

CORRECTION TO A PAPER BY A. G. PAKES

CHRISTIAN BERG

(Received 25 March 2002; revised 8 January 2003)

Communicated by V. T. Stefanov

Abstract

Starting from a probability σ on the half-line with moments of any order A. G. Pakes has defined probabilities σ_r by length biasing of order r and g_r by the stationary-excess operation of order r , $r = 1, 2, \dots$. Examples are given to show that σ can be determined in the Stieltjes sense while σ_1 and g_1 are indeterminate in the Stieltjes sense. This shows that a statement in a recent paper by Pakes does not hold.

2000 *Mathematics subject classification*: primary 44A60; secondary 60E05.

Keywords and phrases: Indeterminate moment problems, length biasing, stationary excess-operation.

1. Introduction

In a recent paper [11] Pakes is considering the criteria of Carleman and Krein together with some converse results. We shall use the notation of [11]. For a measure σ on the half-line \mathbb{R}_+ with moments of any order and distribution function F , Pakes introduces the measure σ_r with distribution function F_r given by

$$F_r(x) = \mu_r^{-1} \int_0^x v^r dF(v),$$

where $\{\mu_n\}$ is the moment sequence of F . The moment sequence of F_r is $\mu_n(r) = \mu_{r+n}/\mu_r$. The construction is called length biasing of order r , and r can be any non-negative integer.

In [11, page 92] Pakes remarks: ‘Obviously $\{\mu_n\}$ is S-determining if and only if $\{\mu_n(r)\}$ is.’

This is not true. While it is clear indeed that S-indeterminacy of σ implies S-indeterminacy of σ_r , the converse is false.

In fact, in our paper with Thill [7] we completely characterized the probabilities σ on the half-line which are S-determinate but for which σ_1 is not S-determinate. This characterization was the starting point for the solution of the Challifour problem solved in [7].

This lead us in [7] to introduce an index of determinacy:

For a measure σ on the half-line with moments of any order and which is S-determinate ($\det(S)$ in short) the *index (of determinacy)* of σ is

$$\text{ind}(\sigma) = \sup\{r \in \mathbb{N}_0 \mid \sigma_r \text{ is } \det(S)\}.$$

Theorems 5.5 and 5.6 of [7] contain a complete characterization of the measures with $\text{ind}(\sigma) = k$. In a continuation [6] we considered the relation between the index and the denseness of the polynomials in L^2 -spaces.

In later papers with Duran [4, 5] we extended this to the Hamburger case, that is, for measures on the real line with moments of any order. For a survey of these results see [2]. It should be added that the remark of Pakes is true if σ is a non-discrete measure, because such a measure is either S-indeterminate or S-determinate with $\text{ind}(\sigma) = \infty$. Our observation has also the consequence that σ can be S-determinate although the *stationary-excess operation of order 1* defined in [11] leads to an S-indeterminate probability density

$$g_1(x) = \bar{F}(x)/\mu_1, \quad \bar{F}(x) = 1 - F(x).$$

In particular, the first part of Theorem 5 in [11] is not true:

THEOREM 1.1. *There exists S-determinate measures σ for which*

$$(1) \quad \int_{x'}^{\infty} x^{-3/2} (-\log \bar{F}(x)) dx < \infty, \quad x' > 0,$$

and the density $g_1(x)$ is S-indeterminate.

We shall explain why the result fails and also give a concrete counterexample in the next section.

2. Counterexamples

For the general theory of the moment problem see [1]. Let us first recall that if σ is S-indeterminate, there are infinitely many solutions to the corresponding Stieltjes moment problem. Among those are the N(evanlinna)-extremal solutions ν_t supported by $[0, \infty[$. Here the parameter t can be any real number in a well-defined interval

$[\alpha, 0]$ where $\alpha < 0$, see [8, page 179] for details. The particular value $t = 0$ gives a measure of the form

$$(2) \quad \nu_0 = \beta_0 \varepsilon_0 + \sum_{n=1}^{\infty} \beta_n \varepsilon_{x_n},$$

where the masses $\beta_n > 0$ sum to 1 and $0 < x_1 < x_2 < \dots$ tend to infinity. If the mass at zero is removed from ν_0 , and we rescale to a probability σ , that is,

$$(3) \quad \sigma = (\nu_0 - \beta_0 \varepsilon_0) / (1 - \beta_0),$$

then σ is S-determinate and determinate even for the corresponding Hamburger moment problem. For different proofs of this see [1, page 115] and [3]. Let as before $\{\mu_n\}$ be the moment sequence of σ .

The probability measure of length biasing of order 1

$$\sigma_1 = \frac{t}{\mu_1} d\sigma(t)$$

is indet(S) because σ_1 is proportional to $t d\nu_0(t)$, which is clearly indet(S) because ν_0 is so.

Let F be the distribution function of σ and define $\bar{F}(x) = 1 - F(x)$, $g_1(x) = \bar{F}(x) / \mu_1$.

Then g_1 is a probability density with moments of any order and moment sequence

$$\bar{\mu}_n(1) = \frac{1}{1+n} \frac{\mu_{n+1}}{\mu_1}.$$

We claim that g_1 is indet(S), because it is the product of the S-indeterminate sequence μ_{n+1} / μ_1 with the moment sequence of Lebesgue measure on $[0, 1]$, see Lemma 2.1 below.

As a preparation for Lemma 2.1 we shall recall the Mellin transformation.

The (open) positive half-line is a locally compact abelian group under multiplication, and the Mellin transformation is the Fourier transformation in the sense of harmonic analysis on such groups.

The corresponding convolution of measures is denoted \diamond , so $\tau \diamond \chi$ is the image measure under $(x, y) \mapsto xy$ of the product measure $\tau \otimes \chi$. The Mellin transformation \mathcal{M} is defined for finite (complex) measures by

$$\mathcal{M}(\tau)(x) = \int_0^{\infty} t^{ix} d\tau(t), \quad x \in \mathbb{R}.$$

The Mellin transform of the convolution product is the ordinary product of the Mellin transforms. Furthermore, for the n 'th moments we have $\mu_n(\tau \diamond \chi) = \mu_n(\tau)\mu_n(\chi)$.

The Mellin transform of the Lebesgue measure m on the unit interval $[0, 1]$ is

$$\mathcal{M}(m)(x) = \frac{1}{1 + ix},$$

hence non-vanishing. The Mellin transformation is one-to-one which implies the first statement of Lemma 2.1.

LEMMA 2.1. *The mapping $\tau \mapsto \tau \diamond m$ is one-to-one. If τ is indet(S), then so is $\tau \diamond m$.*

PROOF. The second statement follows from the first, because if τ and χ are different positive measures with the same moments, then $\tau \diamond m$ and $\chi \diamond m$ are different, and they also have identical moments. □

REMARK 2.2. There exists a measure τ which is det(S) and yet $\tau \diamond m$ is indet(S).

The measure ν_0 from (2) can be written $\nu_0 = \beta_0 \varepsilon_0 + \rho$ and $\nu_0 \diamond m = \beta_0 \varepsilon_0 + \rho \diamond m$ is indet(S) by Lemma 2.1. Since $\rho \diamond m$ is absolutely continuous we can conclude that $\rho \diamond m$ is indeterminate. In fact, if $\rho \diamond m$ was determinate, then the polynomials are dense in $L^2(\rho \diamond m)$ and hence in $L^2(\nu_0 \diamond m)$ by [3, Lemma 2]. Therefore the indeterminate measure $\nu_0 \diamond m$ is N-extremal, but this contradicts the fact that it is non-discrete.

The probability $\tau = \rho / (1 - \beta_0)$ ($= \sigma$ from (3)) satisfies the claim of the remark.

The author does not know if the phenomenon of Remark 2.2 can hold if τ is non-discrete or absolutely continuous.

REMARK 2.3. The Krein condition (1) cannot distinguish between the measures ν_0 and σ given by (2) and (3).

If we let F and G denote the corresponding distribution functions, condition (1) for \bar{F} takes the form

$$(4) \quad 2 \sum_{n=N}^{\infty} -\log(1 - \beta_0 - \dots - \beta_n) \left(\frac{1}{\sqrt{x_n}} - \frac{1}{\sqrt{x_{n+1}}} \right) < \infty,$$

while for \bar{G} it has the form

$$(5) \quad 2 \sum_{n=N}^{\infty} -\log \left(\frac{1 - \beta_0 - \dots - \beta_n}{1 - \beta_0} \right) \left(\frac{1}{\sqrt{x_n}} - \frac{1}{\sqrt{x_{n+1}}} \right) < \infty.$$

Since

$$\sum_{n=N}^{\infty} \left(\frac{1}{\sqrt{x_n}} - \frac{1}{\sqrt{x_{n+1}}} \right) = \frac{1}{\sqrt{x_N}},$$

the two series in (4), (5) converge simultaneously, and we know that ν_0 is $\text{indet}(S)$, but σ is $\text{det}(S)$.

We shall now give a concrete example of a probability of the form (2), which leads to a probability σ which is $\text{det}(S)$ and for which the Krein condition (1) nevertheless holds by direct verification. This gives a concrete example showing that the first part of Theorem 5 in [11] is not correct.

The example comes from a birth and death process with quartic rates studied by Berg and Valent, see [9, 8].

A birth and death process is defined by the sequences $(\lambda_n)_{n \geq 0}$ of birth rates and $(\mu_n)_{n \geq 0}$ of death rates, restricted by $\lambda_n > 0, \mu_{n+1} > 0$ for $n \geq 0$ and $\mu_0 \geq 0$, see for example [10].

In order to solve the so-called Kolmogorov equation, one studies the polynomials $F_n(x)$ defined by the recurrence

$$(\lambda_n + \mu_n - x)F_n(x) = \mu_{n+1}F_{n+1}(x) + \lambda_{n-1}F_{n-1}(x), \quad n \geq 0$$

with the initial conditions $F_{-1}(x) = 0, F_0(x) = 1$.

Defining

$$\pi_0 = 1, \quad \pi_n = \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n}, \quad n \geq 1$$

and

$$a_n = \lambda_n + \mu_n, \quad b_n = \sqrt{\lambda_n \mu_{n+1}}, \quad n \geq 0,$$

it is well known that the polynomials

$$P_n(x) = (-1)^n \frac{1}{\sqrt{\pi_n}} F_n(x)$$

satisfy the three term recurrence relation

$$x P_n(x) = b_n P_{n+1}(x) + a_n P_n(x) + b_{n-1} P_{n-1}(x), \quad n \geq 1$$

together with the initial conditions $P_0(x) = 1, P_1(x) = (x - a_0)/b_0$.

By Favard's Theorem the polynomials $\{P_n\}$ form an orthonormal system with respect to some probability measure on the half-line and the corresponding moment sequence is a Stieltjes moment sequence.

We shall consider the following quartic rates

$$\lambda_n = (4n + 1)(4n + 2)^2(4n + 3), \quad \mu_n = (4n - 1)(4n)^2(4n + 1), \quad n \geq 0$$

initially considered in [14, 12, 13]. Note that $\mu_0 = 0$ and

$$\pi_n = \frac{1}{4n + 1} \left(\frac{(1/2)_n}{n!} \right)^2 \sim \frac{1}{4\pi} \frac{1}{n^2}, \quad \lambda_{n-1} \pi_{n-1} = \mu_n \pi_n \sim \frac{64}{\pi} n^2,$$

and it follows from known criteria that the corresponding moment problem is indet(S), see for example [9].

The N-extremal measure ν_0 is given by

$$\nu_0 = \frac{\pi}{K_0^2} \varepsilon_{x_0} + \frac{4\pi}{K_0^2} \sum_{n=1}^{\infty} \frac{2n\pi}{\sinh(2n\pi)} \varepsilon_{x_n}, \quad x_n = \left(\frac{2n\pi}{K_0} \right)^4,$$

and the constant K_0 is given by an elliptic integral, see [9].

From the general theory mentioned above

$$\sigma = c \sum_{n=1}^{\infty} \frac{2n\pi}{\sinh(2n\pi)} \varepsilon_{x_n}$$

is determinate. The normalization constant c (expressible by K_0) is chosen so that σ is a probability. The function \bar{F} is piecewise constant and to establish (1), we have to prove that

$$(6) \quad \sum_{n=1}^{\infty} -\log(y_n) \left(\frac{1}{\sqrt{x_n}} - \frac{1}{\sqrt{x_{n+1}}} \right) < \infty,$$

where x_n is as above and

$$y_n = c \sum_{k=n+1}^{\infty} \frac{2k\pi}{\sinh(2k\pi)}.$$

Using

$$y_n \geq c \int_{n+1}^{\infty} \frac{2x\pi}{\sinh(2x\pi)} dx \geq 4\pi c \int_{n+1}^{\infty} x e^{-2\pi x} dx \geq 2c(n+1)e^{-2\pi(n+1)},$$

we see that (6) holds.

References

- [1] N. I. Akhiezer, *The classical moment problem and some related questions in analysis* (Oliver and Boyd, Edinburgh, 1965).
- [2] C. Berg, 'Recent results about moment problems', in: *Probability Measures on Groups and Related Structures XI, Proceedings Oberwolfach 1994* (ed. H. Heyer) (World Scientific, Singapore, 1995).
- [3] C. Berg and J. P. R. Christensen, 'Density questions in the classical theory of moments', *Ann. Inst. Fourier* **31** (1981), 99–114.
- [4] C. Berg and A. J. Duran, 'The index of determinacy for measures and the l^2 -norm of orthonormal polynomials', *Trans. Amer. Math. Soc.* **347** (1995), 2795–2811.
- [5] ———, 'When does a discrete differential perturbation of a sequence of orthonormal polynomials belong to ℓ^2 ?', *J. Funct. Anal.* **136** (1996), 127–153.

- [6] C. Berg and M. Thill, 'A density index for the Stieltjes moment problem', in: *Orthogonal polynomials and their applications* (eds. C. Brezinski, L. Gori and A. Ronveaux), IMACS Ann. Comput. Appl. Math. 9 (Baltzer, Basel, 1991) pp. 185–188.
- [7] ———, 'Rotation invariant moment problems', *Acta Math.* **167** (1991), 207–227.
- [8] C. Berg and G. Valent, 'The Nevanlinna parametrization for some indeterminate Stieltjes moment problems associated with birth and death processes', *Methods and Applications of Analysis* **1** (1994), 169–209.
- [9] ———, 'Nevanlinna extremal measures for some orthogonal polynomials related to birth and death processes', *J. Comp. Appl. Math.* **57** (1995), 29–43.
- [10] M. E. H. Ismail, J. Letessier, D. Masson and G. Valent, 'Birth and death processes and orthogonal polynomials', in: *Orthogonal polynomials: theory and practice*, NATO ASI series C 294 (Kluwer Academic Publishers, The Netherlands, 1990) pp. 229–255.
- [11] A. G. Pakes, 'Remarks on the converse Carleman and Krein criteria for the classical moment problem', *J. Austral. Math. Soc.* **71** (2001), 81–104.
- [12] G. Valent, 'Orthogonal polynomials for a quartic birth and death process', in: *Proceedings of the Granada conference 1991*, J. Comput. Appl. Math. 49 (North Holland, Amsterdam, 1993) pp. 281–288.
- [13] ———, 'Asymptotic analysis of some associated orthogonal polynomials connected with elliptic functions', *SIAM J. Math. Anal.* **25** (1994), 749–775.
- [14] ———, 'Exact solutions of some quadratic and quartic birth and death processes and related orthogonal polynomials', *J. Comput. Appl. Math.* **67** (1996), 103–127.

Universitetsparken 5
DK-2100 København Ø
Denmark
e-mail: berg@math.ku.dk

