_*g_\omega = \iota^{-1}(\omega\text{-lim } \varphi_n g_n) \in \Gamma$, so $F_*\Gamma_\omega \subseteq \Gamma$. Now we take $g \in \Gamma$ and we consider the isometries $\psi_n g \in \Gamma_n$ that are defined for sufficiently large n. The sequence $(\psi_n g)$ is admissible and therefore it defines a limit isometry g_ω. For all $y \in X$ we have $F_*(g_\omega)(y) = F g_\omega(y_\omega)$, where $y_\omega = \omega\text{-lim } y_n$ and y_n is a sequence such that $\iota^{-1}(\omega\text{-lim } f_n(y_n)) = y$. Then $F_*(g_\omega)(y) = \iota^{-1}(\omega\text{-lim } f_n(\psi_n(g)y_n))$. Once again ω-a.s. we have

$$\begin{aligned} d(F_*(g_\omega)(y), gy) &\leq d(\iota^{-1}(\omega\text{-lim } f_n(\psi_n(g)y_n)), g f_n(y_n)) + \varepsilon \\ &\leq d(f_n(\psi_n(g)y_n), g f_n(y_n)) + 2\varepsilon. \end{aligned}$$

We conclude that $F_*g_\omega = g$, so $F_*\Gamma_\omega = \Gamma$. □

PROPOSITION 3.11. *Let (X_n, x_n, Γ_n) be a sequence of triples, ω be a non-principal ultrafilter and $(X_\omega, x_\omega, \Gamma_\omega)$ be the ultralimit triple. If X_ω is proper then there exists a subsequence $\{n_k\} \subseteq \mathbb{N}$ such that $(X_{n_k}, x_{n_k}, \Gamma_{n_k}) \underset{\text{eq-pGH}}{\longrightarrow} (X_\omega, x_\omega, \Gamma_\omega)$.*

Remark 3.12. Notice that the subsequence $\{n_k\}$ may not belong to ω.

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Proof. We fix $\varepsilon > 0$. Since X_ω is proper we can select an $\varepsilon/7$-net $S^\varepsilon = \{x_\omega = y_\omega^1, \ldots, y_\omega^{N_\varepsilon}\}$ of $B(x_\omega, 1/\varepsilon)$, where $y_\omega^i = \omega\text{-lim } y_n^i$ for $i = 1, \ldots, N_\varepsilon$. Moreover, Γ_ω is closed by Proposition 3.8, so $\Gamma_{\omega,1/\varepsilon}(x_\omega)$ is relatively compact by the Arzelà–Ascoli theorem [Kel17, Ch. 7 and Theorem 17]. Therefore we can find a finite subset $g_\omega^1, \ldots, g_\omega^{K_\varepsilon} \in \Gamma_{\omega,1/\varepsilon}(x_\omega)$, $g_\omega^i = \omega\text{-lim } g_n^i$, with the property that for every $g_\omega \in \Gamma_{\omega,1/\varepsilon}(x_\omega)$ there exists $1 \le i \le K_\varepsilon$ such that $d(g_\omega y_\omega, g_\omega^i y_\omega) \le \varepsilon/7$ for all $y_\omega \in B(x_\omega, 1/\varepsilon)$.

Now ω-a.s. the following finite set of conditions hold:

- $|d(y_\omega^i, y_\omega^j) - d(y_n^i, y_n^j)| \le \varepsilon/7$ for all $i, j \in \{1, \ldots, N_\varepsilon\}$;
- the set $S_n^\varepsilon = \{y_n^1, \ldots, y_n^{N_\varepsilon}\}$ is a $\frac{2}{7}\varepsilon$-net of $B(x_n, 1/\varepsilon)$;
- $g_n^i \in \Gamma_{n,1/\varepsilon}(x_n)$ for every $i = 1, \ldots, K_\varepsilon$;
- $|d(g_\omega^i y_\omega^j, y_\omega^l) - d(g_n y_n^j, y_n^l)| \le \varepsilon/7$ for all $1 \le i \le K_\varepsilon$ and all $1 \le j, l \le N_\varepsilon$.
- the set $\{g_n^1, \ldots, g_n^{K_\varepsilon}\}$ is a $\frac{2}{7}\varepsilon$-dense subset of $\Gamma_{n,1/\varepsilon}(x_n)$ with respect to the uniform distance.

For the natural numbers n where these conditions hold we define

$$f_n : B\left(x_n, \frac{1}{\varepsilon}\right) \to B\left(x_\omega, \frac{1}{\varepsilon}\right)$$

by sending the point y_n to a point y_ω^i where i is such that $d(y_n, y_n^i) \le \frac{2}{7}\varepsilon$. For $y_n, z_n \in B(x_n, 1/\varepsilon)$ we have

$$|d(f_n(y_n), f_n(z_n)) - d(y_n, z_n)| = |d(y_\omega^{i_1}, y_\omega^{i_2}) - d(y_n, z_n)|$$

for some i_1, i_2. But for these indices n we have $|d(y_\omega^{i_1}, y_\omega^{i_2}) - d(y_n^{i_1}, y_n^{i_2})| \le \frac{2}{7}\varepsilon$, so we get

$$|d(f_n(y_n), f_n(z_n)) - d(y_n, z_n)| \le \tfrac{6}{7}\varepsilon.$$

Moreover, we define

$$\psi_n : \Gamma_{\omega,1/\varepsilon}(x_\omega) \to \Gamma_{n,1/\varepsilon}(x_n)$$

by sending g_ω to g_n^i, where $i \in \{1, \ldots, K_\varepsilon\}$ is such that $d^\infty_{B(x_\omega,1/\varepsilon)}(g_\omega, g_\omega^i) \le \varepsilon/7$.

Let $g_\omega \in \Gamma_{\omega,1/\varepsilon}(x_\omega)$, so $\Psi_n(g_\omega) = g_n^i$ as before. Let $y_n \in B(x_n, 1/\varepsilon)$ such that also $g_n^i y_n \in B(x_n, 1/\varepsilon)$. Let $j, l \in \{1, \ldots, N_\varepsilon\}$ be such that $d(y_n, y_n^j) \le \frac{2}{7}\varepsilon$ and $d(g_n^i y_n, y_n^l) \le \frac{2}{7}\varepsilon$. By definition $f_n(y_n) = y_\omega^i$, while $f_n(\psi_n(g_\omega) y_n) = y_\omega^l$. We have

$$\begin{aligned} d(f_n(\psi_n(g_\omega) y_n), g_\omega f_n(y_n)) &= d(y_\omega^l, g_\omega y_\omega^j) \\ &\le d(y_\omega^l, g_\omega^i y_\omega^j) + \tfrac{2}{7}\varepsilon \\ &\le d(y_n^l, g_n^i y_n^j) + \tfrac{3}{7}\varepsilon \le \varepsilon. \end{aligned}$$

Finally, we define

$$\varphi_n : \Gamma_{n,1/\varepsilon}(x_n) \to \Gamma_{\omega,1/\varepsilon}(x_\omega)$$

as $\varphi_n(g_n) = g_\omega^i$, where $d^\infty_{B(x_n,1/\varepsilon)}(g_n, g_n^i) \le \frac{2}{7}\varepsilon$.

Let $g_n \in \Gamma_{n,1/\varepsilon}(x_n)$ and let $\varphi_n(g_n) = g_\omega^i$. Now let $y_n \in B(x_n, 1/\varepsilon)$ such that also $g_n y_n \in B(x_n, 1/\varepsilon)$. Let $j, l \in \{1, \ldots, N_\varepsilon\}$ such that $d(y_n, y_n^j) \le \frac{2}{7}\varepsilon$ and $d(g_n y_n, y_n^l) \le$

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$\frac{2}{7}\varepsilon$, so that $f_n(y_n) = y_\omega^j$ and $f_n(g_n y_n) = y_\omega^l$. Therefore

$$\begin{aligned} d(f_n(g_n y_n), \varphi_n(g_n) f_n(y_n)) &= d(y_\omega^l, g_\omega^i y_\omega^j) \\ &\leq d(y_n^l, g_n y_n^j) + \tfrac{3}{7}\varepsilon \\ &\leq d(y_n^l, g_n y_n) + \tfrac{5}{7}\varepsilon \leq \varepsilon. \end{aligned}$$

This shows that ω-a.s. we can find an equivariant ε-approximation between (X_n, x_n, Γ_n) and $(X_\omega, x_\omega, \Gamma_\omega)$. For all integers k we set $\varepsilon = 1/k$ and we choose $n_k \in \mathbb{N}$ in the set of indices for which there exists an equivariant $1/k$-approximation as before. The sequence $(X_{n_k}, x_{n_k}, \Gamma_{n_k})$ satisfies the thesis. □

We summarize these properties in the following proposition.

PROPOSITION 3.13. *Let (X_n, x_n, Γ_n) be a sequence of triples and let ω be a non-principal ultrafilter.*

(i) *If $(X_n, x_n, \Gamma_n) \underset{\text{eq-pGH}}{\longrightarrow} (X, x, \Gamma)$ then $(X_\omega, x_\omega, \Gamma_\omega) \cong (X, x, \Gamma)$.*

(ii) *If X_ω is proper then $(X_{n_k}, x_{n_k}, \Gamma_{n_k}) \underset{\text{eq-pGH}}{\longrightarrow} (X_\omega, x_\omega, \Gamma_\omega)$ for some subsequence $\{n_k\}$.*

COROLLARY 3.14. *Let (X_n, x_n, Γ_n) be a sequence of triples and suppose that there is a triple (X, x, Γ), X proper, such that $(X, x, \Gamma) \cong (X_\omega, x_\omega, \Gamma_\omega)$ for every non-principal ultrafilter ω. Then $(X_n, x_n, \Gamma_n) \underset{\text{eq-pGH}}{\longrightarrow} (X, x, \Gamma)$.*

Proof. The equivariant pointed Gromov–Hausdorff convergence is metrizable (cf. [**Fuk86**]), so it is enough to show that every subsequence has a subsequence that converges to (X, x, Γ). Fix a subsequence $\{n_k\}$. The set $\{n_k\}$ is infinite; then there exists a non-principal ultrafilter ω containing it for which $\omega(\{n_k\}) = 1$ (cf. [**Jan17**, Lemma 3.2]). The ultralimit with respect to ω of the sequence $(X_{n_k}, x_{n_k}, \Gamma_{n_k})$ is the same as that of the sequence (X_n, x_n, Γ_n) since $\omega(\{n_k\}) = 1$. By Proposition 3.13 we can extract a further subsequence $\{n_{k_j}\}$ that converges in the equivariant pointed Gromov–Hausdorff sense to (X, x, Γ). □

4. *Gromov hyperbolic metric spaces*

We recall briefly the definition and some properties of Gromov-hyperbolic metric spaces. Good references are, for instance, [**BH13**, **CDP90**].

Let X be a geodesic metric space. Given three points $x, y, z \in X$, the *Gromov product* of y and z with respect to x is defined as

$$(y, z)_x = \tfrac{1}{2}(d(x, y) + d(x, z) - d(y, z)).$$

The space X is said to be δ-hyperbolic, $\delta \geq 0$, if for every four points $x, y, z, w \in X$ the following *four-points condition* holds:

$$(x, z)_w \geq \min\{(x, y)_w, (y, z)_w\} - \delta \tag{4.1}$$

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or, equivalently,

$$d(x, y) + d(z, w) \leq \max\{d(x, z) + d(y, w), d(x, w) + d(y, z)\} + 2\delta. \tag{4.2}$$

The space X is *Gromov hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$.

This formulation of δ-hyperbolicity is convenient when one is interested in taking limits. We will also make use of another classical characterization of δ-hyperbolicity. A *geodesic triangle* in X is the union of three geodesic segments $[x, y], [y, z], [z, x]$ and is denoted by $\Delta(x, y, z)$. For every geodesic triangle there exists a unique *tripod* $\overline{\Delta}$ with vertices $\bar{x}, \bar{y}, \bar{z}$ such that the lengths of $[\bar{x}, \bar{y}], [\bar{y}, \bar{z}], [\bar{z}, \bar{x}]$ equal the lengths of $[x, y], [y, z], [z, x]$, respectively. There exists a unique map f_Δ from $\Delta(x, y, z)$ to the tripod $\overline{\Delta}$ that isometrically identifies the corresponding edges, and there are exactly three points $c_x \in [y, z], c_y \in [x, z], c_z \in [x, y]$ such that $f_\Delta(c_x) = f_\Delta(c_y) = f_\Delta(c_z) = c$, where c is the centre of the tripod $\overline{\Delta}$. By definition of f_Δ, we have

$$d(x, c_z) = d(x, c_y), \quad d(y, c_x) = d(y, c_z), \quad d(z, c_x) = d(z, c_y).$$

The triangle $\Delta(x, y, z)$ is called δ-thin if for every $u, v \in \Delta(x, y, z)$ such that $f_\Delta(u) = f_\Delta(v)$ the inequality $d(u, v) \leq \delta$ holds; in particular, the mutual distances between c_x, c_y and c_z are at most δ. It is well known that every geodesic triangle in a geodesic δ-hyperbolic metric space (as defined above) is 4δ-thin.

Moreover, the last condition is equivalent to the above definition of hyperbolicity, up to slightly increasing the hyperbolicity constant δ in (4.1).

The following is a basic property of Gromov-hyperbolic metric spaces.

LEMMA 4.1. (Projection lemma, cf. [**CDP90**, Lemma 3.2.7]) *Let X be a δ-hyperbolic metric space and let $x, y, z \in X$. For every geodesic segment $[y, z]$ we have $(y, z)_x \geq d(x, [y, z]) - 4\delta$.*

Let X be a proper, δ-hyperbolic metric space and x be a point of X. The *Gromov boundary* of X is defined as the quotient

$$\partial X = \{(z_n)_{n\in\mathbb{N}} \subseteq X \mid \lim_{n,m\to+\infty} (z_n, z_m)_x = +\infty\}/\approx,$$

where $(z_n)_{n\in\mathbb{N}}$ is a sequence of points in X and $\approx$ is the equivalence relation defined by $(z_n)_{n\in\mathbb{N}} \approx (z'_n)_{n\in\mathbb{N}}$ if and only if $\lim_{n,m\to+\infty}(z_n, z'_m)_x = +\infty$. We will write $z = [(z_n)] \in \partial X$ for short, and we say that (z_n) *converges* to z. This definition does not depend on the basepoint x.

There is a natural topology on $X \cup \partial X$ that extends the metric topology of X. The Gromov product can be extended to points $z, z' \in \partial X$ by

$$(z, z')_x = \sup_{(z_n),(z'_n)} \liminf_{n,m\to+\infty} (z_n, z'_m)_x$$

where the supremum is taken among all sequences such that $(z_n) \in z$ and $(z'_n) \in z'$. For every $z, z', z'' \in \partial X$ the inequality

$$(z, z')_x \geq \min\{(z, z'')_x, (z', z'')_x\} - \delta. \tag{4.3}$$

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continues to hold. Moreover, for all sequences (z_n), (z'_n) converging to z, z' respectively, we have

$$(z, z')_x - \delta \leq \liminf_{n,m\to+\infty} (z_n, z'_m)_x \leq (z, z')_x. \tag{4.4}$$

The Gromov product between a point $y \in X$ and a point $z \in \partial X$ is defined in a similar way and satisfies a condition analogous to (4.4).

Every geodesic ray ξ defines a point $\xi^+ = [(\xi(n))_{n\in\mathbb{N}}]$ of the Gromov boundary ∂X: we say that ξ *joins* $\xi(0) = y$ *to* $\xi^+ = z$, and we denote this by $[y, z]$. Moreover, for every $z \in \partial X$ and every $x \in X$ it is possible to find a geodesic ray ξ such that $\xi(0) = x$ and $\xi^+ = z$. Any such geodesic ray is denoted by $\xi_{x,z} = [x, z]$ even if it is possibly not unique. Analogously, given different points $z = [(z_n)], z' = [(z'_n)] \in \partial X$, there always exists a geodesic line γ joining z' to z, that is, such that $\gamma|_{[0,+\infty)}$ and $\gamma|_{(-\infty,0]}$ join $\gamma(0)$ to z, z', respectively (just consider the limit γ of the segments $[z_n, z'_n]$; notice that all these segments intersect a ball of fixed radius centred at x_0, since $(z_n, z'_m)_{x_0}$ is uniformly bounded above). We call z and z' respectively the *positive* and *negative endpoints* of γ, denoted by $\gamma^\pm$. The relation between Gromov product and geodesic ray is highlighted in the following well-known lemma.

LEMMA 4.2. *Let X be a proper, δ-hyperbolic metric space, $z, z' \in \partial X$ and $x \in X$.*

(i) *If $(z, z')_x \geq T$ then $d(\xi_{x,z}(T - \delta), \xi_{x,z'}(T - \delta)) \leq 4\delta$.*

(ii) *If $d(\xi_{x,z}(T), \xi_{x,z'}(T)) < 2b$ then $(z, z')_x > T - b$, for all $b > 0$.*

Proof. Assume $(z, z')_x \geq T$ and suppose $d(\xi_{x,z}(T - \delta), \xi_{x,z'}(T - \delta)) > 4\delta$. Fix $S \geq T - \delta$ and consider the triangle $\Delta(x, \xi_{x,z}(S), \xi_{x,z'}(S))$. There exist $a \in [x, \xi_{x,z}(S)]$, $b \in [x, \xi_{x,z'}(S)]$, $c \in [\xi_{x,z}(S), \xi_{x,z'}(S)]$ such that $d(a, b) < \delta$, $d(b, c) < \delta$, $d(a, c) < \delta$ and $T_\delta := d(x, a) = d(x, b)$, $d(\xi_{x,z}(S), a) = d(\xi_{x,z}(S), c)$, $d(\xi_{x,z'}(S), b) = d(\xi_{x,z'}(S), c)$. Since this triangle is 4δ-thin we conclude that $T - \delta > T_\delta$. Moreover, $d(\xi_{x,z}(S), \xi_{x,z'}(S)) = d(\xi_{x,z}(S), c) + d(c, \xi_{x,z'}(S)) = 2(S - T_\delta)$. Hence

$$(z, z')_x \leq \liminf_{S\to+\infty} \tfrac{1}{2}(2S - d(\xi_{x,z}(S), \xi_{x,z'}(S))) + \delta = T_\delta + \delta < T,$$

where we have used (4.4). This contradiction concludes (i).

Now we assume $d(\xi_{x,z}(T), \xi_{x,z'}(T)) < 2b$. Applying (4.4) again and using $d(\xi_{x,z}(S), \xi_{x,z'}(S)) < 2(S - T) + 2b$ for all $S \geq T$, we obtain

$$(z, z')_x \geq \liminf_{S\to+\infty} \tfrac{1}{2}(2S - d(\xi_{x,z}(S), \xi_{x,z'}(S))) > T + b. \qquad \square$$

Remark 4.3. We remark that the computation above shows also that if $z \in \partial X$, $y \in X$ and $(y, z)_x \geq T$ then $d(x, y) > T - \delta$ and $d(\gamma(T - \delta), \xi_{x,z}(T - \delta)) \leq 4\delta$ for every geodesic segment $\gamma = [x, y]$.

The following is a standard computation; see, for instance, [**BCGS17**].

LEMMA 4.4. *Let X be a proper, δ-hyperbolic metric space. Then every pair of geodesic rays ξ, ξ' with same endpoints at infinity are at distance at most 8δ, that is, there*

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exist $t_1, t_2 \geq 0$ *such that* $t_1 + t_2 = d(\xi(0), \xi'(0))$ *and* $d(\xi(t + t_1), \xi'(t + t_2)) \leq 8\delta$ *for all* $t \in \mathbb{R}$.

A curve $\alpha\colon [a, b] \to X$ is a $(1, \nu)$-quasigeodesic, $\nu \geq 0$, if

$$|s - t| - \nu \leq d(\alpha(s), \alpha(t)) \leq |s - t| + \nu$$

for all $s, t \in [a, b]$. A subset Y of X is said to be λ-quasiconvex if every point of every geodesic segment joining every pair of points y, y' of Y is at distance at most λ from Y. The *quasiconvex hull* of a subset C of ∂X is the union of all the geodesic lines joining two points of C and is denoted by QC-Hull(C). The following lemma justifies this name.

LEMMA 4.5. *Let X be a proper, δ-hyperbolic metric space and let C be a subset of ∂X. Then* QC-Hull(C) *is 36δ-quasiconvex. Moreover, if C is closed then* QC-Hull(C) *is closed.*

Proof. Let $x, y \in$ QC-Hull(C). By definition they belong to geodesics γ_x, γ_y with both endpoints in C. We parametrize γ_x and γ_y in such a way that $d(\gamma_x(0), \gamma_y(0)) = d(\gamma_x, \gamma_y)$ and $x = \gamma_x(t_x)$, $y = \gamma_y(t_y)$ with $t_x, t_y \geq 0$. We take a geodesic $\gamma = [\gamma_x^+, \gamma_y^+] \subseteq$ QC-Hull(C). By Lemma 4.4 there are points $x', y' \in \gamma$ at distance at most 8δ from x and y, respectively. Therefore the path $\alpha = [x, x'] \cup [x', y'] \cup [y', y]$ is a $(1, 16\delta)$-quasigeodesic. By a standard computation in hyperbolic geometry (see, for instance, [**CS20**, Proposition 3.5(a)]) we conclude that any point of $[x, y]$ is at distance at most 28δ from a point of α and so at distance at most 36δ from a point of γ. This concludes the proof of the first part since the points of γ are in the quasiconvex hull of C.

Suppose now that there are points $x_n \in$ QC-Hull(C) converging to $x_\infty \in X$. By definition $x_n \in \gamma_n$, where γ_n is a geodesic line with endpoints $\gamma_n^\pm \in C$. The geodesics γ_n converge uniformly on compact subsets to a geodesic γ_∞ containing x_∞, since X is proper. The sequences $\gamma_n^\pm$ converge to the endpoints of γ_∞ (cf. [**BL12**, Lemma 1.6]). Using the fact that C is closed, we conclude that $\gamma_\infty^\pm \in C$, that is, $x_\infty \in$ QC-Hull(C). □

We need the following approximation result.

LEMMA 4.6. *Let X be a proper, δ-hyperbolic metric space. Let $C \subseteq \partial X$ be a subset with at least two points and $x \in$* QC-Hull(C). *Then for every $z \in C$ there exists a geodesic line γ with endpoints in C such that $d(\xi_{x,z}(t), \gamma(t)) \leq 14\delta$ for every $t \geq 0$. In particular,* $d(\xi_{x,z}(t),$ QC-Hull$(C)) \leq 14\delta$.

Proof. Since $x \in$ QC-Hull(C)), there exists a geodesic line η joining two points $\eta^\pm$ of C such that $x \in \eta$. Of course we have $(\eta^+, \eta^-)_x \leq \delta$, so by (4.3) we get

$$\delta \geq (\eta^+, \eta^-)_x \geq \min\{(\eta^+, z)_x, (\eta^-, z)_x\} - \delta.$$

Therefore one of the two values $(\eta^+, z)_x$, $(\eta^-, z)_x$ is less than or equal to 2δ. Let us suppose it is the former. We consider a geodesic line γ joining η^+ and z. By Lemma 4.1 we get

$$d(x, \gamma([-S, S])) \leq (\gamma(-S), \gamma(S))_x + 4\delta$$

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for every $S \geq 0$. Taking $S \to +\infty$, the points $\gamma(-S)$ and $\gamma(S)$ converge respectively to η^+ and z. Therefore by (4.4) we get $d(x, \gamma) \leq 6\delta$. If we parametrize γ so that $d(x, \gamma(0)) \leq 6\delta$ then $d(\xi_{x,z}(t), \gamma(t)) \leq 14\delta$ for every $t \geq 0$, by Lemma 4.4. □

The Busemann function associated to $z \in \partial X$ with basepoint x is

$$B_z(x, \cdot)\colon X \to \mathbb{R}, \quad y \mapsto \lim_{T \to +\infty} (d(\xi_{x,z}(T), y) - T).$$

It depends on the choice of the geodesic ray $\xi_{x,z} = [x, z]$, but two maps obtained by taking two different geodesic rays are at bounded distance and the bound depends only on δ. Every Busemann function is 1-Lipschitz.

4.1. *Visual metrics.* When X is a proper, δ-hyperbolic metric space it is known that the boundary ∂X is metrizable. A metric $D_{x,a}$ on ∂X is called a *visual metric* of centre $x \in X$ and parameter $a \in (0, 1/(2\delta \cdot \log_2 e))$ if there exists $V > 0$ such that for all $z, z' \in \partial X$ the inequality

$$\frac{1}{V} e^{-a(z,z')_x} \leq D_{x,a}(z, z') \leq V e^{-a(z,z')_x}. \tag{4.5}$$

holds. A visual metric is said to be *standard* if for all $z, z' \in \partial X$, we have

$$(3 - 2e^{a\delta})e^{-a(z,z')_x} \leq D_{x,a}(z, z') \leq e^{-a(z,z')_x}.$$

For all a as before and $x \in X$ there exists always a standard visual metric of centre x and parameter a; see [**Pau96**]. The *generalized visual ball* of centre $z \in \partial X$ and radius $\rho \geq 0$ is

$$B(z, \rho) = \left\{ z' \in \partial X \text{ s.t. } (z, z')_x > \log \frac{1}{\rho} \right\}.$$

It is comparable to the metric balls of the visual metrics on ∂X.

LEMMA 4.7. *Let $D_{x,a}$ be a visual metric of centre x and parameter a on ∂X. Then for all $z \in \partial X$ and for all $\rho > 0$ we have*

$$B_{D_{x,a}}\left(z, \frac{1}{V}\rho^a\right) \subseteq B(z, \rho) \subseteq B_{D_{x,a}}(z, V\rho^a).$$

Proof. If $z' \in B(z, \rho)$ then $(z, z')_x > \log(1/\rho)$, so $D_{x,a}(z, z') \leq V e^{-a(z,z')_x} < V\rho^a$. If $z' \in B_{D_{x,a}}(z, (1/V)\rho^a)$ then $(1/V)e^{-a(z,z')_x} \leq D_{x_0,a}(z, z') < (1/V)\rho^a$, that is, $z' \in B(z, \rho)$. □

It is classical that generalized visual balls are related to shadows, whose definition is as follows. The shadow of radius $r > 0$ cast by a point $y \in X$ with centre $x \in X$ is the set

$$\text{Shad}_x(y, r) = \{z \in \partial X \text{ s.t. } [x, z] \cap B(y, r) \neq \emptyset \text{ for all rays } [x, z]\}.$$

LEMMA 4.8. *Let X be a proper, δ-hyperbolic metric space. Let $z \in \partial X$, $x \in X$ and $T \geq 0$. Then:*

(i) $B(z, e^{-T}) \subseteq \text{Shad}_x(\xi_{x,z}(T), 7\delta)$;

(ii) $\text{Shad}_x(\xi_{x,z}(T), r) \subseteq B(z, e^{-T+r})$ *for all* $r > 0$.

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Proof. Let $z' \in B(z, e^{-T})$, that is, $(z, z')_x > T$. By Lemma 4.2 we know that $d(\xi_{x,z}(T-\delta), \xi_{x,z'}(T-\delta)) \leq 4\delta$. So $d(\xi_{x,z'}(T), \xi_{x,z}(T)) \leq 6\delta < 7\delta$. This implies $z' \in \mathrm{Shad}_x(\xi_{x,z}(T), 7\delta)$, showing (i).

Now we fix $z' \in \mathrm{Shad}_x(\xi_{x,z}(T), r)$, which means that every geodesic ray $\xi_{x,z'}$ passes through $B(\xi_{x,z}(T), r)$, so $d(\xi_{x,z'}(T), \xi_{x,z}(T)) < 2r$. By Lemma 4.2 we conclude $(z, z')_x > T - r$, implying (ii). □

A compact metric space Z is (A, s)-Ahlfors regular if there exists a probability measure μ on Z such that

$$\frac{1}{A}\rho^s \leq \mu(B(z, \rho)) \leq A\rho^s$$

for all $z \in Z$ and all $0 \leq \rho \leq \mathrm{Diam}(Z)$, where $\mathrm{Diam}(Z)$ is the diameter of Z. If $Z = \partial X$ we say that Z is *visually* (A, s)-Ahlfors regular if there exists a probability measure μ on ∂X such that

$$\frac{1}{A}\rho^s \leq \mu(B(z, \rho)) \leq A\rho^s$$

for all $z \in Z$ and all $0 \leq \rho \leq 1$, where $B(z, \rho)$ is the generalized visual ball of centre z and radius ρ. The following result is an immediate consequence of Lemma 4.7.

LEMMA 4.9. *If ∂X is (A, s)-Ahlfors regular with respect to a visual metric of centre x and parameter a, then it is visually (AV^s, as)-Ahlfors regular, where V is the constant of (4.5).*

The packing* number at scale ρ of a subset C of the boundary of a proper Gromov-hyperbolic space ∂X is the maximal number of disjoint generalized visual balls of radius ρ with centre in C. We denote it by $\mathrm{Pack}^*(C, \rho)$. We write $\mathrm{Cov}(C, \rho)$ to denote the minimal number of generalized visual balls of radius ρ needed to cover C.

LEMMA 4.10. *For all $T \geq 0$ we have the inequalities* $\mathrm{Pack}^*(C, e^{-T+\delta}) \leq \mathrm{Cov}(C, e^{-T})$ *and* $\mathrm{Cov}(C, e^{-T+\delta}) \leq \mathrm{Pack}^*(C, e^{-T})$.

Proof. Let $z_1, \ldots, z_N$ be points of C realizing $\mathrm{Cov}(C, e^{-T})$. Suppose there exist points $w_1, \ldots, w_M$ of C such that $B(w_i, e^{-T+\delta})$ are disjoint, in particular $(w_i, w_j)_x \leq T - \delta$ for every $i \neq j$. If $M > N$ then two different points w_i, w_j belong to the same ball $B(z_k, e^{-T})$, that is, $(z_k, w_i)_x > T$ and $(z_k, w_j)_x > T$. By (4.3) we have $(w_i, w_j)_x > T - \delta$, which is a contradiction. This shows the first inequality.

Now let $z_1, \ldots, z_N$ be a maximal collection of points of C such that the $B(z_i, e^{-T})$ are disjoint. Then for every $z \in C$ there exists i such that $B(z, e^{-T}) \cap B(z_i, e^{-T}) \neq \emptyset$. Therefore there exists $w \in \partial X$ such that $(z_i, w)_x > T$ and $(z, w)_x > T$. As before, we get $(z_i, z)_x > T - \delta$, proving the second inequality. □

4.2. *Groups of isometries, limit set and critical exponent.* Let X be a proper, δ-hyperbolic metric space. Every isometry of X acts naturally on ∂X and the resulting map on $X \cup \partial X$ is a homeomorphism. The *limit set* $\Lambda(\Gamma)$ of a discrete group of isometries Γ is the set of accumulation points of the orbit Γx on ∂X, where x is any point of X. It is the smallest Γ-invariant closed set of the Gromov boundary, indeed we have the following proposition.

PROPOSITION 4.11. [**Coo93**, Theorem 5.1] *Let Γ be a discrete group of isometries of a proper, Gromov-hyperbolic metric space. Then $\Lambda(\Gamma)$ is the smallest closed Γ-invariant subset of ∂X, that is, every Γ-invariant, closed subset C of ∂X contains $\Lambda(\Gamma)$.*

The group Γ is called *elementary* if $\#\Lambda(\Gamma) \leq 2$. The limit superior in the definition of the critical exponent of Γ is a true limit.

LEMMA 4.12. [**Cav21b**, Theorem B] *Let X be a proper, δ-hyperbolic metric space and let Γ be a discrete group of isometries of X. Then*

$$h_\Gamma = \lim_{T\to+\infty} \frac{1}{T} \log \#\Gamma x \cap \overline{B}(x, T).$$

The *critical exponent* of Γ can be seen also as

$$h_\Gamma = \inf\left\{s \geq 0 \text{ s.t. } \sum_{g\in\Gamma} e^{-sd(x,gx)} < +\infty\right\}.$$

We remark that for every $s \geq 0$ the series $\sum_{g\in\Gamma} e^{-sd(x,gx)}$, which is called the Poincaré series of Γ, is Γ-invariant. In other words, $\sum_{g\in\Gamma} e^{-sd(x,gx)} = \sum_{g\in\Gamma} e^{-sd(x',gx')}$ for all $x' \in \Gamma x$. There is a canonical way to construct a measure on ∂X starting from the Poincaré series. For every $s > h_\Gamma$ the measure

$$\mu_s = \frac{1}{\sum_{g\in\Gamma} e^{-sd(x,gx)}} \cdot \sum_{g\in\Gamma} e^{-sd(x,gx)} \Delta_{gx},$$

where Δ_{gx} is the Dirac measure at gx, is a probability measure on the compact space $X \cup \partial X$. Then there exists a sequence s_i converging to h_Γ such that μ_{s_i} converges $*$-weakly to a probability measure on $X \cup \partial X$. Any of these limits is called a Patterson–Sullivan measure and is denoted by μ_{PS}.

PROPOSITION 4.13. [**Coo93**, Theorem 5.4] *Let X be a proper, δ-hyperbolic metric space and let Γ be a discrete group of isometries of X with $h_\Gamma < +\infty$. Then every Patterson–Sullivan measure is supported on $\Lambda(\Gamma)$. Moreover, it is a Γ-quasiconformal density of dimension h_Γ, that is, it satisfies*

$$\frac{1}{Q} \cdot e^{h_\Gamma(B_z(x,x)-B_z(x,gx))} \leq \frac{d(g_*\mu_{\mathrm{PS}})}{d\mu_{\mathrm{PS}}}(z) \leq Q \cdot e^{h_\Gamma(B_z(x,x)-B_z(x,gx))}$$

for every $g \in \Gamma$ and every $z \in \Lambda(\Gamma)$, where Q is a constant depending only on δ and an upper bound on h_Γ.

The quantification of Q is not explicated in the original paper, but it follows from the proof therein.

The set $\Lambda(\Gamma)$ is Γ-invariant so it is its quasiconvex hull. We recall that a discrete group of isometries Γ is *quasiconvex-cocompact* if and only if its action on QC-Hull$(\Lambda(\Gamma))$ is cocompact, that is, if there exists $D \geq 0$ such that for all $x, y \in$ QC-Hull$(\Lambda(\Gamma))$ the inequality $d(gx, y) \leq D$ holds for some $g \in \Gamma$. The smallest D satisfying this property is called the *codiameter* of Γ.

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Given two real numbers $\delta \geq 0$ and $D > 0$, we recall that $\mathcal{M}(\delta, D)$ is the class of triples (X, x, Γ), where X is a proper, geodesic, δ-hyperbolic metric space, Γ is a discrete, non-elementary, torsion-free, quasiconvex-cocompact group of isometries with codiameter $\leq D$ and $x \in \text{QC-Hull}(\Lambda(\Gamma))$. For an element (X, x, Γ) of $\mathcal{M}(\delta, D)$ we will use Y to denote $\text{QC-Hull}(\Lambda(\Gamma))$.

5. $\mathcal{M}(\delta, D)$ *is closed under equivariant Gromov–Hausdorff limits*

The purpose of this section is to prove statement (i) of Theorem A. We need to understand better the properties of the spaces belonging to $\mathcal{M}(\delta, D)$.

5.1. *Entropy and systolic estimates on* $\mathcal{M}(\delta, D)$.

The following are straightforward adaptations of results of [**BCGS17**, **BCGS21**].

LEMMA 5.1. *If* $(X, x, \Gamma) \in \mathcal{M}(\delta, D)$ *then* $\Sigma_{2D+72\delta}(x)$ *generates* Γ.

Proof. The proof is classical for geodesic metric spaces. In this setting we need to use the fact that $\text{QC-Hull}(\Lambda(\Gamma))$ is 36δ-quasiconvex. By discreteness we can fix a small $\varepsilon > 0$ such that $d(x, gx) < 2D + 72\delta + \varepsilon$ implies $d(x, gx) \leq 2D + 72\delta$ for all $g \in \Gamma$. We take any $g \in \Gamma$ and we take consecutive points x_i, $i = 0, \ldots, N$, on a geodesic segment $[x, gx]$ such that $x_0 = x$, $x_N = gx$ and $d(x_i, x_{i+1}) < \varepsilon$. By Lemma 4.5 each x_i is at distance at most 36δ from a point $y_i \in \text{QC-Hull}(\Lambda(\Gamma))$, hence there exists some $h_i \in \Gamma$ such that $d(h_i x, x_i) \leq 36\delta + D$. We can choose $h_N = g$ and $h_0 = \text{id}$. We define the elements $g_i = h_{i-1}^{-1} h_i$ for $i = 1, \ldots, N$. Clearly $g_1 \cdots g_{N-1} g_N = g$. Moreover, $d(g_i x, x) < 72\delta + 2D + \varepsilon$ for every i, so $g_i \in \Sigma_{2D+72\delta}(x)$. □

PROPOSITION 5.2. [**BCGS17**, Proposition 5.10] *If* $(X, x, \Gamma) \in \mathcal{M}(\delta, D)$ *then* $h_\Gamma \geq \log 2/(99\delta + 10D)$.

Proof. Using the same proof as for [**BCGS17**, Lemma 5.14], we conclude that there exists a hyperbolic isometry $a \in \Gamma$ such that $\ell(a) \leq 8D + 10\delta$. The remaining part of the proof can be done exactly in the same way as in [**BCGS17**], choosing $y \in \text{Min}(a) \subseteq \text{QC-Hull}(\Lambda(\Gamma))$ and using Lemma 5.1. □

PROPOSITION 5.3. [**BCGS21**, Theorem 3.4] *For every* $H \geq 0$ *there exists* $s = s(\delta, D, H) > 0$ *such that if* $(X, x, \Gamma) \in \mathcal{M}(\delta, D)$ *and if* $h_\Gamma \leq H$ *then* $sys(\Gamma, X) \geq s$.

Proof. The proof is the same as for [**BCGS21**, Theorem 3.4]. The only non-trivial part is the Bishop–Gromov estimate stated in [**BCGS21**, Theorem 3.1] and proved in [**BCGS17**, Theorem 5.1]. It is made in the cocompact case but it extends word for word to the quasiconvex-cocompact setting. □

5.2. *Covering entropy.*

Let Y be any subset of a metric space X.

- A subset S of Y is called r-dense if for all $y \in Y$ there exists $z \in S$ such that $d(y, z) \leq r$.
- A subset S of Y is called r-separated if $d(y, z) > r$ for all $y, z \in S$.

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The packing number of Y at scale r is the maximal cardinality of a $2r$-separated subset of Y and is denoted by $\mathrm{Pack}(Y, r)$. The covering number of Y is the minimal cardinality of an r-dense subset of Y and is denoted by $\mathrm{Cov}(Y, r)$. These two quantities are classically related by

$$\mathrm{Pack}(Y, 2r) \le \mathrm{Cov}(Y, 2r) \le \mathrm{Pack}(Y, r). \tag{5.1}$$

Y is said to be uniformly packed at scales $0 < r \le R$ if

$$\mathrm{Pack}_Y(R, r) := \sup_{x \in Y} \mathrm{Pack}(\overline{B}(x, R) \cap Y, r) < +\infty$$

and uniformly covered at scales $0 < r \le R$ if

$$\mathrm{Cov}_Y(R, r) := \sup_{x \in Y} \mathrm{Cov}(\overline{B}(x, R) \cap Y, r) < +\infty.$$

LEMMA 5.4. *Let $(X, x, \Gamma) \in \mathcal{M}(\delta, D)$. Then $Y = \mathrm{QC\text{-}Hull}(\Lambda(\Gamma))$ is uniformly packed and uniformly covered at any scales.*

Proof. We prove only that Y is uniformly covered since the other case is similar. We fix $0 < r \le R$. The map $y \mapsto \mathrm{Cov}(\overline{B}(y, R) \cap Y, r)$ defined on Y is clearly Γ-invariant. If the thesis is false we could find a sequence of points $x_n \in Y$ such that $\mathrm{Cov}(\overline{B}(x_n, R) \cap Y, r) \ge n$. By Γ-invariance and the compactness of the quotient we can suppose that x_n converges to some point x_∞ that belongs to Y by Lemma 4.5. Clearly we would have $\mathrm{Cov}(\overline{B}(x_\infty, R + 1) \cap Y, r) = \infty$, which is impossible since $\overline{B}(x_\infty \cap Y, R + 1)$ is compact. □

We recall the notion of covering entropy. This was studied by the author in a less general context in [Cav21a].

Definition 5.5. Let X be a proper metric space and $x \in X$. The *upper covering entropy* of X at scale $r > 0$ is the quantity

$$\overline{h}_{\mathrm{Cov}}(X, r) = \limsup_{T \to +\infty} \frac{\log \mathrm{Cov}(\overline{B}(x, T), r)}{T},$$

while the *lower covering entropy* of X at scale r is

$$\underline{h}_{\mathrm{Cov}}(X, r) = \liminf_{T \to +\infty} \frac{\log \mathrm{Cov}(\overline{B}(x, T), r)}{T}.$$

They do not depend on the point $x \in X$ by a standard argument.

LEMMA 5.6. *Let $(X, x, \Gamma) \in \mathcal{M}(\delta, D)$. If there exist $r, P > 0$ such that Pack $_Y(72\delta + 3r, r) \le P$ then Pack $_Y(T, r) \le P \cdot (1 + P)^{(T/r)-1}$ for every $T \ge 0$. In particular, $\overline{h}_{\mathrm{Cov}}(Y, 2r) \le (\log(1 + P))/r$.*

Proof. The proof is the same as for [CS21, Lemma 4.7], except for the fact that Y is not geodesic but only 36δ-quasigeodesic by Lemma 4.5. We proceed by induction on k, where k is the smallest integer such that $T \le 72\delta + 3r + kr$. For $k = 0$ the result is obvious by our assumption. The inductive step goes as follows: by induction

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we can find a maximal $2r$-separated subset $\{y_1, \ldots, y_N\}$ of $\overline{B}(x, T-r) \cap Y$ with $N \leq P(1+P)^{((T-r)/r)-1}$. The key step is to show that $\bigcup_{i=1}^N \overline{B}(y_i, 72\delta + 3r) \supseteq A(x, T-r, T) \cap Y$, where $A(x, T-r, T)$ is the closed annulus centred at x of radii $T-r$ and T. Indeed, for every point $y \in A(x, T-r, T)$ we consider the point y' along a geodesic segment $[x, y]$ at distance $T - r - 36\delta$ from x. By quasiconvexity there is a point $z \in Y$ at distance no greater than 36δ from y'. In particular, $z \in \overline{B}(x, T-r)$, so $d(z, y_i) \leq 2r$ for some $i = 1, \ldots, N$. We conclude that $d(y, y_i) \leq 72\delta + 3r$. The rest of the proof can be done exactly as for [**CS21**, Lemma 4.7], while the estimate on the upper covering entropy follows trivially using (5.1). □

PROPOSITION 5.7. *Let $(X, x, \Gamma) \in \mathcal{M}(\delta, D)$. Then $\overline{h}_{\mathrm{Cov}}(Y, r)$ and $\underline{h}_{\mathrm{Cov}}(Y, r)$ do not depend on r and the same quantities can be defined by replacing the covering function with the packing function. Moreover, they coincide and*

$$h_{\mathrm{Cov}}(Y) := \lim_{T \to +\infty} \frac{\log \mathrm{Cov}(\overline{B}(x, T) \cap Y, r)}{T} = h_\Gamma < +\infty.$$

Proof. Let us fix $0 < r \leq r'$. We have

$$\mathrm{Cov}(\overline{B}(x, T) \cap Y, r') \leq \mathrm{Cov}(\overline{B}(x, T) \cap Y, r)$$

and

$$\mathrm{Cov}(\overline{B}(x, T) \cap Y, r) \leq \mathrm{Cov}(\overline{B}(x, T) \cap Y, r') \cdot \mathrm{Cov}_Y(r', r).$$

The quantity $\mathrm{Cov}_Y(r', r)$ is finite by Lemma 5.4. These inequalities easily imply that $\overline{h}_{\mathrm{Cov}}(Y, r) = \overline{h}_{\mathrm{Cov}}(Y, r')$ and $\underline{h}_{\mathrm{Cov}}(Y, r) = \underline{h}_{\mathrm{Cov}}(Y, r')$. Moreover, by (5.1) these quantities can be defined by replacing the covering function with the packing function. Furthermore, an application of Lemmas 5.4 and 5.6 shows that the upper covering entropy of Y is finite.

Let $2s = \mathrm{sys}(\Gamma, X) > 0$. We have

$$\mathrm{Cov}(\overline{B}(x, T) \cap Y, D) \leq \#\Gamma x \cap \overline{B}(x, T + D)$$

and

$$\mathrm{Pack}(\overline{B}(x, T) \cap Y, s) \geq \#\Gamma x \cap \overline{B}(x, T).$$

Observe that the sequence $\#\Gamma x \cap \overline{B}(x, T)$ converges to h_Γ when T goes to $+\infty$ by Lemma 4.12. Therefore $\overline{h}_{\mathrm{Cov}}(Y, D) \leq h_\Gamma$ and $\underline{h}_{\mathrm{Cov}}(Y, s) \geq h_\Gamma$, implying the last part of the thesis. □

5.3. *Convergence of spaces in $\mathcal{M}(\delta, D)$.* The following situation will be called the *standard setting of convergence*: we have a sequence $(X_n, x_n, \Gamma_n) \in \mathcal{M}(\delta, D)$ such that $(X_n, x_n, \Gamma_n) \underset{\text{eq-pGH}}{\longrightarrow} (X_\infty, x_\infty, \Gamma_\infty)$. Observe that X_∞ is a proper metric space by definition.

LEMMA 5.8. *In the standard setting of convergence* $\sup_{n \in \mathbb{N}} \mathrm{Pack}_{Y_n}(R, r) < +\infty$ *for every* $0 < r \leq R$.

Proof. By the Gromov precompactness theorem [**Gro81**] we know that

$$\sup_{n\in\mathbb{N}} \operatorname{Pack}(\overline{B}(x_n, R + D) \cap Y_n, r) =: P < +\infty.$$

For every $n \in \mathbb{N}$ and every point $y_n \in Y_n$ there is some $g \in \Gamma_n$ such that $d(y_n, g_n x_n) \leq D$. Therefore

$$\begin{aligned}\operatorname{Pack}(\overline{B}(y_n, R) \cap Y_n, r) &\leq \operatorname{Pack}(\overline{B}(gx_n, R + D) \cap Y_n, r)\\ &= \operatorname{Pack}(\overline{B}(x_n, R + D) \cap Y_n, r) \leq P\end{aligned}$$

by the Γ_n-invariance of Y_n. □

COROLLARY 5.9. *In the standard setting of convergence* $\sup_{n\in\mathbb{N}} h_{\Gamma_n} < +\infty$.

Proof. By Lemma 5.8 we have $\sup_{n\in\mathbb{N}} \operatorname{Pack}_{Y_n}(72\delta + 3, 1) =: P < +\infty$. Therefore by Proposition 5.7 and Lemma 5.6 we have

$$h_{\Gamma_n} = h_{\mathrm{Cov}}(Y_n) \leq \log(1 + P).$$ □

COROLLARY 5.10. *In the standard setting of convergence* Γ_∞ *is discrete and torsion-free.*

Proof. By Corollary 5.9 and Proposition 5.3 there is some $s > 0$ such that $\operatorname{sys}(\Gamma_n, X_n) \geq s$ for every $n \in \mathbb{N}$. Let ω be a non-principal ultrafilter. By Proposition 3.13 it is enough to show that Γ_ω is discrete and torsion-free. Let $g_\omega = \omega\text{-lim}\, g_n$ be a non-trivial element of Γ_ω and $y_\omega = \omega\text{-lim}\, y_n$ be a point of X_ω. We know that $d(y_n, g_n y_n) \geq s$ for ω-a.e. (n). This implies $d(y_\omega, g_\omega y_\omega) \geq s$. Since this is true for every $g_\omega \in \Gamma_\omega$ and every $y_\omega \in X_\omega$ we conclude that $\operatorname{sys}(\Gamma_\omega, X_\omega) \geq s$. Since X_ω is proper we conclude that Γ_ω is discrete. Now take an elliptic element $g_\omega = \omega\text{-lim}\, g_n$. It is classical that g_ω must have finite order since Γ_ω is discrete (see, for instance, [**BCGS17**, Remark 8.16]), that is, $g_\omega^k = \mathrm{id}$ for some $k \in \mathbb{Z} \setminus \{0\}$. This means that $\omega\text{-lim}\, d(g_n^k x_n, x_n) = 0$, so $g_n^k = \mathrm{id}$ for ω-a.e. (n). This implies $g_n = \mathrm{id}$ for ω-a.e. (n) and therefore $g_\omega = \mathrm{id}$. In other words, Γ_ω is torsion-free. □

The next step is to show the stability of the boundary under convergence.

PROPOSITION 5.11. *Let* (X_n, x_n) *be a sequence of proper,* δ*-hyperbolic metric spaces and let* $D_{x_n,a}$ *be a standard visual metric of centre* x_n *and parameter* a *on* ∂X_n*. Let* ω *be a non-principal ultrafilter and let* (X_ω, x_ω) *be the ultralimit of the sequence* (X_n, x_n)*. Then there exists a natural map* $\Psi : \omega\text{-lim}(\partial X_n, D_{x_n,a}) \to \partial X_\omega$ *which is a homeomorphism onto the image.*

Remark 5.12. The following observations are in order.

(1) When the spaces ∂X_n are compact with diameter at most 1, the ultralimit $\omega\text{-lim}\, \partial X_n$ does not depend on the basepoints.
(2) In general the map Ψ is not surjective. Let X_n be the closed ball $\overline{B}(o, n)$ inside the hyperbolic plane $\mathbb{H}^2$, where o is a fixed basepoint. Each X_n is proper and δ-hyperbolic for the same δ, but $\partial X_n = \emptyset$. Therefore $\omega\text{-lim}\, \partial X_n = \emptyset$. On the other hand, $X_\omega = \mathbb{H}^2$ and $\partial X_\omega \neq \emptyset$.

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(3) It is possible to prove (but we will not do so because it is not necessary for our purposes) that if for each point y_n of X_n there is a geodesic ray $[x_n, z_n]$ passing at distance no greater than δ from y_n then the map Ψ is surjective. Moreover, when Ψ is surjective, the metric induced on ∂X_ω by Ψ is a visual metric of centre x_ω and parameter a.

Proof. A point of ω-lim ∂X_n is a class of a sequence of points $(z_n) \in \partial X_n$ and for each point z_n there exists a geodesic ray ξ_{x_n,z_n}. The sequence of geodesic rays (ξ_{x_n,z_n}) defines an ultralimit geodesic ray ξ of X_ω with $\xi(0) = x_\omega$ (cf. [**CS21**, Lemma A.7]) which provides a point of ∂X_ω. We denote this point by z_ω and ξ by ξ_{x_ω,z_ω}. We define the map $\Psi\colon \omega\text{-lim}\, \partial X_n \to \partial X_\omega$ as $\Psi((z_n)) = \xi^+_{x_\omega,z_\omega} = z_\omega$.

Good definition. We need to show that Ψ is well defined, that is, it does not depend on the choice of the geodesic ray ξ_{x_n,z_n} or on the choice of the sequence (z_n). Let (z'_n) be a sequence of points equivalent to (z_n), that is, $\omega\text{-lim}\, D_{x_n,a}(z_n, z'_n) = 0$. Choose geodesic rays ξ_{x_n,z_n}, ξ_{x_n,z'_n}. For every n the metric $D_{x_n,a}$ is a standard visual metric. Then for every fixed $\varepsilon > 0$ we have $(z_n, z'_n)_{x_n} > \log(1/\varepsilon) =: T_\varepsilon$ for ω-a.e.(n). Thus $d(\xi_{x_n,z_n}(T_\varepsilon - \delta), \xi_{x_n,z'_n}(T_\varepsilon - \delta)) \le 4\delta$ by Lemma 4.2, ω-a.s. We conclude that $d(\xi_{x_\omega,z_\omega}(T_\varepsilon - \delta), \xi_{x_\omega,z'_\omega}(T_\varepsilon - \delta)) < 6\delta\omega$-a.s. Therefore, again by Lemma 4.2, $(z_\omega, z'_\omega)_{x_\omega} > T_\varepsilon - 4\delta$. Thus $(z_\omega, z'_\omega)_{x_\omega} = +\infty$ by the arbitrariness of ε, that is, $z_\omega = z'_\omega$.

Injectivity. The next step is to show that Ψ is injective. If two sequences of points $(z_n), (z'_n)$ have the same image under Ψ then $(\xi^+_{x_\omega,z_\omega}, \xi^+_{x_\omega,z'_\omega})_{x_\omega} = +\infty$. So $(\xi^+_{x_\omega,z_\omega}, \xi^+_{x_\omega,z'_\omega})_{x_\omega} \ge T$ for every fixed $T \ge 0$. Hence $d(\xi_{x_\omega,z_\omega}(T - \delta), \xi_{x_\omega,z'_\omega}(T - \delta)) \le 4\delta$, by Lemma 4.2. Then $d(\xi_{x_n,z_n}(T - \delta), \xi_{x_n,z'_n}(T - \delta)) < 6\delta\omega$-a.s., that is, $(z_n, z'_n)_{x_n} > T - 4\delta\omega$-a.s., again by Lemma 4.2. Therefore $D_{x_n,a}(z_n, z'_n) \le e^{-a(T-4\delta)}$. Since this is true ω-a.s. we get $\omega\text{-lim}\, D_{x_n,a}(z_n, z'_n) \le e^{-a(T-4\delta)}$ for ω-a.e. (n). By the arbitrariness of T we deduce that $\omega\text{-lim}\, D_{x_n,a}(z_n, z'_n) = 0$, that is, $(z_n) = (z'_n)$ as elements of ω-lim ∂X_n.

Homeomorphism. Let us show Ψ is continuous. As both ω-lim ∂X_n and ∂X_ω are metrizable, it is enough to check the continuity on sequences of points. We take a sequence $(z_n^k)_{k\in\mathbb{N}}$ converging to (z_n^∞) in ω-lim ∂X_n. By definition for every $\varepsilon > 0$ there exists $k_\varepsilon \ge 0$ such that if $k \ge k_\varepsilon$ then $\omega\text{-lim}\, D_{x_n,a}(z_n^k, z_n^\infty) < \varepsilon$. Therefore for every fixed $k \ge k_\varepsilon$ we have $(z_n^k, z_n^\infty)_{x_n} \ge \log(1/\varepsilon) =: T_\varepsilon$ for ω-a.e. (n). As usual, we conclude that $d(\xi_{x_n z_n^k}(T_\varepsilon - \delta), \xi_{x_n z_n^\infty}(T_\varepsilon - \delta)) \le 4\delta\omega$-a.s. Thus $d(\xi_{x_\omega z_\omega^k}(T_\varepsilon - \delta), \xi_{x_\omega,z_\omega^\infty}(T_\varepsilon - \delta)) < 6\delta$ for every fixed $k \ge k_\varepsilon$. Again this implies $(z_\omega^k, z_\omega^\infty)_{x_\omega} > T_\varepsilon - 4\delta$ for all $k \ge k_\varepsilon$. By the arbitrariness of ε we get that z_ω^k converges to z_ω^∞ when k goes to $+\infty$.

The continuity of the inverse map defined on the image of Ψ can be proved in a similar way. □

Proof of Theorem A(i). In order to simplify the notation we fix a non-principal ultrafilter ω. We know that $(X_\omega, x_\omega, \Gamma_\omega)$ is equivariantly isometric to $(X_\infty, x_\infty, \Gamma_\infty)$ by Proposition 3.13 and that X_ω is a proper metric space, so we can prove all the properties for this triple. It is classical that the ultralimit of geodesic, δ-hyperbolic metric spaces is a geodesic and δ-hyperbolic metric space; see, for instance, [**DK18**]. Moreover, by Corollary 5.10 the group Γ_ω is discrete and torsion-free.

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Let Ψ be the homeomorphism onto the image given by Proposition 5.11. We claim that $\Lambda(\Gamma_\omega) = \Psi(\omega\text{-lim}\, \Lambda(\Gamma_n))$. We fix a sequence $z_n \in \Lambda(\Gamma_n)$ and we observe that by Lemma 4.6 and the cocompactness of the action of Γ_n on QC-Hull$(\Lambda(\Gamma_n))$ we can find a sequence $(g_n^k)_{k\in\mathbb{N}} \subseteq \Gamma_n$ such that, denoting one geodesic ray $[x_n, z_n]$ by ξ_{x_n,z_n}, we have:

(a) $g_n^k x_n$ converges to z_n when k tends to $+\infty$;
(b) $g_n^0 = \mathrm{id}$;
(c) $d(g_n^k x_n, g_n^{k+1} x_n) \leq 28\delta + 2D$;
(d) $d(g_n^k x_n, \xi_{x_n,z_n}(k)) \leq 14\delta + D$.

For every $k \in \mathbb{N}$ the sequence g_n^k is admissible by (b) and (c), so it defines a limit isometry $g_\omega^k \in \Gamma_\omega$. Moreover, if ξ_{x_ω,z_ω} is the ultralimit of the sequence of geodesic rays ξ_{x_n,z_n}, we have $d(g_\omega^k x_\omega, \xi_{x_\omega,z_\omega}(k)) \leq 14\delta + D$ for every $k \in \mathbb{N}$. Observe that $\xi^+_{x_\omega,z_\omega} = \Psi((z_n))$ by definition of Ψ. As a consequence the sequence $g_\omega^k x_\omega$ converges to $\Psi((z_n))$, that is, $\Psi(z_n) \in \Lambda(\Gamma_\omega)$. This shows that $\Psi(\omega\text{-lim}\, \Lambda(\Gamma_n)) \subseteq \Lambda(\Gamma_\omega)$. Clearly Γ_ω acts on ω-lim $\Lambda(\Gamma_n)$ by $(g_n)(z_n) = (g_n z_n)$ and this action commutes with Ψ. The set ω-lim $\Lambda(\Gamma_n)$ is Γ_ω-invariant and closed. The Γ_ω-invariance is trivial, so let us check the closure. If $(z_n^k)_{k\in\mathbb{N}} \in \omega\text{-lim}\, \Lambda(\Gamma_n)$ is a sequence converging to (z_n^∞) and $z_n^\infty \notin \omega\text{-lim}\, \Lambda(\Gamma_n)$ then there exists $\varepsilon > 0$ such that $D_{x_n,a}(z_n^\infty, \Lambda(\Gamma_n)) \geq \varepsilon\omega$-a.s. This is a contradiction. Therefore the set $\Psi(\omega\text{-lim}\, \Lambda(\Gamma_n))$ is also closed and Γ_ω-invariant. By Proposition 4.11 we conclude that $\Lambda(\Gamma_\omega) = \Psi(\omega\text{-lim}\, \Lambda(\Gamma_n))$. This also implies that ω-lim QC-Hull$(\Lambda(\Gamma_n))$ = QC-Hull$(\Lambda(\Gamma_\omega))$ and so $x_\omega \in$ QC-Hull$(\Lambda(\Gamma_\omega))$. For every pair of points $y_\omega, y'_\omega \in$ QC-Hull$(\Lambda(\Gamma_\omega))$ there exist sequences of points $y_n, y'_n \in$ QC-Hull$(\Lambda(\Gamma_n))$ such that $y_\omega = \omega\text{-lim}\, y_n$ and $y'_\omega = \omega\text{-lim}\, y'_n$. So there exists $g_n \in \Gamma_n$ such that $d(g_n y_n, y'_n) \leq D$. The sequence g_n is clearly admissible so it defines an element $g_\omega = \omega\text{-lim}\, g_n$ of Γ_ω and $d(g_\omega y_\omega, y'_\omega) \leq D$, implying that the action of Γ_ω on QC-Hull$(\Lambda(\Gamma_\omega))$ is cocompact with codiameter less than or equal to D. It remains only to show that Γ_ω is non-elementary. If Γ_ω is elementary then QC-Hull$(\Lambda(\Gamma_\omega)) = \mathbb{R}$ and Γ_ω acts on $\mathbb{R}$ as $\mathbb{Z}_\tau$, the group generated by the translation of length τ, for some $\tau > 0$. Denote by $g_\omega = \omega\text{-lim}\, g_n$ the element corresponding to this translation. For every $k \in \mathbb{N}$ we notice that

$$A_\omega(k) = \{h_\omega \in \Gamma_\omega \text{ s.t. } d(x_\omega, h_\omega x_\omega) < (k+1)\cdot\tau\} = \{g_\omega^{\pm m}\} : m = 0, \ldots, k.$$

In particular, $A_\omega(k)$ has cardinality $2k+1$. We define also the sets

$$A_n(k) = \left\{h_n \in \Gamma_n \text{ s.t. } d(x_n, h_n x_n) \leq \left(k + \tfrac{1}{2}\right)\cdot\tau\right\}.$$

Since we have a uniform bound on the systole and the action is torsion-free, we have that $\#A_n(k) \leq \#A_\omega(k)\omega$-a.s., for every fixed $k \in \mathbb{N}$. We apply this property to $k_0 = (72\delta + 2D)/\tau$. Clearly $g_n^{\pm m} \in A_n(k_0)$ for every $m = 0, \ldots, k_0$, ω-a.s. Therefore $A_n(k_0) = \{g_n^{\pm m}\}_{m=0,\ldots,k_0}\omega$-a.s. By Lemma 5.1 we conclude that

$$\Gamma_n = \langle A_n(k_0)\rangle = \langle g_n\rangle,$$

that is, Γ_n is elementary ω-a.s., which is a contradiction. □

It is interesting to compare our convergence with the Gromov–Hausdorff convergence of the quotient spaces.

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THEOREM 5.13. *Let* $(X_n, x_n, \Gamma_n), (X_\infty, x_\infty, \Gamma_\infty) \in \mathcal{M}(\delta, D)$.

(i) *If* $(X_n, x_n, \Gamma_n) \underset{\text{eq-pGH}}{\longrightarrow} (X_\infty, x_\infty, \Gamma_\infty)$ *then* $(\Gamma_n \backslash X_n, \bar{x}_n) \underset{\text{pGH}}{\longrightarrow} (\Gamma_\infty \backslash X_\infty, \bar{x}_\infty)$ *and* $\sup_{n\in\mathbb{N}} h_{\Gamma_n} < +\infty$.

(ii) *If* $(\Gamma_n \backslash X_n, \bar{x}_n) \underset{\text{pGH}}{\longrightarrow} (Y, y)$ *and* $\sup_{n\in\mathbb{N}} h_{\Gamma_n} < +\infty$ *then there exists a subsequence* $\{n_k\}$ *such that* $(X_{n_k}, x_{n_k}, \Gamma_{n_k}) \underset{\text{eq-pGH}}{\longrightarrow} (X_\infty, x_\infty, \Gamma_\infty)$ *and* (Y, y) *is isometric to* $(\Gamma_\infty \backslash X_\infty, \bar{x}_\infty)$.

Proof. If $(X_n, x_n, \Gamma_n) \underset{\text{eq-pGH}}{\longrightarrow} (X_\infty, x_\infty, \Gamma_\infty)$ then $\sup_{n\in\mathbb{N}} h_{\Gamma_n} < +\infty$ by Corollary 5.9. The second part of the first statement is true once we show that the ultralimit $(\bar{X}_\omega, \bar{x}_\omega)$ of the sequence $(\Gamma_n \backslash X_n =: \bar{X}_n, \bar{x}_n)$ is isometric to $(\Gamma_\infty \backslash X_\infty, \bar{x}_\infty)$ for every non-principal ultrafilter, by Corollary 3.14. We fix a non-principal ultrafilter ω. By Proposition 3.13 the triple $(X_\omega, x_\omega, \Gamma_\omega)$ is equivariantly isometric to $(X_\infty, x_\infty, \Gamma_\infty)$.

The projections $p_n \colon X_n \to \bar{X}_n$ form an admissible sequence of 1-Lipschitz maps and then, by [**CS21**, Proposition A.5], they yield a limit map $p_\omega \colon X_\omega \to \bar{X}_\omega$ defined as $p_\omega(y_\omega) = \omega\text{-lim}\, p_n(y_n)$, for $\omega\text{-lim}\, y_n = y_\omega$. The map p_ω is clearly surjective. It is also Γ_ω-equivariant: indeed,

$$p_\omega(\gamma_\omega y_\omega) = \omega\text{-lim}\, p_n(\gamma_n y_n) = \omega\text{-lim}\, p_n(y_n) = p_\omega(y_\omega)$$

for every $g_\omega = \omega\text{-lim}\, g_n \in \Gamma_\omega$ and $y_\omega = \omega\text{-lim}\, y_n \in X_\omega$. Therefore we have a well-defined, surjective quotient map $\bar{p}_\omega \colon \Gamma_\omega \backslash X_\omega \to \bar{X}_\omega$. The next step is to show it is a local isometry. We fix an arbitrary point $y_\omega = \omega\text{-lim}\, y_n \in X_\omega$ and we consider its class $[y_\omega] \in \Gamma_\omega \backslash X_\omega$. By Proposition 5.3 there exists $s > 0$ such that $\mathrm{sys}(\Gamma_n, X_n) \geq s$ for every n. So, as in the proof of Corollary 5.10, the systole of Γ_ω is at least s. Therefore the quotient map $X_\omega \to \Gamma_\omega \backslash X_\omega$ is an isometry between $\overline{B}(y_\omega, s/2)$ and $\overline{B}([y_\omega], s/2)$. Moreover, $\overline{B}(p_n(y_n), s/2)$ is isometric to $\overline{B}(y_n, s/2)$ for every n. By [**CS21**, Lemma A.8] we know that $\omega\text{-lim}\, \overline{B}(p_n(y_n), s/2)$ is isometric to $\overline{B}(p_\omega(y_\omega), s/2) = \overline{B}(\bar{p}_\omega([y_\omega]), s/2)$ and that $\omega\text{-lim}\, \overline{B}(y_n, s/2)$ is isometric to $\overline{B}(y_\omega, s/2)$. Therefore $\overline{B}(\bar{p}_\omega([y_\omega]), s/2)$ is isometric to $\overline{B}([y_\omega], s/2)$, that is, $\bar{p}_\omega$ is a local isometry.

Now we prove that $\bar{p}_\omega$ is injective. Let $[z_\omega], [y_\omega] \in \Gamma_\omega \backslash X_\omega$. Clearly $\bar{p}_\omega([z_\omega]) = \bar{p}_\omega([y_\omega])$ if and only if $p_\omega(z_\omega) = p_\omega(y_\omega)$. This means $\omega\text{-lim}\, d(p_n(z_n), p_n(y_n)) = 0$ and, as the systole of Γ_n is greater than or equal to $s > 0$, we have $\omega\text{-lim}\, d(z_n, g_n y_n) = 0$ for some $g_n \in \Gamma_n$, ω-a.s. The sequence (g_n) is admissible, hence it defines an element $g_\omega = \omega\text{-lim}\, g_n \in \Gamma_\omega$ satisfying $d(z_\omega, g_\omega y_\omega) = 0$. This implies $[z_\omega] = [y_\omega]$.

The map $\bar{p}_\omega \colon \Gamma_\omega \backslash X_\omega \to \bar{X}_\omega$ is a bijective local isometry between two length spaces. If its inverse is continuous then it is an isometry. We take points $\bar{y}_\omega^k = \omega\text{-lim}\, \bar{y}_n^k \in \bar{X}_\omega$ converging to $\bar{y}_\omega^\infty = \omega\text{-lim}\, \bar{y}_n^\infty \in \bar{X}_\omega$ as $k \to +\infty$. We have $\bar{y}_n^k = p_n(y_n^k)$, $\bar{y}_n^\infty = p_n(y_n^\infty)$ for some $y_n^k, y_n^\infty \in X_n$. We can suppose that y_n^k, y_n^∞ belong to a fixed ball around x_n. We consider the points $y_\omega^k = \omega\text{-lim}\, y_n^k$ and $y_\omega^\infty = \omega\text{-lim}\, y_n^\infty$ of X_ω and their images $[y_\omega^k], [y_\omega^\infty] \in \Gamma_\omega \backslash X_\omega$. It is straightforward to show that $\bar{p}_\omega([y_\omega^k]) = \bar{y}_\omega^k$ and $\bar{p}_\omega([y_\omega^\infty]) = \bar{y}_\omega^\infty$. Now it is not difficult to check that the sequence $[y_\omega^k]$ converges to $[y_\omega^\infty]$ when $k \to +\infty$, proving that the inverse of $\bar{p}_\omega$ is continuous.

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Therefore $(\bar{X}_\omega, \bar{x}_\omega)$ is isometric to $(\Gamma_\omega \backslash X_\omega, p_\omega x_\omega)$ which is clearly isometric to $(\Gamma_\infty \backslash X_\infty, \bar{x}_\infty)$. The proof of (i) is then finished since this is true for every non-principal ultrafilter ω.

Suppose now that $(\Gamma_n \backslash X_n, \bar{x}_n) \underset{\text{pGH}}{\longrightarrow} (Y, y)$ and $\sup_{n\in\mathbb{N}} h_{\Gamma_n} < +\infty$. Again by Proposition 5.3 there exists $s > 0$ such that $\operatorname{sys}(\Gamma_n, X_n) \geq s$ for every n. We fix a non-principal ultrafilter ω and we consider the ultralimit triple $(X_\omega, x_\omega, \Gamma_\omega)$. As usual, we get $\operatorname{sys}(\Gamma_\omega, X_\omega) \geq s$. We can apply the same argument above to show that $\Gamma_\omega \backslash X_\omega$ is isometric to Y. Moreover, by the condition on the systole of Γ_ω we know that X_ω is locally isometric to $\Gamma_\omega \backslash X_\omega$. Since Y is compact we conclude that X_ω is a geodesic, complete, locally compact metric space. Therefore it is proper by the Hopf–Rinow theorem; see [**BH13**, Corollary I.3.8]. Then there exists a subsequence $\{n_k\}$ such that $(X_{n_k}, x_{n_k}, \Gamma_{n_k}) \underset{\text{eq-pGH}}{\longrightarrow} (X_\omega, x_\omega, \Gamma_\omega)$ by Proposition 3.13. □

6. *Continuity of the critical exponent*

Let Γ be a discrete, quasiconvex-cocompact group of isometries of a proper, δ-hyperbolic metric space X. Then it is proved in [**Coo93**] that the Patterson–Sullivan measure on $\Lambda(\Gamma)$ is (A, h_Γ)-Ahlfors regular for some $A > 0$. Our goal is to quantify the constant A.

THEOREM 6.1. *Let $\delta, D, H \geq 0$. There exists $A = A(\delta, D, H) \geq 1$ such that for all $(X, x, \Gamma) \in \mathcal{M}(\delta, D)$ with $h_\Gamma \leq H$ the subset $\Lambda(\Gamma)$ is visually (A, h_Γ)-Ahlfors regular with respect to every Patterson–Sullivan measure.*

Proof. We divide the proof into steps.

Step 1: for all $z \in \partial X$ and for all $\rho > 0$ we have $\mu_{\text{PS}}(B(z, \rho)) \leq e^{h_\Gamma(55\delta + 3D)} \rho^{h_\Gamma}$.

We suppose first $z \in \Lambda(\Gamma)$ and we take the set

$$\tilde{B}(z, \rho) = \left\{ y \in X \cup \partial X \text{ s.t. } (y, z)_x > \log \frac{1}{\rho} \right\}.$$

It is open (cf. [**DSU17**, Observation 4.5.2]) and $\tilde{B}(z, \rho) \cap \partial X = B(z, \rho)$, so $\mu_{\text{PS}}(\tilde{B}(z, \rho)) = \mu_{\text{PS}}(B(z, \rho))$ since μ_{PS} is supported on $\Lambda(\Gamma) \subseteq \partial X$. Let $T = \log(1/\rho)$ and $\xi_{x,z}$ be a geodesic ray $[x, z]$. For every $y \in \Gamma x \cap \tilde{B}(z, \rho)$ we have

$$d(x, y) \geq T - \delta \quad \text{and} \quad d(x, y) \geq d(x, \xi_{x,z}(T)) + d(\xi_{x,z}(T), y) - 12\delta. \tag{6.1}$$

The first inequality is given by Remark 4.3. Let γ be any geodesic segment $[x, y]$. Again by Remark 4.3 we have $d(\xi_{x,z}(T), \gamma(T)) \leq 6\delta$, therefore

$$d(x, y) = d(x, \gamma(T)) + d(\gamma(T), y) \geq d(x, \xi_{x,z}(T)) + d(\xi_{x,z}(T), y) - 12\delta.$$

Moreover, we have $d(\xi_{x,z}(T), \text{QC-Hull}(\Lambda(\Gamma))) \leq 14\delta$ by Lemma 4.6, since $x \in \text{QC-Hull}(\Lambda(\Gamma))$. By the cocompactness of the action on QC-Hull$(\Lambda(\Gamma))$ we can find a point $x_1 \in \Gamma x$ such that $d(\xi_{x,z}(T), x_1) \leq 14\delta + D$. This implies

$$d(x, y) \geq d(x, x_1) + d(x_1, y) - 40\delta - 2D$$

for every $y \in \Gamma x \cap \tilde{B}(z, \rho)$. Therefore

$$\begin{aligned}\sum_{y\in\Gamma x\cap\tilde{B}(z,\rho)} e^{-sd(x,y)} &\leq \sum_{y\in\Gamma x\cap\tilde{B}(z,\rho)} e^{-s(d(x,x_1)+d(x_1,y)-40\delta-2D)}\\ &= e^{s(40\delta+2D)}e^{-sd(x,x_1)}\cdot \sum_{y\in\Gamma x\cap\tilde{B}(z,\rho)} e^{-sd(x_1,y)}\\ &\leq e^{s(54\delta+3D)}e^{-sd(x,\xi_{xz}(T))}\cdot\sum_{g\in\Gamma} e^{-sd(x_1,gx_1)}\\ &= e^{s(54\delta+3D)}\cdot\rho^s\cdot\sum_{g\in\Gamma} e^{-sd(x,gx)}.\end{aligned}$$

In other words, we have $\mu_s(\tilde{B}(z, \rho)) \leq e^{s(54\delta+3D)}\rho^s$, and by $*$-weak convergence we conclude that

$$\mu_{\mathrm{PS}}(B(z, \rho)) = \mu_{\mathrm{PS}}(\tilde{B}(z, \rho)) \leq \liminf_{i\to+\infty} \mu_{s_i}(\tilde{B}(z, \rho)) \leq e^{h_\Gamma(54\delta+3D)}\rho^{h_\Gamma}.$$

If $z \in \partial X$ we observe that if $B(z, \rho) \cap \Lambda(\Gamma) = \emptyset$ then the thesis is obviously true since μ_{PS} is supported on $\Lambda(\Gamma)$. Otherwise there exists $w \in \Lambda(\Gamma)$ such that $(z, w)_x > \log(1/\rho)$. It is straightforward to check that $B(w, \rho) \subseteq B(z, \rho e^\delta)$ by (4.3). Then $\mu_{\mathrm{PS}}(B(z, \rho)) \leq e^{h_\Gamma(55\delta+3D)}\rho^{h_\Gamma}$.

Step 2: for every $R \geq R_0 := (\log 2/h_\Gamma) + 55\delta + 3D + 5\delta$ and for every $g \in \Gamma$ we have $\mu_{\mathrm{PS}}(\mathrm{Shad}_x(gx, R)) \geq (1/2Q)e^{-h_\Gamma d(x,gx)}$, where Q is the constant of Proposition 4.13 that depends only on δ and H.

From the first step we know that for every $\rho \leq \rho_0 := 2^{-(1/h_\Gamma)}e^{-(55\delta+3D)}$ and for every $z \in \partial X$ the inequality $\mu_{\mathrm{PS}}(B(z, \rho)) \leq \frac{1}{2}$ holds. A direct computation shows that $R_0 = \log(1/\rho_0) + 5\delta$. We claim that for every $R \geq R_0$ and every $g \in \Gamma$ the set $\partial X \setminus g(\mathrm{Shad}_x(g^{-1}x, R))$ is contained in a generalized visual ball of radius at most ρ_0. Indeed, if $z, w \in \partial X \setminus g(\mathrm{Shad}_x(g^{-1}x, R))$ then there are geodesic rays $\xi = [gx, z]$, $\xi' = [gx, w]$ that do not intersect the ball $B(x, R)$. Therefore we get $(\xi(T), gx)_x \geq d(x, [gx, \xi(T)]) - 4\delta \geq R - 4\delta$ by Lemma 4.1, so $(z, gx)_x \geq \liminf_{T\to+\infty}(\xi(T), gx)_x \geq R - 4\delta$. The same holds for w. Thus by (4.3) we get $(z, w)_x \geq R - 5\delta$, proving the claim. By Proposition 4.13 we get

$$\frac{\mu_{\mathrm{PS}}(\mathrm{Shad}_x(gx, R))}{\mu_{\mathrm{PS}}(g^{-1}(\mathrm{Shad}_x(gx, R)))} \geq \frac{1}{Q}e^{-h_\Gamma(B_z(x,x)-B_z(x,g^{-1}x))}.$$

Since $R \geq R_0$ the measure of $g^{-1}(\mathrm{Shad}_x(gx, R))$ is at least $\frac{1}{2}$. Moreover, the Busemann function is 1-Lipschitz, so

$$\mu_{\mathrm{PS}}(\mathrm{Shad}_x(gx, R)) \geq \frac{1}{2Q}e^{-h_\Gamma d(x,g^{-1}x)} = \frac{1}{2Q}e^{-h_\Gamma d(x,gx)}.$$

Step 3. $\mu_{\mathrm{PS}}(B(z, \rho)) \geq (1/2Q)e^{-h_\Gamma(R_0+28\delta+2D)}\rho^{h_\Gamma}$ for every $z \in \Lambda(\Gamma)$ and every $\rho > 0$.

For every $\rho > 0$ we set $T = \log(1/\rho)$. If $z \in \partial X$ and $R \geq 0$ then by Lemma 4.8 we get $\mathrm{Shad}_x(\xi_{x,z}(T + R), R) \subseteq B(z, e^{-T})$. We take $R = R_0 + 14\delta + D$, where R_0 is the constant of the second step, and we conclude that $\mathrm{Shad}_x(\xi_{x,z}(T + R), R)$ is contained in $B(z, \rho)$. Again applying Lemma 4.6 and the cocompactness of the action, we can find $g \in \Gamma$ such that $d(gx, \xi_{x,z}(T+R)) \leq 14\delta + D$, implying $\mathrm{Shad}_x(gx, R_0) \subseteq \mathrm{Shad}_x(\xi_{x,z}(T+R), R) \subseteq B(z,\rho)$. From the second step we obtain $\mu_{\mathrm{PS}}(B(z,\rho)) \geq (1/2Q)e^{-h_\Gamma d(x,gx)}$. Furthermore, $d(x,gx) \leq T + R_0 + 28\delta + 2D$, so finally

$$\mu_{\mathrm{PS}}(B(z,\rho)) \geq \frac{1}{2Q} e^{-h_\Gamma(R_0+28\delta+2D)} \rho^{h_\Gamma}.$$

The explicit description of the constants shows that they depend only on δ, H, D and on a lower bound on h_Γ, which is given in terms of δ and D by Proposition 5.2. $\square$

As a consequence, applying Corollary 5.9, we have the following result.

COROLLARY 6.2. *In the standard setting of convergence there exists some $A > 0$ such that every visual boundary ∂X_n is visually (A, h_{Γ_n})-Ahlfors-regular with respect to any Patterson–Sullivan measure.*

With this result it is possible to show the continuity of the critical exponent under the standard setting of convergence. However, we prefer to use the equidistribution of the orbits, following again the ideas of [**Coo93**].

Proof of Theorem B. By Corollary 5.9 we have $\sup_{n\in\mathbb{N}} h_{\Gamma_n} =: H < +\infty$. By Proposition 5.3 there exists $s > 0$ such that $\mathrm{sys}(\Gamma_n, X_n) \geq s$ for every n. Let $R_0 = R_0(\delta, D, H)$ be the number from Step 2 of Theorem 6.1 and Q be the constant from Proposition 4.13. By Lemma 5.8 we have

$$\sup_{n\in\mathbb{N}} \mathrm{Pack}_{Y_n}\left(4R_0 + 1, \frac{s}{2}\right) =: N < +\infty.$$

We fix $n \in \mathbb{N}$. It is easy to check that if $[x_n, z_n] \cap B(y_n, R_0) \neq \emptyset$ and $[x_n, z_n] \cap B(y'_n, R_0) \neq \emptyset$, where $z_n \in \partial X$ and y_n, y'_n are points of X_n with $|d(x_n, y_n) - d(x_n, y'_n)| \leq 1$, then $d(y_n, y'_n) \leq 4R_0 + 1$. Thus for every $j \in \mathbb{N}$ we have $\#\{y_n \in \Gamma_n x_n \text{ s.t. } y_n \in A(x_n, j, j+1) \text{ and } z_n \in \mathrm{Shad}_{x_n}(y_n, R_0)\} \leq N$.

Step 1. For all $k \in \mathbb{N}$ we have $\#\Gamma_n x_n \cap \overline{B}(x_n, k) \leq 4QNe^{h_{\Gamma_n}k}$. Let $A_{n,j} = \Gamma_n x_n \cap A(x_n, j, j+1)$. By the observation made before, we conclude that among the set of shadows $\{\mathrm{Shad}_{x_n}(y_n, R_0)\}_{y_n \in A_j}$ there are at least $\#A_j/N$ disjoint sets. Thus

$$1 \geq \mu_{\mathrm{PS}}\left(\bigcup_{y_n \in A_{n,j}} \mathrm{Shad}_{x_n}(y_n, R_0)\right) \geq \frac{\#A_{n,j}}{N} \cdot \frac{1}{2Q} e^{-h_{\Gamma_n}(j+1)},$$

where we used Step 2 of Theorem 6.1. This implies $\#A_{n,j} \leq 2QNe^{h_{\Gamma_n}(j+1)}$ for every $j \in \mathbb{N}$. Finally, we have

$$\#\Gamma_n x_n \cap \overline{B}(x_n, k) \leq \sum_{j=0}^{k-1} \#A_{n,j} \leq 4QNe^{h_{\Gamma_n}k}.$$

Step 2. For all $T \geq 0$ we have $\#\Gamma_n x_n \cap \overline{B}(x_n, T) \geq e^{-h_{\Gamma_n}(84\delta+5D+1)} e^{h_{\Gamma_n}T}$. We fix $z_1^n, \ldots, z_{K_n}^n \in \Lambda(\Gamma_n)$, realizing $\mathrm{Pack}^*(\Lambda(\Gamma_n), e^{-T+28\delta+2D+1})$: in particular, $(z_i^n, z_j^n)_{x_n} \leq T - 28\delta - 2D - 1$ for all $1 \leq i \neq j \leq K_n$. By Lemma 4.2 we deduce that $d(\xi_{x_n,z_i^n}(T - 14\delta - D), \xi_{x_n,z_j^n}(T - 14\delta - D)) \geq 28\delta + 2D + 1 > 28\delta + 2D$. Moreover,

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for every $1 \leq i \leq K_n$ we can find a point $y_i^n \in \Gamma x$ such that $d(\xi_{x_n,z_i^n}(T - 14\delta - D), y_i^n) \leq 14\delta + D$ by Lemma 4.6. Therefore we have $d(x_n, y_i^n) \leq T$ and $d(y_i^n, y_j^n) > 0$ for every $1 \leq i \neq j \leq K_n$. So

$$\begin{aligned}\#\Gamma_n x_n \cap \overline{B}(x_n, T) &\geq \mathrm{Pack}^*(\Lambda(\Gamma_n), e^{-T+28\delta+2D+1}) \\ &\geq \mathrm{Cov}(\Lambda(\Gamma_n), e^{-T+29\delta+2D+1}) \\ &\geq e^{-h_{\Gamma_n}(84\delta+5D+1)} e^{h_{\Gamma_n} T}.\end{aligned}$$

The first inequality follows from the discussion above, while the second is Lemma 4.10. The last inequality follows by Step 1 of Theorem 6.1. Indeed, we get

$$\begin{aligned}\mathrm{Cov}(\Lambda(\Gamma_n), e^{-T+29\delta+2D+1}) &\geq e^{-h_{\Gamma_n}(55\delta+3D)} e^{-h_{\Gamma_n}(-T+29\delta+2D+1)} \\ &= e^{-h_{\Gamma_n}(84\delta+5D+1)} e^{h_{\Gamma_n} T}.\end{aligned}$$

The thesis follows by the bounded quantification of all the constants involved in terms of δ, D, H, N and the lower bound on the critical exponent given in terms of δ and D by Proposition 5.2. □

We can conclude now the proof of Theorem A.

Proof of Theorem A(ii). Let ω be a non-principal ultrafilter. By Proposition 3.13 the triple $(X_\omega, x_\omega, \Gamma_\omega)$ is equivariantly isometric to $(X_\infty, x_\infty, \Gamma_\infty)$. By Theorem A(i) the triple $(X_\omega, x_\omega, \Gamma_\omega)$ belongs to $\mathcal{M}(\delta, D)$. By Proposition 5.7 the critical exponent of Γ_ω is finite, so for every $\varepsilon > 0$ there exists $T_\varepsilon \geq 0$ such that if $T \geq T_\varepsilon$ then

$$e^{T(h_{\Gamma_\omega} - \varepsilon)} \leq \#\Gamma_\omega x_\omega \cap \overline{B}(x_\omega, T) \leq e^{T(h_{\Gamma_\omega} + \varepsilon)}, \tag{6.2}$$

by Lemma 4.12. We fix K as in Theorem B and we set $T := \max\{T_\varepsilon, \log K \cdot e/\varepsilon\}$. It is not difficult to show that

$$\#\Gamma_n x_n \cap \overline{B}(x_n, T-1) \leq \#\Gamma_\omega x_\omega \cap \overline{B}(x_\omega, T) \leq \#\Gamma_n x_n \cap \overline{B}(x_n, T+1)$$

ω-a.s., so

$$\frac{1}{K} \cdot e^{-1} \cdot e^{T \cdot h_{\Gamma_n}} \leq \#\Gamma_\omega x_\omega \cap \overline{B}(x_\omega, T) \leq K \cdot e \cdot e^{T \cdot h_{\Gamma_n}} \tag{6.3}$$

ω-a.s. Putting together (6.2) and (6.3) and using the definition of T, we get

$$h_{\Gamma_n} - 2\varepsilon \leq h_{\Gamma_\omega} \leq h_{\Gamma_n} + 2\varepsilon,$$

ω-a.s. This means ω-$\lim h_{\Gamma_n} = h_{\Gamma_\omega}$, by definition. This is true for every non-principal ultrafilter, hence the continuity under equivariant pointed Gromov–Hausdorff convergence follows by the next result. □

LEMMA 6.3. *Let a_n be a bounded sequence of real numbers.*

(i) *If a_{n_j} is a subsequence converging to $\tilde{a}$ then there exists a non-principal ultrafilter ω such that ω-$\lim a_n = \tilde{a}$.*

(ii) *If there exists $a \in \mathbb{R}$ such that ω-$\lim a_n = a$ for every non-principal ultrafilter ω, then there exists $\lim_{n\to+\infty} a_n = a$.*

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Proof. Let us start with (i). The set $\{n_j\}_j$ is infinite, so there exists a non-principal ultrafilter ω containing $\{n_j\}_j$ (cf. [**Jan17**, Lemma 3.2]). Moreover, for every $\varepsilon > 0$ there exists j_ε such that for all $j \geq j_\varepsilon$ the inequality $|a_{n_j} - \tilde{a}| < \varepsilon$ holds. The set of indices where the inequality is true belongs to ω since the complementary is finite. This implies exactly that $\tilde{a} = \omega\text{-lim}\, a_n$.

The proof of (ii) is now a direct consequence. We take subsequences $\{n_j\}_{j\in J}$, $\{n_k\}_{k\in K}$ converging respectively to the limit inferior and limit superior of the sequence. By (i) there are two non-principal ultrafilters ω_J, ω_K such that $\omega_J\text{-lim}\, a_{n_j} = \liminf_{n\to+\infty} a_n$ and $\omega_K\text{-lim}\, a_{n_k} = \limsup_{n\to+\infty} a_n$. By assumption these two ultralimits coincide, so $\liminf_{n\to+\infty} a_n = \limsup_{n\to+\infty} a_n$. □

7. *Algebraic and equivariant Gromov–Hausdorff convergence*

Let X be a proper metric space and G be a topological group. We denote by $\text{Act}\,(G, X)$ the set of homomorphisms $\varphi\colon G \to \text{Isom}(X)$.

Definition 7.1. Let $\varphi_n, \varphi_\infty \in \text{Act}(G, X)$. We say φ_n converges in the algebraic sense to φ_∞ if φ_n converges to φ_∞ in the compact-open topology. In this case we write $\varphi_n \underset{\text{alg}}{\longrightarrow} \varphi_\infty$. The compact-open topology is Hausdorff since $\text{Isom}(X)$ is, so the algebraic limit is unique, if it exists.

If G has the discrete topology then the algebraic convergence is equivalent to the isometries $\varphi_n(g)$ converging to $\varphi_\infty(g)$ uniformly on compact subsets of X for every $g \in G$.

If G has the discrete topology and is finitely generated by $\{g_1, \ldots, g_\ell\}$ then the algebraic convergence is equivalent to the isometries $\varphi_n(g_i)$ converging to $\varphi_\infty(g_i)$ uniformly on compact subsets of X for every $i = 1, \ldots, \ell$.

The algebraic limit is always contained in the ultralimit group in the following sense.

PROPOSITION 7.2. *Let $\varphi_n, \varphi_\infty \in \text{Act}(G, X)$ and suppose $\varphi_n \underset{\text{alg}}{\longrightarrow} \varphi_\infty$. Let ω be a non-principal ultrafilter. Then $\varphi_\infty(G) \subseteq (\varphi_n(G))_\omega$, where we use Lemma 3.9 to identify the ultralimit group of the sequence $\varphi_n(G)$ with a group of isometries of X.*

Proof. For every $g \in G$ the sequence $\varphi_n(g)$ converges uniformly on compact subsets of X to $\varphi_\infty(g)$ by assumption. It is easy to see that the ultralimit element $\omega\text{-lim}\, \varphi_n(g)$ coincides with $\varphi_\infty(g)$, so $\varphi_\infty(G) \subseteq (\varphi_n(G))_\omega$. □

In general the inclusion is strict.

Example 7.3. Let $X = \mathbb{R}$, $G = \mathbb{Z}$ and $\varphi_n\colon \mathbb{Z} \to \text{Isom}(\mathbb{R})$ defined by sending 1 to the translation of length $1/n$. Clearly the sequence φ_n converges algebraically to the trivial homomorphism. On the other hand, $(\varphi_n(\mathbb{Z}))_\omega$ is the group Γ of all translations of $\mathbb{R}$ for every non-principal ultrafilter ω.

However, the two limits coincide when restricted to the class $\mathcal{M}(\delta, D)$.

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THEOREM 7.4. *Let* $(X, x, \Gamma_n) \in \mathcal{M}(\delta, D)$.

(i) *If* $(X, x, \Gamma_n) \underset{\text{eq-pGH}}{\longrightarrow} (X, x, \Gamma_\infty)$ *then there exists a group* G *such that, for every sufficiently large* n*, we have* $\Gamma_n = \varphi_n(G)$*,* $\Gamma_\infty = \varphi_\infty(G)$ *with* $\varphi_n, \varphi_\infty$ *isomorphisms and* $\varphi_n \underset{\text{alg}}{\longrightarrow} \varphi_\infty$.

(ii) *Conversely, if there exist a group* G *and isomorphisms* $\varphi_n \colon G \to \Gamma_n$ *and if* $\varphi_n \underset{\text{alg}}{\longrightarrow} \varphi_\infty$ *then* $(X, x, \Gamma_n) \underset{\text{eq-pGH}}{\longrightarrow} (X, x, \varphi_\infty(G))$.

Before the proof we need two results. Given a group Γ and a finite set of generators Σ of Γ, the word-metric d_Σ is classically defined on Γ as

$$d_\Sigma(g, h) := \inf\{\ell \in \mathbb{N} \text{ s.t. } g = h \cdot \sigma_1 \cdots \sigma_\ell, \text{ where } \sigma_i \in \Sigma\}.$$

By definition d_Σ takes values in the set of natural numbers and $d_\Sigma(g, h) = d_\Sigma(h^{-1}g, \mathrm{id})$. The couple (Γ, Σ) is called a marked group.

LEMMA 7.5. [**BCGS21**, Lemma 4.6] *Let* $(X, x, \Gamma) \in \mathcal{M}(\delta, D)$ *and let* $R > 2D + 72\delta$. *Set* $\Sigma := \Sigma_R(\Gamma, x)$ *and denote by* d_Σ *the associated word-metric on* Γ *(observe that* Σ *is a generating set of* Γ *by Lemma 5.1). Then*

$$(R - 2D - 72\delta) \cdot d_\Sigma(g, h) \le d(gx, hx) \le R \cdot d_\Sigma(g, h)$$

for all $g, h \in \Gamma$.

Proof. It is enough to check the inequalities for $g \in \Gamma$ and $h = \mathrm{id}$. We write $g = \sigma_1 \cdots \sigma_\ell$, with $\sigma_i \in \Sigma$, $\ell = d_\Sigma(g, \mathrm{id})$. The right inequality follows by the triangle inequality on X, indeed $d(gx, x) \le \ell \cdot R$.

We now take consecutive points x_i, $i = 0, \ldots, \ell$, along a geodesic segment $[x, gx]$ with $x_0 = x$, $x_\ell = gx$, $d(x_{i-1}, x_i) = R - 2D - 72\delta$ for $i = 1, \ldots, \ell - 1$ and $d(x_{\ell-1}, gx) \le R - 2D - 72\delta$. This implies $\ell \le d(x, gx)/(R - 2D - 72\delta)$. By Lemma 4.5 and by cocompactness we can find an element $g_i \in \Gamma$ such that $d(x_i, g_i x) \le 36\delta + D$ for every $i = 0, \ldots, \ell$. We choose $g_0 = \mathrm{id}$ and $g_\ell = g$. Clearly $d(g_i x, g_{i-1} x) \le R$ for every $i = 1, \ldots, \ell$. This shows that $\sigma_i = g_{i-1}^{-1} g_i \in \Sigma$. Moreover, $g = \sigma_1 \cdots \sigma_\ell$, that is, $d_\Sigma(g, \mathrm{id}) \le \ell \le d(x, gx)/(R - 2D - 72\delta)$. □

In the following proposition we make the metric in the pointed Gromov–Hausdorff convergence explicit for the sake of clarity.

PROPOSITION 7.6. *In the standard setting of convergence let* R *be a real number satisfying:*

(i) $2D + 72\delta < R \le 2D + 72\delta + 1$*;*

(ii) *for every* $g \in \Gamma_\infty$ *such that* $d(x_\infty, gx_\infty) \le R$*, we have* $d(x_\infty, gx_\infty) < R$.

Let $\Sigma_n := \Sigma_R(\Gamma_n, x_n)$ *and* $\Sigma_\infty = \Sigma_R(\Gamma_\infty, x_\infty)$ *be generating sets of* Γ_n *and* Γ_∞*, respectively (by Lemma 5.1). Equip* Γ_n *and* Γ_∞ *with the word-metrics* d_{Σ_n}, d_{Σ_∞}*, respectively. Then* $(\Gamma_n, d_{\Sigma_n}, \mathrm{id}) \underset{\text{pGH}}{\longrightarrow} (\Gamma_\infty, d_{\Sigma_\infty}, \mathrm{id})$.

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Proof. We fix a non-principal ultrafilter ω, we take the ultralimit triple $(X_\omega, x_\omega, \Gamma_\omega)$ and we set $\Sigma_\omega := \Sigma_R(\Gamma_\omega, x_\omega)$. We have that $(X_\omega, x_\omega, \Gamma_\omega)$ is equivariantly isometric to $(X_\infty, x_\infty, \Gamma_\infty)$ by Proposition 3.13, so $(\Gamma_\omega, d_{\Sigma_\omega}, \mathrm{id})$ is isometric to $(\Gamma_\infty, d_{\Sigma_\infty}, \mathrm{id})$, and they are proper. If we show that the ultralimit of the sequence of spaces $(\Gamma_n, d_{\Sigma_n}, \mathrm{id})$ is isometric to $(\Gamma_\omega, d_{\Sigma_\omega}, \mathrm{id})$ we conclude by Corollary 3.14 that $(\Gamma_n, d_{\Sigma_n}, \mathrm{id}) \underset{\mathrm{pGH}}{\longrightarrow} (\Gamma_\infty, d_{\Sigma_\infty}, \mathrm{id})$.

We denote by ω-$\lim(\Gamma_n, d_{\Sigma_n}, \mathrm{id})$ the ultralimit space of this sequence. Observe that each element is represented by a sequence (g_n) with $g_n \in \Gamma_n$ and $d_{\Sigma_n}(\mathrm{id}, g_n) \leq M\omega$-a.s., for some M. We define $\Phi \colon \omega$-$\lim(\Gamma_n, d_{\Sigma_n}, \mathrm{id}) \to (\Gamma_\omega, d_{\Sigma_\omega}, \mathrm{id})$ by sending a point (g_n) to the isometry of Γ_ω defined by ω-$\lim g_n$. We have to show that Φ is well defined. First of all the condition $d_{\Sigma_n}(\mathrm{id}, g_n) \leq M$ implies $d(x_n, g_n x_n) \leq R \cdot M\omega$-a.s. by Lemma 7.5, so the sequence (g_n) is admissible and defines a limit isometry belonging to Γ_ω. Suppose that (h_n) is another sequence of isometries of Γ_n such that ω-$\lim d_{\Sigma_n}(g_n, h_n) = 0$. Then $d_{\Sigma_n}(g_n, h_n) < 1\omega$-a.s, thus $d_{\Sigma_n}(g_n, h_n) = 0\omega$-a.s., that is, $g_n = h_n\omega$-a.s. This shows that the map Φ is well defined.

It remains to show that it is an isometry. Let $(g_n), (h_n) \in \omega$-$\lim(\Gamma_n, d_{\Sigma_n}, \mathrm{id})$ and set $\ell = \omega$-$\lim d_{\Sigma_n}(g_n, h_n)$. Since any word-metric takes values in $\mathbb{N}$ we get $d_{\Sigma_n}(g_n, h_n) = \ell\omega$-a.s. For these indices we can write $g_n = h_n \cdot a_n^1 \cdots a_n^\ell$ with $a_n^1, \ldots, a_n^\ell \in \Sigma_n$. The sequence of isometries (a_n^i) are admissible by definition, so they define elements $a_\omega^i \in \Sigma_\omega$. We have $g_\omega = h_\omega \cdot a_\omega^1 \cdots a_\omega^\ell$, showing that $d_{\Sigma_\omega}(g_\omega, h_\omega) \leq \ell = \omega$-$\lim d_{\Sigma_n}(g_n, h_n)$.

We now take isometries $g_\omega = \omega$-$\lim g_n$, $h_\omega = \omega$-$\lim h_n$ of Γ_ω. By definition $d(x_n, g_n x_n) \leq M$ and $d(x_n, h_n x_n) \leq M$ ω-a.s., for some M. By Lemma 7.5 we get $d_{\Sigma_n}(g_n, \mathrm{id}) \leq M'$ and $d_{\Sigma_n}(h_n, \mathrm{id}) \leq M'$ ω-a.s., for $M' = M/(R - 2D - 72\delta)$. Therefore the sequences $(g_n), (h_n)$ defines point of ω-$\lim(\Gamma_n, d_{\Sigma_n}, \mathrm{id})$. Observe that this shows also that Φ is surjective. We set $d_{\Sigma_\omega}(g_\omega, h_\omega) = \ell$. Then we can write $g_\omega = h_\omega \cdot a_\omega^1 \cdots a_\omega^\ell$, for some $a_\omega^i = \omega$-$\lim a_n^i \in \Sigma_\omega$. This means that $d(x_\omega, a_\omega^i x_\omega) \leq R$, so $d(x_\omega, a_\omega^i x_\omega) < R$ by our assumptions on R. Therefore the following finite set of conditions holds ω-a.s.: $d(x_n, a_n^i x_n) \leq R$ for every $i = 1, \ldots, \ell$, that is, $a_n^i \in \Sigma_n\omega$-a.s. Now observe that if $g_n \neq h_n \cdot a_n^1 \cdots a_n^\ell =: b_n\omega$-a.s. then $d(g_n x_n, b_n x_n) \geq s > 0$ ω-a.s. Indeed, by Corollary 5.9 and Proposition 5.3 it is enough to take s smaller than a uniform lower bound of the systole of all the groups Γ_n. Hence we get $d(g_\omega x_\omega, b_\omega x_\omega) > 0$, which is clearly false. This shows that ω-$\lim d_{\Sigma_n}(g_n, h_n) \leq \ell$, that is, $d_{\Sigma_\omega}(g_\omega, h_\omega) \geq \omega$-$\lim d_{\Sigma_n}(g_n, h_n)$. Therefore Φ is an isometry. □

Proof of Theorem 7.4. We first prove (i). We can always find R as in the assumptions of Proposition 7.6 since Γ_∞ is discrete by Corollary 5.10. So, with the same notation as in Proposition 7.6, $(\Gamma_n, d_{\Sigma_n}, \mathrm{id}) \underset{\mathrm{pGH}}{\longrightarrow} (\Gamma_\infty, d_{\Sigma_\infty}, \mathrm{id})$. Applying word for word the proof of Theorem 4.4 of [**BCGS21**], using Lemma 7.5 instead of Lemma 4.6 therein, we get only a finite number of isomorphic types of the marked groups (Γ_n, Σ_n). This implies that for sufficiently large n all the marked groups (Γ_n, Σ_n) are pairwise isomorphic, and in particular isomorphic to $(\Gamma_\infty, \Sigma_\infty)$. We set $G = \Gamma_\infty$, $\varphi_\infty = \mathrm{id}$ and φ_n' a fixed marked isomorphism between $(\Gamma_\infty, \Sigma_\infty)$ and (Γ_n, Σ_n). By Corollary 5.9 and Proposition 5.3 we can find $s > 0$ such that $\mathrm{sys}(\Gamma_n, X) \geq 2s$ for every n. By definition of equivariant pointed

Gromov–Hausdorff convergence for each element $g \in \Sigma_\infty$ there exists $g_n \in \Gamma_n$ such that $d(g_n x, gx) < s$, if n is sufficiently large. By the condition on the systole we deduce that g_n is unique. Finally, by the definition of R, if n is taken possibly larger, every such g_n belongs to Σ_n. Clearly this correspondence $g \mapsto g_n$ is one-to-one. This means that there exists a permutation $\mathcal{P}_n$ of the set Σ_n such that $\varphi_n = \mathcal{P}_n \circ \varphi_n'$ is again a marked isomorphism between $(\Gamma_\infty, \Sigma_\infty)$ and (Γ_n, Σ_n) such that $\varphi_n(g) = g_n$. It is now trivial to show that $\varphi_n(g)$ converges uniformly on compact subsets of X to g for every $g \in \Sigma_\infty$, and so that φ_n converges algebraically to φ_∞.

We now show (ii). By Corollary 3.14 it is enough to show that $\Gamma_\omega = \varphi_\infty(G)$ for every non-principal ultrafilter ω, where Γ_ω is the ultralimit group of the sequence Γ_n. Here we are using Lemma 3.9 to identify the ultralimit group of the sequence (X, x, Γ_n) with a group of isometries of X. By Proposition 3.13 we know that there is a subsequence $\{n_k\}$ such that $(X, x, \Gamma_{n_k}) \underset{\text{eq-pGH}}{\longrightarrow} (X, x, \Gamma_\omega)$ because X is proper. Therefore by the first part of the theorem there exists a homomorphism $\psi\colon G \to \mathrm{Isom}(X)$ with $\psi(G) = \Gamma_\omega$ and $\varphi_{n_k} \underset{\text{alg}}{\longrightarrow} \psi$. So $\psi = \varphi_\infty$ by the uniqueness of the algebraic limit. This shows that $\Gamma_\omega = \varphi_\infty(G)$ and concludes the proof. □

We observe that the first part of the argument above shows the following corollary.

COROLLARY 7.7. *In the standard setting of convergence the groups Γ_n are eventually isomorphic to Γ_∞.*

8. *Examples*

We show that each assumption on the class $\mathcal{M}(\delta, D)$ is necessary in order to have the discreteness of the limit group.

Example 8.1. (Non-elementarity of the group) We take $X_n = \mathbb{R}$, $x_n = 0$ and $\Gamma_n = \mathbb{Z}_{1/n}$, the group generated by the translation of length $1/n$. It is easy to show that $(X_\omega, x_\omega) = (\mathbb{R}, 0)$ and Γ_ω is the group of all translations of $\mathbb{R}$, for every non-principal ultrafilter ω. Clearly Γ_ω, and therefore any possible limit under equivariant pointed Gromov–Hausdorff convergence, is not discrete. Observe that each X_n is a proper, geodesic, 0-hyperbolic metric space and each Γ_n is discrete, torsion-free and cocompact with codiameter $\leq 1/n$.

Example 8.2. (Non-uniform bound on the diameter) For every n we take a hyperbolic surface of genus 2 with systole equal to $1/n$. Its fundamental group acts cocompactly on $X_n = \mathbb{H}^2$ as a subgroup Γ_n of isometries. We take a basepoint $x_n \in \mathbb{H}^2$ which belongs to the axis of an isometry of Γ_n with translation length $1/n$. As in the example above Γ_ω contains all the possible translations along an axis of $X_\omega = \mathbb{H}^2$ and so it is not discrete, for every non-principal ultrafilter ω. Observe that each X_n is a proper, geodesic, log 3-hyperbolic metric space and each Γ_n is discrete, non-elementary, torsion-free and cocompact. However, the codiameter of Γ_n is not uniformly bounded above.

Example 8.3. (Groups with torsion) Let Y be the wedge of a hyperbolic surface S of genus 2 and a sphere $\mathbb{S}^2$ and let X be its universal cover, which is Gromov-hyperbolic. Denote by G_n the group of isometries of Y generated by the isometry that fixes S and acts as a rotation of angle $2\pi/n$ on $\mathbb{S}^2$ fixing the wedging point. Let Γ_n be the covering group of G_n acting on $X_n := X$ by isometries. The action of Γ_n is clearly discrete with bounded codiameter. However, Γ_ω is not discrete.

Example 8.4. (Gromov-hyperbolicity) Let $X = \mathbb{R}^2$, $x = (0, 0)$ and Γ_n be the cocompact, discrete, torsion-free group generated by the translations of vectors $(1/n, 0)$ and $(0, 1)$. It is clear that Γ_ω is not discrete for every non-principal ultrafilter ω.

References

[BCGS17] G. Besson, G. Courtois, S. Gallot and A. Sambusetti. Curvature-free Margulis lemma for Gromov-hyperbolic spaces. *Preprint*, 2020, arXiv:1712.08386.

[BCGS21] G. Besson, G. Courtois, S. Gallot and A. Sambusetti. Finiteness theorems for Gromov-hyperbolic spaces and groups. *Preprint*, 2021, arXiv:2109.13025.

[BH13] M. Bridson and A. Haefliger. *Metric Spaces of Non-positive Curvature (Grundlehren der mathematischen Wissenschaften, 319)*. Springer, Berlin, 2013.

[BJ97] C. Bishop and P. Jones. Hausdorff dimension and Kleinian groups. *Acta Math.* **179**(1) (1997), 1–39.

[BL12] A. Bartels and W. Lück. Geodesic flow for CAT(0)-groups. *Geom. Topol.* **16**(3) (2012), 1345–1391.

[Cav21a] N. Cavallucci. Entropies of non positively curved metric spaces. *Preprint*, 2021, arXiv:2102.07502.

[Cav21b] N. Cavallucci. Topological entropy of the geodesic flow of non-positively curved metric spaces. *Preprint*, 2021, arXiv:2105.11774.

[CDP90] M. Coornaert, T. Delzant and A. Papadopoulos. *Géométrie et théorie des groupes: Les groupes hyperboliques de Gromov (Lecture Notes in Mathematics, 1441)*. Springer, Berlin, 1990.

[Coo93] M. Coornaert. Mesures de Patterson–Sullivan sur le bord d'un espace hyperbolique au sens de Gromov. *Pacific J. Math.* **159**(2) (1993), 241–270.

[CS20] N. Cavallucci and A. Sambusetti. Discrete groups of packed, non-positively curved, Gromov hyperbolic metric spaces. *Preprint*, 2020, arXiv:2102.09829.

[CS21] N. Cavallucci and A. Sambusetti. Packing and doubling in metric spaces with curvature bounded above. *Math. Z.* (2021). https://doi.org/10.1007/s00209-021-02905-5

[DK18] C. Druţu and M. Kapovich. *Geometric Group Theory (Colloquium Publications, 63)*. American Mathematical Society, Providence, RI, 2018.

[DSU17] T. Das, D. Simmons and M. Urbański. *Geometry and Dynamics in Gromov Hyperbolic Metric Spaces (Mathematical Surveys and Monographs, 218)*. American Mathematical Society, Providence, RI, 2017.

[Fuk86] K. Fukaya. Theory of convergence for Riemannian orbifolds. *Jpn. J. Math. New Ser.* **12**(1) (1986), 121–160.

[Gro81] M. Gromov. Groups of polynomial growth and expanding maps (with an appendix by Jacques Tits). *Publ. Math. Inst. Hautes Études Sci.* **53** (1981), 53–78.

[Her16] D. A. Herron. Gromov–Hausdorff distance for pointed metric spaces. *J. Anal.* **24**(1) (2016), 1–38.

[Jan17] D. Jansen. Notes on pointed Gromov–Hausdorff convergence. *Preprint*, 2017, arXiv:1703.09595.

[Kel17] J. L. Kelley. *General Topology*. Dover Publications, Mineola, NY, 2017.

[Pau96] F. Paulin. Un groupe hyperbolique est déterminé par son bord. *J. Lond. Math. Soc. (2)* **54**(1) (1996), 50–74.

[Pau97] F. Paulin. On the critical exponent of a discrete group of hyperbolic isometries. *Differential Geom. Appl.* **7**(3) (1997), 231–236.

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