7

Euler–Arnold Theory: PDEs via Geometry

7.1 Introduction

In this chapter we shall give an introduction to the Euler–Arnold theory for partial differential equations (PDEs). The main idea of this theory is to reinterpret certain PDEs as smooth ordinary differential equations (ODEs) on infinite-dimensional manifolds. One advantage of this idea is that the usual solution theory for ODEs can be used to establish properties for the PDE under consideration. This principle has been successfully applied to a variety of PDEs arising, for example, in hydrodynamics. Among these are the Euler equations for an ideal fluid, the Camassa–Holm equation, the Hunter–Saxton equation and the inviscid Burgers equation. We refer to Khesin and Wendt (2009, p. 34) for a much longer list of physically relevant PDEs which fit into this setting.

As in the rest of the book, we shall only work with smooth functions. This is rather unnatural for solutions of partial differential equations but allows us to avoid spaces and manifolds of finitely often differentiable mappings. From the theoretical point of view, this is problematic (at least considering the results we are after) and we will comment on the 'correct setting' at the end of this chapter. Before we begin, recall the relation between the energy of a curve and it being a geodesic (see §4.2).

7.1 Definition Let M be a manifold and for $x, y \in M$ we define the closed submanifold

$$C^\infty_{x,y}([0,1], M) := \{c \in C^\infty([0,1], M) \mid c(0) = x, c(1) = y\} \subseteq C^\infty([0,1], M)$$

of curves from x to y (see Exercise 7.1.1). A smooth map $p\colon\,]-\delta,\delta[\times[0,1] \to M$ is a *smooth variation* of $c \in C^\infty_{x,y}([0,1], M)$ if $p(0,t) = c(t)$, for all $t \in [0,1]$ and $p(s,\cdot) \in C^\infty_{x,y}([0,1], M)$, for all $s \in]-\delta,\delta[$.

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With the notation in place, we can now formulate the following standard result from Riemannian geometry for weak Riemannian metrics which admit a metric spray.

7.2 Proposition *Let (M, g) be a (weak) Riemannian manifold which admits a metric spray S. Then $c\colon [0,1] \to M$ is a geodesic if and only if it extremises the energy*

$$\mathrm{En}(c) = \frac{1}{2}\int_a^b g_{c(t)}(\dot{c}(t), \dot{c}(t))\mathrm{d}t,$$

that is, if for each smooth variation $p\colon\,]-\delta, \delta[\times[0,1] \to M$ of c we have

$$\left.\frac{d}{ds}\right|_{s=0} \mathrm{En}(p(s,\cdot)) = 0.$$

Proof Pick a smooth variation q of a curve c with $h(t) := \left.\frac{\partial}{\partial s}\right|_{s=0} q(s,t)$. We compute the derivative by exploiting the formula for the derivative of the energy from Lemma C.17 in a local chart (U, φ). Suppressing again the identifications in the notation we find for $\left.\frac{d}{ds}\right|_{s=0} \mathrm{En}(q(s))$ the formula

$$\begin{aligned}
&\int_0^1 \frac{1}{2} d_1 g_U(c, c'(t), c'(t); h) - d_1 g_U(c(t), h(t), c'(t); c'(t)) \\
&\qquad - g_U(c(t), h(t), c''(t))\mathrm{d}t \\
&\overset{(4.6)}{=} \int_0^1 g_U(c(t), B_U(c(t), c'(t)), c'(t), h(t)) - g_U(c(t), h(t), c''(t))\mathrm{d}t \\
&= \int_0^1 g_U(c(t), B_U(c(t), c'(t), c'(t)) - c''(t), h(t))\mathrm{d}t \\
&\overset{(4.13)}{=} \int_0^1 -g_U(h(t), \nabla_{\dot{c}(t)}\dot{c}(t), h(t))\mathrm{d}t.
\end{aligned}$$

Hence $g_U(h(t), \nabla_{\dot{c}(t)}\dot{c}(t), h(t))$ needs to vanish for every h. Since the only element which gets annulled by all $g(h, \cdot)$ is 0, we conclude that $\left.\frac{d}{ds}\right|_{s=0} \mathrm{En}(p(s,\cdot)) = 0$ holds for all smooth variations if and only if $\nabla_{\dot{c}}\dot{c} = 0$, that is, c is a geodesic. □

7.3 Remark Proposition 7.2 shows that geodesics are critical points of the energy. In §4.3 we have taken the perspective that geodesics are locally length minimising, that is, critical points of the length. However, the energy depends on the parametrisation while the length does not. To reconcile this, note that the Cauchy–Schwarz inequality yields equality of energy and length if $g_c(\dot{c}, \dot{c})$ is constant. In Exercise 7.1.2 we will see that every geodesic satisfies this property, whence our definition of geodesic comes with a preferred parametrisation which makes both points of view equivalent.

The variational approach will enable us to identify the geodesic equations of infinite-dimensional Riemannian metrics. Before we turn to these results, let us fix some notation for this and later sections.

Concerning Partial Derivatives Let $p\colon]-\delta,\delta[\times[0,1] \to M$ be a smooth variation. Assume that ∇ is the metric derivative of the Riemannian manifold (M,g). For $s \in]-\delta,\delta[$ and $t \in [0,1]$ we denote by ∂s and ∂t the constant unit vector field (on $]-\delta,\delta[$ resp. on $[0,1]$). For p, we denote by $\frac{\partial}{\partial s}p(s,t) := Tp(s,t)(1,0)$ and $\frac{\partial}{\partial t}p(s,t) := Tp(s,t)(0,1)$ the partial derivatives as vector fields along p. Hence we can consider, for example, $\frac{\nabla}{\partial s}\frac{\partial}{\partial t}p$ as a vector field along p (in the sense of Definition 5.4). Moreover, we note that as ∇ is the metric derivative (hence torsion free), Klingenberg (1995, Proposition 1.5.8.i) implies that

$$\frac{\nabla}{\partial s}\frac{\partial}{\partial t}p = \frac{\nabla}{\partial t}\frac{\partial}{\partial s}p. \tag{7.1}$$

7.4 Example (The inviscid Burgers equation) Recall from Corollary 2.8 that the diffeomorphism group $\mathrm{Diff}(\mathbb{S}^1) \Subset C^\infty(\mathbb{S}^1,\mathbb{S}^1)$ is an open submanifold. Fix some Riemannian metric g on $\mathbb{S}^1$. We exploit that $\mathbb{S}^1 \subseteq \mathbb{R}^2$ is an embedded submanifold and apply Remark 5.9(c) to deduce from Proposition 5.8 that the L^2-metric on $C^\infty(\mathbb{S}^1,\mathbb{S}^1)$ admits a metric spray. The same then holds for its restriction to $\mathrm{Diff}(\mathbb{S}^1)$:

$$g^{L^2}(X,Y) = \int_{\mathbb{S}^1}^1 g_{\varphi(\theta)}(X(\theta),Y(\theta))\mathrm{d}\theta,$$

where we exploited the identification $T_\varphi \mathrm{Diff}(\mathbb{S}^1) \cong \{X \circ \varphi \mid X \in \mathcal{V}(\mathbb{S}^1)\}$. Now pick a smooth variation $c\colon]-\delta,\delta[\times[0,1] \to \mathrm{Diff}(\mathbb{S}^1)$ of some curve (which we also denote by c). Set $v(t) := \frac{\partial}{\partial s}\big|_{s=0} c(s,t)$ and note that taking the derivative with respect to s coincides with taking the derivative with respect to the unit vector field ∂s. We can now compute the variation of the energy

$$\begin{aligned}
\frac{d}{ds}\bigg|_{s=0}\mathrm{En}(c(s,\cdot)) &= \frac{1}{2}\int_0^1\int_{\mathbb{S}^1}^1 \frac{\partial}{\partial s}\bigg|_{s=0} g\left(\frac{\partial}{\partial t}c(s,t)(x),\frac{\partial}{\partial t}c(s,t)(x)\right)\mathrm{d}\theta\mathrm{d}t\\
&\overset{(5.3)}{=} \int_0^1\int_{\mathbb{S}^1}^1 g\left(\frac{\nabla}{\partial s}\frac{\partial}{\partial t}c(s,t)(x),\frac{\partial}{\partial t}c(s,t)(x)\right)\bigg|_{s=0}\mathrm{d}\theta\mathrm{d}t\\
&\overset{(7.1)}{=} \int_0^1 g^{L^2}\left(\frac{\nabla}{\partial t}\frac{\partial}{\partial s}c(s,t)(x),\frac{\partial}{\partial t}c(s,t)(x)\right)\bigg|_{s=0}\mathrm{d}t\\
&= -\int_0^1 g^{L^2}\left(v(t),\frac{\nabla}{\partial t}\frac{\partial}{\partial t}c(t)\right)\mathrm{d}t,
\end{aligned}$$

where the last equality is due to a usual integration by parts argument. In particular, we see that if $c\colon [0,1] \to \mathrm{Diff}(\mathbb{S}^1)$ should be a geodesic, it must satisfy the pointwise equation

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$$\frac{\nabla}{\partial t}\frac{\partial}{\partial t}c^\wedge(t,\theta) = 0, \quad \text{for all } \theta \in \mathbb{S}^1. \tag{7.2}$$

To connect this equation to objects on the finite-dimensional manifold $\mathbb{S}^1$, we construct a (time-dependent) vector field on $\mathbb{S}^1$ from the curve of diffeomorphisms $c(t)$ by setting

$$u(t,\theta) = \frac{\partial}{\partial t}c^\wedge(t, c^{-1}(t,\theta)), \tag{7.3}$$

where the inverse is the inverse in $\mathrm{Diff}(M)$. In other words, $c(t)(\theta)$ is the flow of the time-dependent vector field u, that is, $\frac{\partial}{\partial t}c(t)(\theta) = u(t, c(t)(\theta))$. Plugging this definition into the left-hand side of (7.2), the chain rule and a quick computation yield the following statement.

7.5 Lemma *Let* $c\colon [0,1] \to \mathrm{Diff}(\mathbb{S}^1)$ *be smooth and the flow of the time-dependent vector field* u *on* $\mathbb{S}^1$*, cf.* (7.3). *Then*

$$\frac{\nabla}{\partial t}\frac{\partial}{\partial t}c^\wedge(t,\theta) = \frac{\nabla}{\partial t}(u(t,c^\wedge(t,\theta))) = \frac{\partial u}{\partial t}(t,c^\wedge(t,\theta)) + \nabla_u u(t,c^\wedge(t,\theta)), \tag{7.4}$$

where $\nabla_u u$ *denotes the covariant derivative of* u *against itself for fixed* t *and we interpret* $\frac{\partial u}{\partial t}(t,\theta)$ *as a partial derivative of* $u(\cdot,\theta)$ *in* $T_\theta\mathbb{S}^1$ *for every fixed* θ.

Note that (7.4) is central to the idea of the Euler–Arnold theory (whence we promoted it to its own lemma) and holds in similar form if one replaces $\mathbb{S}^1$ by an arbitrary smooth compact manifold M. To distinguish the interpretation of $\frac{\partial u}{\partial t}(t,\theta)$ from the usual partial derivative of a smooth variation, let us write $\partial_t u$ for this derivative. We conclude from Lemma 7.5 and (7.2) that c is a geodesic of the L^2-metric if and only if the associated vector field u solves the inviscid Burgers (or Hopf) equation

$$\partial_t u + \nabla_u u = 0. \tag{7.5}$$

Burgers' equation is a partial differential equation ($\nabla_u u$ takes derivatives of u) which is miraculously equivalent to an ordinary differential equation (the geodesic equation) on the infinite-dimensional group $\mathrm{Diff}(\mathbb{S}^1)$. Note that (7.5) is connected to the classical Burgers equation $u_t + 3uu_x = 0$ (subscripts denoting partial derivatives) from Example 4.15. This will be made explicit in Exercise 7.1.3: Since $\mathbb{S}^1 \subseteq \mathbb{R}^2$ is an embedded submanifold, we endow $\mathbb{S}^1$ with the pullback metric induced by the Euclidean inner product $g_x(v,w) \coloneqq \langle v,w\rangle$ (by identifying $T_x\mathbb{S}^1 \subseteq \mathbb{R}^2$). Working out the covariant derivatives, a canonical identification shows that the Burgers equation (7.4) coincides with the classical Burgers equation $u_t + uu_x = 0$ (which is up to scaling equivalent to Example 7.4).

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7.6 Remark The derivation of Burgers' equation as a geodesic equation on $\mathbb{S}^1$ did not exploit any special structure of $\mathbb{S}^1$. It just enabled us to make sense of the integrals without recourse to integration against volume forms. Thus the same argument carries over without any change to an arbitrary compact manifold M. There the inviscid Burgers equation

$$\partial_t u + \nabla_u u = 0$$

makes sense (with respect to the covariant derivative induced by the metric) and is the geodesic equation of the L^2-metric (cf. (7.8) below) on $\operatorname{Diff}(M)$.

In the next section we will systematically investigate the mechanism to associate a geodesic equation to certain partial differential equations.

Exercises

7.1.1 Let (M, g) be a (weak) Riemannian manifold (with metric derivative ∇) and denote by $C^\infty([0,1], M)$ the space of smooth curves with the manifold structure from Appendix C.4. Fix $x, y \in M$. Show that:

(a) $C^\infty_{x,y}([0,1], M) = \{c \in C^\infty([0,1], M) \mid c(0) = x, c(1) = y\}$ is a closed submanifold of $C^\infty([0,1], M)$.
Hint: Consider a canonical chart for the manifold of mappings. Show that the model space splits for every $c \in C^\infty_{x,y}([0,1], M)$.

(b) $p\colon]-\delta, \delta[\times [0,1] \to M$ is a smooth variation of $c \in C^\infty_{x,y}([0,1], M)$ if and only if $p^\vee\colon]-\delta,\delta[\to C^\infty_{x,y}([0,1], M)$ is smooth.

(c) The energy En restricts to a smooth function on $C^\infty_{x,y}([0,1], M)$ and prove the following analogue of Proposition 7.2: A curve c is a geodesic connecting x and y if and only if $d\mathrm{En}(c;\cdot)$ vanishes on $C^\infty_{x,y}([0,1], M)$.

7.1.2 Let (M, g) be a weak Riemannian manifold which allows a metric spray S and an associated metric derivative ∇. Show that a geodesic $c\colon [0,1] \to M$ is a curve of constant speed, that is, $g_c(\dot{c}, \dot{c})$ is constant.
Hint: Use 4.29 to show that $\frac{d}{dt} g_c(\dot{c}, \dot{c})$ vanishes.

7.1.3 Show that in Example 7.4 we can rewrite (7.5) in traditional notation as

$$u_t + u u_x = 0, \quad \text{where } u\colon]-\delta,\delta[\times [0, 2\pi] \to \mathbb{R}.$$

In addition, show that in this setting the flow η of u then satisfies $\frac{\partial^2}{\partial t^2}\eta = 0$.
Hint: Use the idea that the tangent bundle of $\mathbb{S}^1$ is trivial together with Example 4.33.

7.1.4 Prove Lemma 7.5.

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7.2 The Euler Equation for an Ideal Fluid

We shall exhibit the general principle first for the classical example considered by Arnold (1966): the Euler equation for an inviscid incompressible fluid on a Riemannian manifold. It describes the development of a fluid occupying the manifold M under certain assumptions. Let us first fix some notation.

Conventions In this section we will denote by (M, g) a compact (thus finite-dimensional) orientable Riemannian manifold.

- For a (time-dependent) vector field u, we write $\partial_t u(t,x)$ for the partial t-derivative in $T_x M$ (thus not taking values in $T(TM)$!).
- Since M is orientable, it admits a volume form μ induced by g. Further, we denote by $\operatorname{div} X$ the *divergence* of a vector field $X \in \mathcal{V}(M)$ and by $\operatorname{grad} f$ the *gradient* of a smooth function $f\colon M \to \mathbb{R}$ (see Appendix E.3).

We will not derive Euler's equations here from first principles, but refer to Modin (2019) for an account together with a history of the problem.

7.7 (Euler equation) The *Euler equation* for an incompressible fluid is

$$\begin{cases} \partial_t u(t,m) + \nabla_u u(t,m) = -\operatorname{grad} p, & \\ \operatorname{div} u(t,\cdot) = 0 & \text{for all } t, \\ u(0,\cdot) = u_0 & \text{with } \operatorname{div} u_0 = 0, \end{cases} \tag{7.6}$$

where the function $p\colon \mathbb{R}\times M \to \mathbb{R}$ is interpreted as 'pressure'. Euler's equation searches for a (time-dependent) vector field u on M. The condition, $\operatorname{div} u(t,\cdot) = 0$, that is, that u is divergence-free, is the condition enforcing the incompressibility of the fluid.

In (7.6) we seek a vector field, whence one says that the equation is in *Eulerian form*.

7.8 Remark Apart from the incompressibility condition, Euler's equation is similar to Burgers' equation. Indeed the only difference on the PDE level is the right-hand side which is given by the gradient of a pressure function. We will see later that the gradient acts as a Lagrange multiplier enforcing the incompressibility condition (in general, the term $\nabla_u u$ will not be divergence-free).

Again Euler's equation, just like Burgers' equation, is formulated on a finite-dimensional manifold and has a priori nothing to do with infinite-dimensional geometry. However, we will change the perspective to uncover the connection to infinite-dimensional geometry. The idea is similar to what we did for Burgers' equation (but we will now start with the vector field rather than a flow).

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From the Eulerian to the Lagrangian Perspective

Let us consider a time-dependent smooth vector field $u\colon I \times M \to TM$ on a compact interval I. Then recall (e.g. from Lang, 1999, §IV.1, where we exploit M being compact) that the flow for u is a mapping $\eta\colon I \times M \to M$ such that

$$u(t,\eta(t,m)) = \frac{\partial}{\partial t}\eta(t,m), \text{ for all } t \in I,\ m \in M. \tag{7.7}$$

Furthermore, it is well known that for each $t \in I$ the flow $u(t,\cdot)$ is a diffeomorphism of M. The equation (7.7) or the equivalent equation (7.3) is sometimes called the *reconstruction equation*. Observe now that instead of constructing a vector field u which solves the Euler equation (7.6), we can construct its flow. Searching for the flow whose associated vector field solves the PDE is called the *Lagrangian perspective* on the PDE. If u is now divergence-free, $\operatorname{div} u = 0$, this implies that $\eta_*\mu = \mu$, that is, for every t, the diffeomorphism $\eta(t,\cdot)$ leaves the volume form invariant. As η is smooth, the exponential law, Theorem 2.12, allows us to reinterpret the flow $\eta\colon I \times M \to M$ as a smooth curve $\eta^\vee\colon I \to \operatorname{Diff}(M) \Subset C^\infty(M,M)$. Now the incompressibility condition shows that $\eta^\vee$ takes its values in the closed Lie subgroup $\operatorname{Diff}_\mu(M) \subseteq \operatorname{Diff}(M)$ of volume-preserving diffeomorphisms.

Our aim is again to connect the finite-dimensional PDE to the infinite-dimensional Riemannian geometry induced by the L^2-metric. Let us briefly recall its definition.

7.9 Definition (L^2-metric on the diffeomorphism group) Let (M,g) be a compact Riemannian manifold[1]. We define a weak Riemannian metric on $\operatorname{Diff}(M)$ via

$$g^{L_2}_\varphi(X \circ \varphi, Y \circ \varphi) = \int_M g_{\varphi(m)}(X(\varphi(m)), Y(\varphi(m)))\,\mathrm{d}\mu(m). \tag{7.8}$$

Here $X,Y \in \mathcal{V}(M), \varphi \in \operatorname{Diff}(M)$ and we exploited $\operatorname{Diff}(M)$ being a Lie group, whence its tangent bundle is trivial, that is, $T\operatorname{Diff}(M) \cong \operatorname{Diff}(M) \times \mathcal{V}(M)$ (where the diffeomorphism is induced by right translation). It follows directly from the rules of integration that g^{L^2} is a right-invariant Riemannian metric (see 4.13) on the subgroup $\operatorname{Diff}_\mu(M)$ of volume-preserving diffeomorphisms (but not on $\operatorname{Diff}(M)$!). Moreover, in Exercise 7.3.2 we will see that the weak Riemannian metric admits a metric spray and a covariant derivative. With more work, one can also establish this for the restriction of the L^2-metric to the closed Lie subgroup $\operatorname{Diff}_\mu(M)$ (see Ebin and Marsden, 1970, Theorem 9.6).

[1] Readers unfamiliar with integration on manifolds may safely replace M in the following with $\mathbb{S}^1$. However, this does not simplify any of the argument.

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We have seen above that for vector fields solving Euler's equations, the associated flow yields a curve into the group of volume-preserving diffeomorphisms. Since g^{L^2} induces a weak Riemannian metric on $\operatorname{Diff}_\mu(M)$ which admits a metric spray, we can compute geodesics as curves which extremise the energy. This allows us to derive a differential equation for the flow corresponding to vector field solutions of (7.6). As in the Burgers case, we need that $g^{L^2}(h, \frac{\nabla}{\partial t}\frac{\partial}{\partial t}\eta)$ vanishes for every $h \in T\operatorname{Diff}_\mu(M)$. We know that the tangent space of $\operatorname{Diff}_\mu(M)$ is (up to a shift) given by divergence-free vector fields. Now due to the Helmholtz decomposition, Proposition E.17, elements which are L^2-orthogonal for every h are gradients of functions. Thus if $\frac{\nabla}{\partial t}\frac{\partial}{\partial t}\eta$ is a gradient then the inner product vanishes and the curve η extremises the energy. Conversely, if we assume that u solves (7.6) and denote by η its flow, we compute the derivative as follows:

$$\frac{\nabla}{\partial t}\frac{\partial}{\partial t}\eta(t,m) = ((\partial_t u) + \nabla_u u))(t,\eta(t)) = -\operatorname{grad} p(t,\eta(t,m)). \tag{7.9}$$

In other words, the Helmholtz decomposition shows that a flow η extremises the energy if and only if its associated vector field solves Euler's equation (7.6). Hence the Euler equation can equivalently be formulated as the following set of differential equations on the volume-preserving diffeomorphisms.

7.10 (Lagrangian formulation of the Euler problem) Find $\eta(t,\cdot) \in \operatorname{Diff}(M)$ for all t on some interval containing 0 such that

$$\begin{cases} \frac{\nabla}{\partial t}\frac{\partial}{\partial t}\eta(t,x) &= -\operatorname{grad} p(t,\eta(t,x)), \\ \eta(t,\cdot)_*\mu &= \mu \qquad \text{for all } t, \\ \eta(0,\cdot) &= \mathrm{id}_M\,. \end{cases} \tag{7.10}$$

These equations, (7.10), are called the *Euler equations in Lagrangian form.*

We achieved our goal to rewrite Euler's equation as a differential equation on an infinite-dimensional manifold. However, we have not yet exploited that the metric and the equation are right invariant (with respect to the group multiplication). In the next section we will investigate these properties and connect the Lagrangian formulation to the geometry of the Lie group at hand. This will lead (among other things) to another derivation of the geodesic equation as the Euler equations. While this might on first sight look like a superfluous exercise (after all we already know that Euler's equations can be rewritten as the geodesic equation) we wish to point out that this property is crucial for the investigation of PDEs in the Euler–Arnold framework we present here.

7.3 Euler–Poincaré Equations on a Lie Group

Comparing the Lagrangian version of Euler's equations (7.10) and the L^2-metric, it is immediate that all terms arise by right-shifting objects. Moreover, as the tangent of the right shift with a diffeomorphism is just precomposition with the diffeomorphism, we obtain for a curve c with values in the volume-preserving diffeomorphisms, the formula

$$\mathrm{En}(c) = \int_0^1 \int_M g_m(\dot{c}(t) \circ c(t)^{-1}(m), \dot{c}(t) \circ c(t)^{-1}(m))\mathrm{d}\mu(m)\mathrm{d}t. \quad (7.11)$$

Hence we can compute the energy using the L^2-inner product on the Lie algebra. We shall see that the geometry of the Lie group is tightly connected to the geodesics by virtue of the Riemannian metric being right invariant. To state Euler's equations as a geodesic equation, we need to understand derivatives of right-shifted variations.

7.11 Lemma *Let G be a Lie group and $p\colon]-\delta,\delta[\times[0,1] \to G$ a smooth variation. We identify $TG \cong G\times\mathbf{L}(G)$ by right multiplication (see Lemma* 3.12*) and define*

$$X_p\colon]-\delta,\delta[\times[0,1] \to \mathbf{L}(G), \quad (s,t) \mapsto \mathrm{pr}_2\left(T\rho_{p(s,t)^{-1}}\frac{\partial}{\partial t}p(s,t)\right),$$

$$Y_p\colon]-\delta,\delta[\times[0,1] \to \mathbf{L}(G), \quad (s,t) \mapsto \mathrm{pr}_2\left(T\rho_{p(s,t)^{-1}}\frac{\partial}{\partial s}p(s,t)\right).$$

Then the mixed derivatives of the right-shifted variations *are related as follows:*

$$\frac{\partial}{\partial s}X_p - \frac{\partial}{\partial t}Y_p = -[X_p,Y_p]. \quad (7.12)$$

Proof It suffices to establish (7.12) pointwise for every pair (s_0,t_0). Define the smooth map $\tilde{p}(s,t) := p(s,t)\cdot p(s_0,t_0)^{-1}$. Then $\tilde{p}(s_0,t_0) = \mathbf{1}_G$ and a quick calculation shows that we have $\mathrm{pr}_2\left(T\rho_{\tilde{p}(s,t)^{-1}}\left(\frac{\partial}{\partial t}\tilde{p}(s,t)\right)\right) = X_p(s,t)$. A similar identity holds for Y_p and we may thus assume without loss of generality for the proof that $p(s_0,t_0) = \mathbf{1}_G$.

We work locally and pick a chart $\varphi\colon G \supseteq U \to V \subseteq \mathbf{L}(G)$ such that $\varphi(\mathbf{1}_G) = 0$ and $T_{\mathbf{1}_G}\varphi = \mathrm{id}_{\mathbf{L}(G)}$. As in 3.21 we define a local multiplication $v * w := \varphi(\varphi^{-1}(v)\varphi^{-1}(w))$ for all elements $v,w \in \mathbf{L}(G)$ near enough to zero. For an element v, we define (if it is close enough to 0) the inverse $v^{-1} := \varphi(\varphi^{-1}(v)^{-1})$. Choose an open 0-neighbourhood Ω such that $*$ is defined on $\Omega\times\Omega$ and Ω is symmetric (i.e. $v \in \Omega$ implies $v^{-1} \in \Omega$).

By construction, we have for all (s,t) in a neighbourhood of (s_0,t_0) that $q := \varphi \circ p$ makes sense and takes values in Ω. Note that also $q^{-1} = \varphi(p^{-1})$

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takes values in Ω for all such (s,t). Employing the rule on partial differentials, Proposition 1.20 shows that $X_p(s,t) = d_1 * \left(q(s,t), q^{-1}(s,t); \frac{\partial}{\partial t} q(s,t)\right)$. Specialising to (s_0,t_0), we see that $X_p(s_0,t_0) = \frac{\partial}{\partial t} q(s_0,t_0)$. Similar identities hold for Y_p by exchanging t and s. Compute the second derivative using the rule on partial differentials twice (where again the situation is symmetric in s and t):

$$\begin{aligned}\frac{\partial}{\partial s} X_p &= \frac{\partial}{\partial s} d_1 * \left(q, q^{-1}; \frac{\partial}{\partial t} q\right) \\ &= d_1^2 * \left(q, q^{-1}; \frac{\partial}{\partial t} q, \frac{\partial}{\partial s} q\right) + d_2(d_1 *) \left(q, q^{-1}; \frac{\partial}{\partial t} q, \frac{\partial}{\partial s} q^{-1}\right) \\ &\quad + d_1 * \left(q, q^{-1}; \frac{\partial^2}{\partial t \partial s} q\right).\end{aligned}$$

Due to Schwarz' rule, we see that the first and third terms in the above formula are completely symmetric in t and s. Hence these terms will not contribute to the difference (7.12). Let us now compute the differential of the inverse:

$$\begin{aligned}\frac{\partial}{\partial s} q^{-1}(s,t) &= \mathrm{pr}_2 T\varphi \left(\frac{\partial}{\partial s} p(s,t)^{-1}\right) \\ &\overset{(3.3)}{=} -\mathrm{pr}_2 T\varphi T\lambda_{p(s,t)^{-1}} T\rho_{p(s,t)^{-1}} \left(\frac{\partial}{\partial s} p(s,t)\right). \qquad (7.13)\end{aligned}$$

In particular, (7.13) reduces for (s_0,t_0) to $Y_p(s_0,t_0)$. Likewise for $\frac{\partial}{\partial t} q^{-1}(s_0,t_0)$ we obtain $X_p(s_0,t_0)$. Note that by construction $q(s_0,t_0) = 0 = q^{-1}(s_0,t_0)$. Hence we can now deduce that the difference $\left(\frac{\partial}{\partial s} X_p - \frac{\partial}{\partial t} Y_p\right)(s_0,t_0)$ is given as

$$\begin{aligned}&\left(d_2(d_1 *) \left(q, q^{-1}; \frac{\partial}{\partial t} q, \frac{\partial}{\partial s} q^{-1}\right) - d_2(d_1 *) \left(q, q^{-1}; \frac{\partial}{\partial s} q, \frac{\partial}{\partial t} q^{-1}\right)\right)(s_0,t_0) \\ &\overset{(7.13)}{=} -\Big(d_2(d_1 *) \left(0, 0; X_p(s_0,t_0), Y_p(s_0,t_0)\right) \\ &\qquad + d_2(d_1 *) \left(0, 0; Y_p(s_0,t_0), X_p(s_0,t_0)\right)\Big) \\ &= \left.\frac{\partial^2}{\partial s \partial t}\right|_{s,t=0} \left(tY_p(s_0,t_0) * sX_p(s_0,t_0) - sX_p(s_0,t_0) * tY_p(s_0,t_0)\right) \\ &\overset{(3.21)}{=} -[X_p(s_0,t_0), Y_p(s_0,t_0)].\end{aligned}$$

□

We are now in a position to establish Arnold's classical result on the Euler equation via geometry on the Lie group of volume-preserving diffeomorphisms.

7.12 Theorem (Arnold) *Let (M,g) be a compact Riemannian manifold and consider a curve $\varphi\colon [0,1] \to \mathrm{Diff}_\mu(M)$. Then φ is a geodesic of the L^2-metric (i.e. the restriction of* (7.8) *to* $\mathrm{Diff}_\mu(M)$*) if and only if*

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$$u := \dot{\varphi} \circ \varphi^{-1} \in \mathcal{V}_\mu(M)$$

solves the Euler equations (7.6)*, that is, for some function* $p\colon [0,1] \times M \to \mathbb{R}$*, we have*

$$\partial_t u + \nabla_u u = -\operatorname{grad} p.$$

Proof Let $\varphi(s,t)$ be a smooth variation of φ and $u(s,t) := \frac{\partial}{\partial t}\varphi(s,t)\circ\varphi(s,t)^{-1}$, that is, $u(t) = u(0,t)$. Set $h(t) = \frac{\partial}{\partial s}\varphi(s,t) \circ \varphi(s,t)^{-1}\big|_{s=0}$ and note that by picking different smooth variations, $h(t)$ can be chosen to be an arbitrary curve in $\mathcal{V}_\mu(M)$ with vanishing endpoints. We now take the derivative of the energy and compute with (7.11):

$$\begin{aligned} \frac{d}{ds}\bigg|_{s=0} \operatorname{En}(\varphi(s,\cdot)) &= \int_0^1 \int_M g_m\left(u(t)(m), \frac{\partial}{\partial s}\bigg|_{s=0} u(s,t)(m)\right) \mathrm{d}\mu(m)\mathrm{d}t \\ &\overset{(7.12)}{=} \int_0^1 g^{L^2}\left(u(t), \frac{d}{dt}h(t) - [u(t),h(t)]\right) \mathrm{d}t. \end{aligned} \tag{7.14}$$

Recall that the Lie bracket in (7.14) is the Lie bracket of $\mathcal{V}_\mu(M)$. As this is a subalgebra of the Lie algebra of $\operatorname{Diff}(M)$, Example 3.25 implies that it is the negative of the commutator bracket of vector fields, whence $\nabla_u h - \nabla_h u = -[u,h]$. Now replace the Lie bracket and apply Exercise E.3.5 to $g(u, -\nabla_h u)$ to see that (7.14) yields

$$\begin{aligned} &\int_0^1 g^{L^2}\left(u(t), \frac{d}{dt}h(t) + \nabla_{u(t)}h(t) - \nabla_{h(t)}u(t)\right) \mathrm{d}t \\ &= \int_0^1 \left(g^{L^2}\left(u(t), \frac{d}{dt}h(t) + \nabla_{u(t)}h(t)\right) \underbrace{-\frac{1}{2} g^{L^2}(\operatorname{grad} g(u(t),u(t)), h(t))}_{=0 \text{ by Prop. E.17 as } h(t) \in \mathcal{V}_\mu(M)} \right) \mathrm{d}t \\ &= \int_0^1 g^{L^2}\left(u(t), \frac{d}{dt}h(t) + \nabla_{u(t)}h(t)\right) \mathrm{d}t. \end{aligned}$$

We continue with integration by parts with respect to t and the identity $g(u, \nabla_u h) = g(u, \operatorname{grad} g(u,h)) - g(\nabla_u u, h)$ (see Exercise E.3.5). Together with the Helmholtz decomposition, Proposition E.17, the above equation is equal to

$$-\int_0^1 g^{L^2}\left(\frac{d}{dt}u(t) + \nabla_{u(t)}u(t), h(t)\right) \mathrm{d}t.$$

Hence if φ extremises the energy, we see that $-(\partial_t u + \nabla_u u)$ must be L^2-orthogonal to every curve (with vanishing endpoints) in $\mathcal{V}_\mu(M)$. By the Helmholtz decomposition, this happens if and only if there is some function p (determined up to a constant) such that $\partial_t u + \nabla_u u = -\operatorname{grad} p$. □

We have now seen that the geometry of the group of volume-preserving diffeomorphisms can be exploited to identify the Euler equation as a geodesic equation. This connection is typical for PDEs and their associated geodesic equations which are amenable to Arnold's approach. Indeed there is one last reformulation of the Euler equation on the Lie group $\mathrm{Diff}_\mu(M)$ which needs to be mentioned here as it exhibits the connection between the invariant Riemannian metric and Lie group more explicitly.

7.13 Remark (The Euler equation as an Euler–Poincaré equation on $\mathrm{Diff}_\mu(M)$) Our aim is to identify the geodesic equation as a so-called *Euler–Poincaré equation* on $\mathrm{Diff}_\mu(M)$. For this, let us start more generally: Let G be a regular Lie group with Lie algebra $(\mathbf{L}(G), [\,\cdot\,,\cdot\,])$ and $\langle\cdot,\cdot\rangle$ a continuous inner product on $\mathbf{L}(G)$. Assume that we wish to compute geodesics for the right-invariant metric induced by the choice of inner product. Arguing as for the Euler equation, we see that a curve $\varpi\colon [0,1] \to G$ extremises the energy if and only if the expression (7.14) vanishes, that is, in the notation of Exercise 3.2.11 we must have

$$0 = \int_0^1 \left\langle u(t), \frac{d}{dt}h(t) - [u(t), h(t)]\right\rangle \mathrm{d}t = \int_0^1 \left\langle u(t), \frac{d}{dt}h(t) - \mathrm{ad}_{u(t)}(h(t))\right\rangle \mathrm{d}t,$$

where $u(t) = \delta^r\varphi(t)$ (the right logarithmic derivative of φ). Now assume that for all $x \in \mathbf{L}(G)$, there exists an adjoint $\mathrm{ad}_x^\top$ for the linear operator ad_x with respect to the inner product $\langle\cdot,\cdot\rangle$, that is, $\langle \mathrm{ad}_x^\top(y), z\rangle = \langle y, \mathrm{ad}_x(z)\rangle$. Applying again integration by parts we see that φ is a geodesic if and only if its right logarithmic derivative satisfies the *Euler–Poincaré equation*[2]

$$\frac{d}{dt}\delta^r\varphi = -\mathrm{ad}^\top_{\delta^r\varphi}(\delta^r\varphi).$$

Thus we have derived yet another expression which is equivalent to the geodesic equation and by the previous results also to the Euler equation of an incompressible fluid if $G = \mathrm{Diff}_\mu(M)$ and the inner product is the L^2-inner product. Observe that the Euler–Poincaré equation is a differential equation on the Lie algebra. The Euler–Poincare equation reduces the geodesic equation to the Lie algebra and shows that the geometry of the Lie group and the Riemannian geometry of a right-invariant metric are closely intertwined. We will not discuss this fruitful perspective on the Euler equation. However, there are accounts of the general mechanism with many examples available in the literature. The interested reader is referred, for example, to Vizman (2008), Modin (2019) and Khesin and Wendt (2009, II.3).

[2] The name goes back to honor Henri Poincaré who formulated in Poincaré (1901) differential equations for mechanical systems on (finite-dimensional) Lie groups in the presented form.

Exercises

7.3.1 Let (M, g) be a compact Riemannian manifold. Show that the L^2-metric (7.8) on $\operatorname{Diff}(M)$ restricts to a right-invariant Riemannian metric on $\operatorname{Diff}_\mu(M)$.

7.3.2 We again let (M, g) be a compact Riemannian manifold and consider the L^2-metric g^{L^2} (7.8) on $\operatorname{Diff}(M)$. Let S be the metric spray of g and K the associated connector. The aim of this exercise is to prove that g^{L^2} admits a metric spray, connector and covariant derivative by exploiting the right invariance of the metric.
Remark: The proof for this statement for the L^2-metric on $C^\infty(\mathbb{S}^1, M)$ from §5.1 can be adapted to the present situation. However, we will follow in this exercise the classical argument of Ebin and Marsden (1970) which highlights the use of invariance properties.

(a) Show that the pushforward S_* is a spray on $\operatorname{Diff}(M)$ and the pushforward K_* is a connector on $\operatorname{Diff}(M)$.

(b) Define the covariant derivative $\nabla^{L^2}_X Y = K_* \circ TY \circ X$ associated to the connector K (i.e. X, Y are vector fields on $\operatorname{Diff}(M)$). Work out $\nabla^{L^2}_X Y$ for right-invariant vector fields on $\operatorname{Diff}(M)$ (i.e. vector fields $X(\varphi) = X(\mathrm{id}) \circ \varphi$, where $X(\mathrm{id}) \in \mathcal{V}(M) = \mathbf{L}(\operatorname{Diff}(M))$. Then verify that ∇^{L^2} satisfies the properties of the metric derivative associated to g^{L^2} for all right-invariant vector fields.

(c) Establish that ∇^{L^2} is the metric derivative of g^{L^2} and deduce that K_* is the connector and S_* the metric spray associated to g^{L^2}.
Hint: Exploit the idea that for every vector field $X \in \mathcal{V}(\operatorname{Diff}(M))$ and $\varphi \in \operatorname{Diff}(M)$ there exists a right-invariant vector field X^R such that $X^R(\varphi) = X(\varphi)$.

7.3.3 Supply the necessary details for the proof of Lemma 7.11. Show in particular that $\operatorname{pr}_2\Big(T\rho_{\tilde{p}(s,t)^{-1}}\big(\frac{\partial}{\partial s}\tilde{p}(s,t)\big)\Big) = X_p(s,t)$, $X_p = d_1 * (q, q^{-1}; \frac{\partial}{\partial s} q)$ and $X_p(s_0, t_0) = \frac{\partial}{\partial s} q(s_0, t_0)$ and (7.13) reduces to $X_p(s_0, t_0)$.

7.4 An Outlook on Euler–Arnold Theory

In the last section we saw how the Euler equations of an incompressible fluid can be identified as a geodesic equation of a right-invariant metric on an infinite-dimensional Lie group. This equation can in turn be rephrased as the Euler–Poincaré equation on the Lie algebra, highlighting the close connection of Lie group and Riemannian geometry. In the present section we will discuss several

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applications of the theory developed so far. The section concludes with a discussion of the problems one faces related to carrying out the program sketched.

7.14 (Applications of Euler–Arnold theory) As we have seen, certain PDEs, such as the Euler equation of an incompressible fluid, can be rewritten as geodesic equations on an infinite-dimensional manifold. Moreover, the mechanism is reversible, that is, solutions to the original PDE correspond to geodesics. Vice versa, solutions to the geodesic equation of certain (weak) Riemannian metrics yield solutions to partial differential equations. This has the following immediate applications.

(a) *Existence, uniqueness and parameter dependence of solutions*. Geodesics are solutions to ordinary differential equations. To establish properties of their solutions (such as local existence of unique solutions) one can hope to apply the usual toolbox for ordinary differential equations. Transporting solutions back to the finite-dimensional world, this will yield local existence and uniqueness for solutions of the PDE. Historically, this was how the existence and uniqueness problem for Euler's equations of an incompressible fluid was first solved in the general case in Ebin and Marsden (1970).[3] The caveat here is that ODE tools break down beyond Banach manifolds and one requires a technical analysis to make the program work (see 7.18).

Before we continue let us recall the concept of sectional curvature.

7.15 Let (M,g) be a (weak) Riemannian manifold with covariant derivative ∇ and curvature R. Then for two linearly independent vectors $u,v \in T_m M$ spanning a 2-dimensional subspace σ the *sectional curvature* is defined as

$$K(\sigma) := \frac{g_x(R(v,w)w,v)}{g_x(v,v)g_x(w,w) - (g_x(v,w))^2}$$

(where we actually evaluate the curvature R in (local) vector fields V,W with $V(x) = v$ and $W(x) = w$).

As a concrete example, endow the diffeomorphism group $\mathrm{Diff}(M)$ with the weak L^2-metric. Every vector $V \in T_\varphi \mathrm{Diff}(M)$ can be expressed as the value of a right-invariant vector field X_V on $\mathrm{Diff}(M)$. Hence it suffices to compute the sectional curvature using right-invariant vector fields. The formula for the covariant derivative of the L^2-metric then shows that for a subspace σ

[3] The emphasis here is on 'general'. Some results were known for special cases previous to the treatment in loc. cit.

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generated by the orthonormal elements $\{X_V(\varphi), Y_V(\varphi)\}$, the sectional curvature is given as

$$K^{L^2}(\sigma)=g^{L^2}(R^{L^2}(X_V,Y_V)Y_V,X_V)=\int_M K(X_V(\mathrm{id})(m),Y_V(\mathrm{id})(m))\mathrm{d}\varphi^*(\mu)(m),$$

where K is the sectional curvature on M (computed with respect to the space spanned by the vectors $X_V(\mathrm{id})(m)$ and $Y_V(\mathrm{id})(m)$; Smolentsev, 2007, 6.4).

Continuing with our review of Euler–Arnold theory from 7.14:

(b) *Geometric tools for PDE analysis*. It is well known from finite-dimensional Riemannian geometry that curvature controls the behaviour of geodesics (see e.g. do Carmo, 1992, Chapter 5, and also note the connection to the Hopf–Rinow theorem in Remark 4.44). The point is that for positive sectional curvature, geodesics starting at the same point with slight variation of the initial velocity tend to converge towards each other, while for negative sectional curvature they diverge (this can be made explicit as in the finite-dimensional case, but we will not discuss the details here). In the context of partial differential equations these properties can be interpreted as stability of solutions under perturbations of initial conditions.

Indeed Arnold (1966) showed that the sectional curvature of the L^2-metric on $\mathrm{Diff}_\mu(M)$ is negative in almost all directions. So nearby fluid regions will typically diverge exponentially fast from each other. This analysis applies, in particular, to partial differential equations employed in weather forecasts. So infinite-dimensional Riemannian geometry shows that reliable long-term weather forecasts are practically impossible.

7.16 Remark There is also a beautiful connection of the Euler–Arnold equations to ideas from Hamiltonian mechanics on $\mathrm{Diff}(M)$. The differential geometric context for this is (weakly) symplectic structures on $\mathrm{Diff}(M)$ and we refer to Smolentsev (2007, §§6 and 7) for a discussion.

In the present chapter we have only seen the mechanism applied to the Burgers and Euler equations. There are many more PDEs which typically arise in hydrodynamics and can be treated in the same framework. Equations which are amenable to this treatment are nowadays called *Euler–Arnold equations*. We refer to Khesin and Wendt (2009) for an extensive list but mention explicitly the Camassa–Holm equation, the Hunter–Saxton equation and the Korteweg–deVries (KdV) equation as PDEs belonging to this class. Since the Hunter–Saxton equation admits a beautiful geometric interpretation, we will now briefly discuss a few more details related to this equation and its geometric treatment.

7.17 Example (Hunter–Saxton equation) We consider the *periodic Hunter–Saxton equation* on the circle $\mathbb{S}^1$. The task is to find a time-dependent vector field $u\colon [0,T[\times \mathbb{S}^1 \to \mathbb{R}$ which satisfies the following equation:

$$u_{txx} + 2u_x u_{xx} + u u_{xxx} = 0, \tag{7.15}$$

where subscripts again denote partial derivatives with respect to time (t) or $x \in \mathbb{S}^1$. We will see in Exercise 7.4.1 that the Hunter–Saxton equation is the geodesic equation of the right-invariant $\dot{H}^1$-semimetric. To describe this semimetric, we recall from §5.1 the notation $U' := T_\theta U(1)$ and define the inner product

$$g_{\mathrm{id}}^{\dot{H}^1}(U,V) = \frac{1}{4}\int_{\mathbb{S}^1} U'(\theta)V'(\theta)\mathrm{d}\theta \qquad \text{on } C^\infty(\mathbb{S}^1,\mathbb{R}). \tag{7.16}$$

Then the $\dot{H}^1$-semimetric is the right-invariant semimetric induced by (7.16). Note that it is a semimetric as constant vector fields are annihilated. In Example 3.42 we saw that the constant vector fields generate the group of rotations $\mathrm{Rot}(\mathbb{S}^1)$. Hence there are two possibilities to obtain a (weak) Riemannian metric: One can work with the quotient manifold $\mathrm{Diff}(\mathbb{S}^1)/\mathrm{Rot}(\mathbb{S}^1)$ (see Lenells, 2008 for a detailed discussion) or one has to fix a subgroup containing only the trivial rotation. We consider the induced weak Riemannian metric on the Lie subgroup

$$D_0 := \{\phi \in \mathrm{Diff}(\mathbb{S}^1) \mid \phi(\theta(0)) = \theta(0)\}, \tag{7.17}$$

where $\theta\colon [0,2\pi] \to \mathbb{R}^2$, $t \mapsto (\cos(t),\sin(t))$ is the canonical parametrisation of the circle. To ease the computations, we follow Lenells (2008) and will in the following always identify diffeomorphisms and vector fields of $\mathbb{S}^1$ as periodic mappings $[0,2\pi] \to \mathbb{R}$. This allows one to prove that the Hunter–Saxton equation exhibits a fascinating geometric feature discovered in Lenells (2007): The group D_0 can be identified as a convex subset of a sphere and this embedding is a Riemannian isometry relating the $\dot{H}^1$-metric to the L^2-metric. Indeed this embedding is surprisingly simple, as it is given by

$$\Psi\colon D_0 \to C^\infty(\mathbb{S}^1,\mathbb{R}), \quad \Psi(\varphi) = \sqrt{\varphi'}.$$

Its image becomes the convex set

$$\{f(\theta) > 0, \text{ for all } \theta \in \mathbb{S}^1, \int_{\mathbb{S}^1} |f(\theta)|^2 \mathrm{d}\theta = 1\} \subseteq C^\infty(\mathbb{S}^1,\mathbb{R}).$$

We omit the details here and refer instead to the exposition in Lenells (2007).

The geometric content of this observation is that we can transform the Hunter–Saxton equation to a geodesic equation on the L^2-sphere in $C^\infty(\mathbb{S}^1,\mathbb{R})$. Now solutions to the geodesic equation on the L^2-sphere (with respect to the natural L^2-metric) can be explicitly computed: We have seen in Example 4.43

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that geodesics of the L^2-sphere (seen as the unit sphere of the Hilbert space $L^2(\mathbb{S}^1,\mathbb{R})$) are given by great circles.

While the Hunter–Saxton equation can be interpreted as the geodesic equation on an infinite-dimensional sphere in a Hilbert space, our approach so far has been to consider this equation as an equation on the space $C^\infty(\mathbb{S}^1,\mathbb{R})$ which is not a Hilbert space. This is quite unnatural for two reasons: From the perspective of the PDEs this approach will only allow solutions which are smooth in space, whereas it is often of interest to have much less regular solutions, for example, solutions which are only finitely often differentiable in space. The geometric perspective allows us to connect the PDE to an ODE which we then have to solve. However, on $\mathrm{Diff}(M)$ this presents a problem, as we will discuss now.

7.18 (Returning to the Banach and Hilbert setting) In 7.14 we listed as an advantage of the Euler–Arnold approach to PDEs that (local) existence and uniqueness of solutions to these PDEs can be obtained by methods for ordinary differential equations (ODEs) on infinite-dimensional manifolds. Unfortunately the manifolds we have been working in this chapter are submanifolds of the manifolds of mappings $C^\infty(K,M)$ (where K is a compact manifold). These manifolds are never (except in trivial cases) Banach manifolds, so there are no black-box techniques for ODEs as Appendix A.6 shows. Thus the elegant theory developed so far misses an essential analytic ingredient to solve the ODEs occurring.

The solution to this problem is, in principle, simple (if one glosses over the technical details): Replace C^∞ functions by finitely often differentiable ones. Note that this dovetails nicely with the problem statement from the PDE side we mentioned earlier. By going to finitely often differentiable functions, we allow solutions to the PDE which are much less regular in space. From the perspective of infinite-dimensional manifolds, one can prove that

$$C^\infty(K,M) = \bigcap_{k\in\mathbb{N}_0} C^k(K,M) = \bigcap_{s\in\mathbb{N}_0} H^s(K,M),$$

where the manifolds in the middle are Banach manifolds and the spaces on the right even Hilbert manifolds. Here $H^s(K,M)$ denotes all mappings of Sobolev H^s-type (meaning that their weak derivatives are in L^2). We refrain from defining Sobolev spaces on manifolds as there are several subtle points involved in their construction. Instead we remark that they admit a manifold structure similar to the manifolds of mappings we constructed in Chapter 2. For more information we refer the reader to the detailed exposition in Inci et al. (2013).

Conveniently, Sobolev type groups of diffeomorphisms $\mathrm{Diff}^{H^s}(K)$ also exist and one can even prove that $\mathrm{Diff}(K) = \lim_{s\to\infty} \mathrm{Diff}^{H^s}(K)$ as a projective

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limit in the category of manifolds. This structure is called an ILH–Lie group (see Omori, 1974), and coincides with the Lie group structure of $\mathrm{Diff}(K)$ constructed in Example 3.5. The point of the construction is of course that one can work on the Hilbert manifold of Sobolev morphisms. Unfortunately, because of Omori's theorem (Omori, 1978), the groups $\mathrm{Diff}^{H^s}(K)$ cannot be Lie groups. They are, however, manifolds and topological groups such that the left multiplication is continuous but not differentiable – see (2.6) – to see that the derivative loses orders of differentiability. However, right multiplication is still smooth and one obtains a so-called *half-Lie group* (Marquis and Neeb, 2018). This leads to several analytic problems which need to be solved to establish smoothness of the associated metric sprays (see e.g. Ebin and Marsden, 1970). We refer the reader to the literature for more details as these problems are beyond the scope of this chapter.

By this point, the reader should be suitably equipped to understand the classical research literature on these topics. For example, Ebin and Marsden (1970) as well as Ebin (2015) present the theory for the Euler equation of an incompressible fluid. Also the monograph of Khesin and Wendt (2009) provides an excellent overview of the theory together with many pointers towards the literature. This chapter concludes with a short remark on some more recent developments in Euler–Arnold theory.

7.19 Remark Euler–Arnold theory is still an active area of research. Among the many recent results, I like to point out several which I find particularly interesting:

(a) Classical Euler–Arnold theory works with right-invariant Riemannian metrics on (subgroups of) diffeomorphism groups. In Bauer and Modin (2020) it was shown that the approach also works for Riemannian metrics on $\mathrm{Diff}(M)$ which are only invariant with respect to $\mathrm{Diff}_\mu(M)$. Thus a whole new family of PDEs, such as certain shallow water equations, can be treated by Euler–Arnold methods.

(b) Instead of a purely deterministic PDE one can apply the mechanism to a stochastic partial differential equation. Stochastic versions of Euler's equations for an incompressible fluid have recently been considered as models for data-driven hydrodynamics (modelling uncertainty in the data). For the Euler equation of an incompressible fluid, Maurelli et al. (2019) work out the necessary details to make the Euler–Arnold machinery work in the stochastic setting.

(c) There is a connection between Euler–Arnold theory, the differential geometry of diffeomorphism groups and optimal mass transport. This is based

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on the observation that there is a submersion $\pi\colon \mathrm{Diff}(M) \to \mathrm{Dens}(M)$ from the diffeomorphism group to the manifold of densities on M. This links the Riemannian metrics on $\mathrm{Diff}(M)$ to the Wasserstein metric from optimal transport. An introduction to these topics can be found in Khesin and Wendt (2009, Appendix A.5).

Exercises

7.4.1 We consider the group $\mathrm{Diff}(\mathbb{S}^1)$ and the $\dot{H}^1$-semimetric (7.16). Show that:

(a) Identifying diffeomorphisms with periodic mappings we can identify D_0 with

$$\{u + \mathrm{id} \mid u\colon [0,2\pi] \to \mathbb{R}, \quad \frac{d}{dx}u > -1, u(0) = 0 = u(2\pi)\}.$$

(b) $E = \{u \in C^\infty([0,2\pi],\mathbb{R}) \mid u(0) = 0\}$ is a closed subspace of $C^\infty([0,2\pi],\mathbb{R})$ and D_0 is diffeomorphic to an open subset of $\mathrm{id} + E \subseteq E$.

(c) The continuous linear operator $A(u) := -u''$ induces an isomorphism from E to $F := \{f \in C^\infty([0,2\pi],\mathbb{R}) \mid \int_0^{2\pi} f(x)\mathrm{d}x = 0\}$. Moreover, show that (up to identification) $g_{\mathrm{id}}^{\dot{H}^1}(U,V) = \int_{\mathbb{S}^1} UA(V)\mathrm{d}\theta$.

(d) The group D_0 from (7.17) is a Lie subgroup of $\mathrm{Diff}(\mathbb{S}^1)$ and its Lie algebra can be identified as

$$\mathbf{L}(D_0) = \{f \in C^\infty(\mathbb{S}^1,\mathbb{R}) \mid \int_{\mathbb{S}^1} f(\theta)\mathrm{d}\theta = 0\}.$$

(e) (7.16) induces a right-invariant weak Riemannian metric on D_0.

(f) A curve φ extremises the energy of the $\dot{H}^1$-semimetric (7.16) if and only if $u = \frac{\partial}{\partial t}\varphi \circ \varphi^{-1}$ satisfies the Hunter–Saxton equation (7.15).

7.4.2 Prove the claims on the sectional curvature of the L^2-metric on $\mathrm{Diff}(M)$ from (7.15).