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CONGRUENCES FOR SUMS OF MACMAHON'S q-CATALAN POLYNOMIALS

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Abstract

One variant of the q-Catalan polynomials is defined in terms of Gaussian polynomials by

$$C_k(q) = \begin{bmatrix} 2k \\ k \end{bmatrix}_q - q\begin{bmatrix} 2k \\ k+1 \end{bmatrix}_q.$$

Liu ['On a congruence involving q-Catalan numbers', *C. R. Math. Acad. Sci. Paris* **358** (2020), 211–215] studied congruences of the form $\sum_{k=0}^{n-1} q^k C_k$ modulo the cyclotomic polynomial $\Phi_n(q)^2$, provided that $n \equiv \pm 1 \pmod 3$. Apparently, the case $n \equiv 0 \pmod 3$ has been missing from the literature. Our primary purpose is to fill this gap. In addition, we discuss a certain fascinating link to Dirichlet character sum identities.

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1. Introduction

There are several possible q-analogues of the Catalan numbers $(k+1)^{-1}\binom{2k}{k}$. Here, we consider MacMahon's q-Catalan polynomials which are defined by

$$C_k(q) := \frac{1-q}{1-q^{k+1}}\begin{bmatrix} 2k \\ k \end{bmatrix}_q = \begin{bmatrix} 2k \\ k \end{bmatrix}_q - q\begin{bmatrix} 2k \\ k+1 \end{bmatrix}_q,$$

where $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$ denotes the q-binomial coefficient recalled in Section 2. The first few q-Catalan polynomials are

$$C_0(q) = C_1(q) = 1, \quad C_2(q) = 1 + q^2 \quad \text{and} \quad C_3(q) = 1 + q^2 + q^3 + q^4 + q^6.$$

Notice that $C_k(q)$ is a polynomial in q and it reduces to the ordinary Catalan number as $q \to 1$. Moreover, $C_k(q)$ has a natural enumerative meaning. Indeed, MacMahon [4, Volume 2, page 214] established that

$$C_k(q) = \sum_w q^{\mathrm{maj}(w)},$$

where w ranges over all ballot sequences $a_1a_2\dots a_{2k}$ (that is, any permutation of the multiset $\{0^k, 1^k\}$ such that in any subword $a_1a_2\dots a_i$, there are at least as many 0s as there are 1s) and

$$\mathrm{maj}(w) := \sum_{\{i:a_i>a_{i+1}\}} i$$

is the *major index* of w (see also the survey [1, Section 3] and [7, Problem A43]).

In the present work, however, we focus on a problem of number-theoretic interest. The second author in [8, Theorem 6.1] proved that, modulo $\Phi_n(q)$,

$$\sum_{k=0}^{n-1} q^k C_k(q) \equiv \begin{cases} q^{\lfloor n/3 \rfloor} & \text{if } n \equiv 0, 1 \pmod 3, \\ -1 - q^{(2n-1)/3} & \text{if } n \equiv 2 \pmod 3, \end{cases}$$

where

$$\Phi_n(q) = \prod_{\substack{1\le k\le n \\ (n,k)=1}} (q - e^{2k\pi i/n})$$

denotes the cyclotomic polynomial of order n. Afterwards, a stronger version was proved by Liu in [2, Theorem 1]: modulo $\Phi_n(q)^2$,

$$\sum_{k=0}^{n-1} q^k C_k(q) \equiv \begin{cases} q^{(n^2-1)/3} - \dfrac{n-1}{3}(q^n - 1) & \text{if } n \equiv 1 \pmod 3, \\ -q^{(n^2-1)/3} - q^{n(2n-1)/3} & \text{if } n \equiv 2 \pmod 3, \end{cases}$$

where the case $n \equiv 0 \pmod 3$ is *not* covered. Our main aim is to fill this gap as stated next.

THEOREM 1.1. *If n is a positive integer divisible by 3, then*

$$\sum_{k=0}^{n-1} q^k C_k(q) \equiv q^{n(2n+1)/3} + \frac{1}{3}(q^n - 1)(2 + (n+1)q^{2n/3}) \pmod{\Phi_n(q)^2}.$$

As we will explain in more detail below, this theorem holds as soon as we prove the following more manageable identity, which is of interest in its own right.

THEOREM 1.2. *If n is a positive integer divisible by 3 and q is a primitive nth root of unity, then*

$$\sum_{k=1}^{n/3} \frac{(-1)^k q^{k(3k-1)/2}}{1 - q^{3k-1}} + \sum_{k=1}^{n/3-1} \frac{(-1)^k q^{k(3k+5)/2}}{1 - q^{3k}} = \frac{1}{6}(2 + (n+1)q^{2n/3}). \tag{1.1}$$

Notice that, according to [2, Lemma 3], our Theorem 1.2 mirrors

$$\sum_{k=1}^{\lfloor n/3 \rfloor} \frac{(-1)^k q^{k(3k-1)/2}}{1 - q^{3k-1}} + \sum_{k=1}^{\lfloor (n-1)/3 \rfloor} \frac{(-1)^k q^{k(3k+5)/2}}{1 - q^{3k}} \equiv \begin{cases} -\dfrac{n-1}{6} & \text{if } n \equiv 1 \pmod 3, \\ 0 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

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The remainder of the paper is organised as follows. In Section 2, we present a reduction of our main result Theorem 1.1 to Theorem 1.2. Section 3 contains preliminary results which we need towards the proof of Theorem 1.2. Sections 4 and 5 split up Theorem 1.2 according to the parity of n and contain the corresponding proofs. Finally, in Section 6, we consider a conversion of one particular identity coming from (1.1) into a trigonometric format and a remarkable implication in the language of character sums.

2. Reducing Theorem 1.1 to Theorem 1.2

We recall that the Gaussian q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \dfrac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} & \text{if } 0 \le k \le n, \\ 0 & \text{otherwise,} \end{cases}$$

where the q-shifted factorial is given by $(a;q)_n = (1-a)(1-aq)\dots(1-aq^{n-1})$ for $n \ge 1$ and $(a;q)_0 = 1$.

By [3, Theorem 1.2],

$$\sum_{k=0}^{n-1} q^k \begin{bmatrix} 2k \\ k \end{bmatrix}_q \equiv \left(\frac{n}{3}\right) q^{(n^2-1)/3} \pmod{\Phi_n(q)^2}, \tag{2.1}$$

where $\left(\frac{\cdot}{\cdot}\right)$ denotes the Legendre symbol. In the same vein, we also recall the identity [9, Theorem 4.2],

$$\sum_{k=0}^{n-1} q^{k+1} \begin{bmatrix} 2k \\ k+1 \end{bmatrix}_q = \sum_{k=1}^{n} \left(\frac{k-1}{3}\right) q^{(2k^2-k(k-1)/3)/3} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q.$$

Let $1 \le k \le n-1$. Then, the q-analogue [6, Theorem 2.2]

$$\begin{bmatrix} an+b \\ cn+d \end{bmatrix}_q \equiv \binom{a}{c} \begin{bmatrix} b \\ d \end{bmatrix}_q \pmod{\Phi_n(q)}$$

of Lucas' classical binomial congruence combined with $(1-q^n) \equiv 0 \pmod{\Phi_n(q)}$, and the fact that

$$\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q = \prod_{j=1}^{k-1} \frac{1-q^{n-j}}{1-q^j} = q^{-k(k-1)/2} \prod_{j=1}^{k-1} \frac{q^j - q^n}{1-q^j} \equiv (-1)^{k-1} q^{-k(k-1)/2} \pmod{\Phi_n(q)},$$

immediately imply that

$$\begin{aligned} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q &= \frac{1-q^{2n}}{1-q^{n+k}} \begin{bmatrix} 2n-1 \\ n+k-1 \end{bmatrix}_q \equiv (1-q^n) \cdot \frac{2}{1-q^k} \binom{1}{1} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \\ &\equiv (q^n - 1) \cdot \frac{2(-1)^k q^{-k(k-1)/2}}{1-q^k} \pmod{\Phi_n(q)^2}. \end{aligned}$$

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Therefore, modulo $\Phi_n(q)^2$,

$$\begin{aligned}
\sum_{k=0}^{n-1} q^{k+1}\begin{bmatrix}2k\\ k+1\end{bmatrix}_q &\equiv \sum_{k=0}^{n-1}\left(\frac{k-1}{3}\right)q^{(2k^2-k(k-1)/3)/3}\begin{bmatrix}2n\\ n+k\end{bmatrix}_q\\
&\equiv 2(q^n-1)\sum_{k=1}^{n-1}\left(\frac{k-1}{3}\right)q^{(2k^2-k(k-1)/3)/3}\frac{(-1)^k q^{-k(k-1)/2}}{1-q^k}\\
&\equiv -2(q^n-1)\left(\sum_{k=1}^{\lfloor n/3\rfloor}\frac{(-1)^k q^{k(3k-1)/2}}{1-q^{3k-1}}+\sum_{k=1}^{\lfloor (n-1)/3\rfloor}\frac{(-1)^k q^{k(3k+5)/2}}{1-q^{3k}}\right).
\end{aligned}$$

By substituting this congruence together with (2.1) into the definition of $C_k(q)$, we easily see that, when n is divisible by 3, Theorem 1.1 is indeed equivalent to Theorem 1.2.

3. Preparing our proof of Theorem 1.2

Henceforth, we replace n with $3n$ so that our target in (1.1) amounts to proving

$$\sum_{k=1}^{n}\frac{(-1)^k q^{k(3k-1)/2}}{1-q^{3k-1}}+\sum_{k=1}^{n-1}\frac{(-1)^k q^{k(3k+5)/2}}{1-q^{3k}}=\frac{1}{3}+\frac{3n+1}{6}q^{2n}. \tag{3.1}$$

To establish this identity, we need the next two results.

LEMMA 3.1. *For any complex number z,*

$$\begin{aligned}
&\sum_{k=1}^{n}\frac{(-1)^k z^{k(3k-1)/2}}{1-z^{3k-1}}+\sum_{k=1}^{n-1}\frac{(-1)^k z^{k(3k+5)/2}}{1-z^{3k}}\\
&=\frac{(-1)^{n-1}}{2}\sum_{k=1}^{n-1}\frac{z^{k(3n+2)/2}}{1+z^{3k/2}}+\frac{1}{2}\sum_{k=1}^{n-1}\frac{(-1)^k z^{k(3n+2)/2}}{1-z^{3k/2}}\\
&\quad+\sum_{k=1}^{n}\frac{1}{1-z^{3k-1}}-\sum_{k=1}^{\lfloor (n+1)/2\rfloor}\frac{1}{1-z^{3k-2}}-\frac{2n-1+(-1)^n}{4}.
\end{aligned} \tag{3.2}$$

PROOF. Employing partial fractions and after further rearrangement, we obtain

$$\begin{aligned}
\sum_{k=1}^{n}\frac{(-1)^k z^{k(3k-1)/2}}{1-z^{3k-1}} &= \frac{1}{2}\sum_{k=1}^{n}\frac{(-1)^k z^{k(3k-1)/2}}{1-z^{(3k-1)/2}}+\frac{1}{2}\sum_{k=1}^{n}\frac{(-1)^k z^{k(3k-1)/2}}{1+z^{(3k-1)/2}}\\
&=\frac{1}{2}\sum_{k=1}^{n}\frac{(-1)^k((z^{(3k-1)/2})^k-1+1)}{1-z^{(3k-1)/2}}-\frac{1}{2}\sum_{k=1}^{n}\frac{-(-z^{(3k-1)/2})^k+1-1}{1-(-z^{(3k-1)/2})}
\end{aligned}$$

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$$= -\frac{1}{2}\sum_{k=1}^{n}\sum_{j=0}^{k-1}((-1)^k + (-1)^j)z^{j(3k-1)/2} + \frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{1+z^{(3k-1)/2}} + \frac{(-1)^k}{1-z^{(3k-1)/2}}\right)$$
$$= -\frac{1}{2}\sum_{k=1}^{n}\sum_{j=0}^{k-1}((-1)^k + (-1)^j)z^{j(3k-1)/2} + \sum_{k=1}^{n}\frac{1}{1-z^{3k-1}} - \sum_{k=1}^{\lfloor (n+1)/2\rfloor}\frac{1}{1-z^{3k-2}}.$$

Continuing with additional algebraic manipulation leads to

$$\sum_{k=1}^{n}\sum_{j=0}^{k-1}((-1)^k + (-1)^j)z^{j(3k-1)/2} = \sum_{j=0}^{n-1}\sum_{k=j+1}^{n}((-1)^k + (-1)^j)z^{j(3k-1)/2}$$
$$= \frac{2n-1+(-1)^n}{2} + 2\sum_{j=1}^{n-1}\frac{(-1)^j z^{j(3j+5)/2}}{1-z^{3j}} + \sum_{j=1}^{n-1}\left(\frac{(-1)^n z^{j(3n+2)/2}}{1+z^{3j/2}} - \frac{(-1)^j z^{j(3n+2)/2}}{1-z^{3j/2}}\right).$$

Combining the last two calculations, we find (3.2). □

LEMMA 3.2. *If α is a primitive mth root of unity, then*

$$\sum_{k=1}^{m}\frac{1}{1-z^{-1}\alpha^k} = \frac{m}{1-z^{-m}}. \tag{3.3}$$

PROOF. We introduce the function $f(z) := z^m - 1 = \prod_{k=1}^{m}(z-\alpha^k)$. Then, taking the logarithmic derivative, we obtain

$$\frac{f'(z)}{f(z)} = \sum_{k=1}^{m}\frac{1}{z-\alpha^k},$$

which means

$$\frac{m}{1-z^{-m}} = \sum_{k=1}^{m}\frac{1}{1-z^{-1}\alpha^k}.$$

□

We set $q = \exp(2\pi ij/3n)$ with $\gcd(j, 3n) = 1$. Applying (3.3) with $\alpha = q^3$, and $z = q$ and $m = n$,

$$\sum_{k=1}^{n}\frac{1}{1-q^{3k-1}} = \frac{n}{1-q^{-n}} = \frac{n}{3}(1-q^n). \tag{3.4}$$

By substituting (3.4) in the right-hand side of (3.2) with $z = q$, we can put the target (3.1) in a form that is more convenient for our method of proof:

$$\frac{(-1)^{n-1}}{2}\sum_{k=1}^{n-1}\frac{q^{k(3n+2)/2}}{1+q^{3k/2}} + \frac{1}{2}\sum_{k=1}^{n-1}\frac{(-1)^k q^{k(3n+2)/2}}{1-q^{3k/2}} + \frac{n}{3}(1-q^n)$$
$$- \sum_{k=1}^{\lfloor (n+1)/2\rfloor}\frac{1}{1-q^{3k-2}} - \frac{2n-1+(-1)^n}{4} = \frac{1}{3} + \frac{3n+1}{6}q^{2n}. \tag{3.5}$$

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Next, we proceed to study (3.5) by distinguishing two cases: $n = 2N$ and $n = 2N - 1$. This allows us to circumvent fractional powers of q.

(a) If $n = 2N$, then $q^{3N} = (-1)^j = -1$ because j is odd. With some algebraic simplifications, (3.5) yields

$$q^{2N}\sum_{k=1}^{N-1}\frac{q^k}{1-q^{6k}}+\sum_{k=1}^{N}\frac{q^{2k-1}}{1-q^{6k-3}}+\frac{2N}{3}(1-q^{2N})-\sum_{k=1}^{N}\frac{1}{1-q^{3k-2}}-N = \frac{1}{3}-\Big(N+\frac{1}{6}\Big)q^N.$$

(b) If $n = 2N - 1$, then we determine that (3.5) is equivalent to

$$q^{2N-1}\sum_{k=1}^{N-1}\frac{q^k}{1-q^{6k}}+\sum_{k=1}^{N-1}\frac{q^{2k}}{1-q^{6k}}+\frac{2N-1}{3}(1-q^{2N-1}) -\sum_{k=1}^{N}\frac{1}{1-q^{3k-2}}-(N-1)=\frac{1}{3}+\Big(N-\frac{1}{3}\Big)q^{2(2N-1)}.$$

In the next two sections, we furnish the proofs for these two cases.

4. Proof of the case $n = 2N$

The condition $\gcd(j, 6N) = 1$ forces $j = \pm 1 \pmod 6$. We set $\omega := q^N$ so that we have $1 - \omega + \omega^2 = 0$ and $\omega^3 = -1$. Therefore, it suffices to show the following result.

LEMMA 4.1. *We have*

$$\omega^2\sum_{k=1}^{N-1}\frac{q^k}{1-q^{6k}}+\sum_{k=1}^{N}\frac{q^{2k-1}}{1-q^{6k-3}}-\sum_{k=1}^{N}\frac{1}{1-q^{3k-2}}=-\frac{N}{3}(1+\omega)+\frac{1}{3}-\frac{\omega}{6}. \tag{4.1}$$

PROOF. We find it convenient to express our claim in terms of the quantities

$$A_1=\sum_{k=1}^{N-1}\frac{1}{1-q^k},\quad A_2=\sum_{k=1}^{N-1}\frac{1}{1-\omega q^k},\quad A_3=\sum_{k=1}^{N-1}\frac{1}{1-\omega^2 q^k},$$
$$A_4=\sum_{k=1}^{N-1}\frac{1}{1+q^k},\quad A_5=\sum_{k=1}^{N-1}\frac{1}{1+\omega q^k},\quad A_6=\sum_{k=1}^{N-1}\frac{1}{1+\omega^2 q^k}.$$

(i) By partial fraction decomposition,

$$\frac{6x}{1-x^6}=\frac{1}{1-x}-\frac{\omega}{1-\omega^2 x}+\frac{\omega^2}{1+\omega x}-\frac{1}{1+x}+\frac{\omega}{1+\omega^2 x}-\frac{\omega^2}{1-\omega x}. \tag{4.2}$$

Hence, taking $x = q^k$ results in

$$6\sum_{k=1}^{N-1}\frac{q^k}{1-q^{6k}}=A_1-\omega^2A_2-\omega A_3-A_4+\omega^2A_5+\omega A_6.$$

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(ii) Again by partial fraction decomposition,

$$\frac{3x}{1-x^3} = \frac{1}{1-x} - \frac{\omega}{1-\omega^2 x} + \frac{\omega^2}{1+\omega x}. \tag{4.3}$$

Thus, the choice $x = q^{2k-1}$ gives

$$3\sum_{k=1}^{N} \frac{q^{2k-1}}{1-q^{6k-3}} = B_1 - \omega B_2 + \omega^2 B_3,$$

where

$$B_1 = \sum_{k=1}^{N} \frac{1}{1-q^{2k-1}}, \quad B_2 = \sum_{k=1}^{N} \frac{1}{1-\omega^2 q^{2k-1}}, \quad B_3 = \sum_{k=1}^{N} \frac{1}{1+\omega q^{2k-1}}.$$

It is easy to check that $B_2 = N/2$ and $B_1 + B_3 = N$ directly from

$$\begin{aligned} 2\,\mathrm{Re}(B_2) = B_2 + \overline{B_2} &= \sum_{k=1}^{N} \frac{1}{1+\omega^{-1} q^{2k-1}} + \sum_{k=1}^{N} \frac{1}{1+\omega q^{1-2k}}, \\ B_1 + B_3 &= \sum_{k=1}^{N} \frac{1}{1-q^{2k-1}} + \sum_{k=1}^{N} \frac{1}{1+q^N q^{2N+2-2k-1}}. \end{aligned}$$

Consequently,

$$3\sum_{k=1}^{N} \frac{q^{2k-1}}{1-q^{6k-3}} = B_1 - \frac{\omega N}{2} + \omega^2 (N - B_1) = (2-\omega)\Big(B_1 - \frac{N}{2}\Big).$$

Moreover, we recognise that

$$B_1 = \sum_{k=1}^{2N-1} \frac{1}{1-q^k} - \sum_{k=1}^{N-1} \frac{1}{1-q^{2k}} = A_1 + \frac{1}{1-q^N} + A_2 - \frac{A_1 + A_4}{2} = \frac{A_1}{2} + A_2 - \frac{A_4}{2} + \omega.$$

(iii) Introducing the values

$$C_1 := \sum_{k=1}^{N} \frac{1}{1-q^{3k-1}}, \quad C_2 := \sum_{k=1}^{N} \frac{1}{1-q^{3k-2}}$$

and using a partial fraction decomposition of $1/(1-x^3)$,

$$\begin{aligned} C_1 + C_2 &= \sum_{k=1}^{3N-1} \frac{1}{1-q^k} - \sum_{k=1}^{N-1} \frac{1}{1-q^{3k}} \\ &= A_1 + \frac{1}{1-q^N} + A_2 + \frac{1}{1-q^{2N}} + A_3 - \frac{A_1 + A_3 + A_5}{3} \\ &= \frac{2A_1}{3} + A_2 + \frac{2A_3}{3} - \frac{A_5}{3} + \frac{4\omega + 1}{3}. \end{aligned}$$

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Taking advantage of (3.4) yields

$$\frac{2N}{3}(1-q^{2N}) = \sum_{k=1}^{2N} \frac{1}{1-q^{3k-1}} = \sum_{k=1}^{N} \frac{1}{1-q^{3k-1}} + \sum_{k=1}^{N} \frac{1}{1-q^{3(2N+1-k)-1}}$$

$$= C_1 + \sum_{k=1}^{N} \frac{1}{1-q^{-3k+2}} = C_1 - \sum_{k=1}^{N} \frac{q^{3k-2}}{1-q^{3k-2}} = C_1 + N - C_2.$$

The last two evaluations lead to

$$C_2 = \frac{A_1}{3} + \frac{A_2}{2} + \frac{A_3}{3} - \frac{A_5}{6} + \frac{1+4\omega}{6} + \frac{N(2\omega-1)}{6}.$$

Finally, by using items (i), (ii) and (iii), we reduce (4.1) to

$$\tfrac{1}{6}(-(A_1+A_6)+(A_2+A_5)-(A_3+A_4)+N-1-\omega(A_2+A_5-(N-1)))=0,$$

which holds because of the symmetry $A_\ell + A_{7-\ell} = N-1$ for $\ell = 1, 2$ and 3. □

5. Proof of the case $n = 2N - 1$

Let $\omega := -q^{2(2N-1)} = e^{\pi i/3}$ so that $\omega^2 = q^{2N-1}$, $1-\omega+\omega^2 = 0$ and $\omega^3 = -1$.

LEMMA 5.1. *We have*

$$\omega^2 \sum_{k=1}^{N-1} \frac{q^k}{1-q^{6k}} + \sum_{k=1}^{N-1} \frac{q^{2k}}{1-q^{6k}} - \sum_{k=1}^{N} \frac{1}{1-q^{3k-2}} = -\frac{N}{3}(1+\omega). \tag{5.1}$$

PROOF. We adopt the notation A_i from the previous section.

(i) By the partial fraction decomposition (4.2),

$$6q^{2N-1} \sum_{k=1}^{N-1} \frac{q^k}{1-q^{6k}} = \omega^2(A_1 - A_4 - \omega(A_3 - A_6) + \omega^2(A_5 - A_2)).$$

(ii) By the partial fraction decomposition (4.3),

$$6 \sum_{k=1}^{N-1} \frac{q^{2k}}{1-q^{6k}} = A_1 + A_4 - \omega(A_2 + A_5) + \omega^2(A_3 + A_6).$$

(iii) We have

$$\sum_{k=1}^{N-1} \frac{1}{1-q^{3k-1}} + \sum_{k=1}^{N} \frac{1}{1-q^{3k-2}} = \sum_{k=1}^{3N-2} \frac{1}{1-q^k} - \sum_{k=1}^{N-1} \frac{1}{1-q^{3k}}$$

$$= A_1 + \sum_{k=0}^{N-1} \frac{1}{1-q^{N+k}} + A_3 - \sum_{k=1}^{N-1} \frac{1}{1-q^{3k}}$$

$$= A_1 + \sum_{k=0}^{N-1} \frac{1}{1-q^{2N-1-k}} + A_3 - \sum_{k=1}^{N-1} \frac{1}{1-q^{3k}}$$

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$$= A_1 + \left(N - \frac{1}{1+\omega} - A_5\right) + A_3 - \frac{A_1 + A_3 + A_5}{3}$$
$$= \frac{2A_1}{3} + \frac{2A_3}{3} - \frac{4A_5}{3} + N - \frac{2-\omega}{3}$$

and invoking (3.4) yields

$$\frac{2N-1}{3}(1-q^{2N-1}) = \sum_{k=1}^{2N-1} \frac{1}{1-q^{3k-1}} = \sum_{k=1}^{N-1} \frac{1}{1-q^{3k-1}} + \sum_{k=1}^{N} \frac{1}{1-q^{3(2N-1+1-k)-1}}$$
$$= \sum_{k=1}^{N-1} \frac{1}{1-q^{3k-1}} + \sum_{k=1}^{N} \frac{1}{1-q^{-3k+2}}$$
$$= \sum_{k=1}^{N-1} \frac{1}{1-q^{3k-1}} - \sum_{k=1}^{N} \frac{q^{3k-2}}{1-q^{3k-2}}$$
$$= \sum_{k=1}^{N-1} \frac{1}{1-q^{3k-1}} + N - \sum_{k=1}^{N} \frac{1}{1-q^{3k-2}}.$$

The last two results imply that

$$6\sum_{k=1}^{N} \frac{1}{1-q^{3k-2}} = 2A_1 + 2A_3 - 4A_5 + 6N + \omega - 2 + (2N-1)(w^2-1).$$

Now, by items (i), (ii) and (iii), we are able to restate (5.1) in terms of A_i. So, the problem reduces to exhibiting a proof for the relation

$$A_1 + A_3 - A_4 - 2A_5 + A_6 = 0. \tag{5.2}$$

Since

$$\frac{x(1-x)}{1+x^3} = -\frac{2}{1+x} + \frac{1}{1-\omega x} + \frac{1}{1+\omega^2 x},$$

we have

$$\sum_{k=1}^{N-1} \frac{q^k(1-q^k)}{1+(q^k)^3} = A_6 + A_2 - 2A_4,$$
$$\sum_{k=1}^{N-1} \frac{\omega q^k(1-\omega q^k)}{1+(\omega q^k)^3} = A_1 + A_3 - 2A_5,$$
$$\sum_{k=1}^{N-1} \frac{\omega^2 q^k(1-\omega^2 q^k)}{1+(\omega^2 q^k)^3} = A_2 + A_4 - 2A_6.$$

Therefore, we arrive at the following equivalent form of (5.2):

$$\sum_{k=1}^{N-1}\left(\frac{q^k(1-q^k)}{1+(q^k)^3} + 3\,\frac{\omega q^k(1-\omega q^k)}{1+(\omega q^k)^3} - \frac{\omega^2 q^k(1-\omega^2 q^k)}{1+(\omega^2 q^k)^3}\right) = 0. \tag{5.3}$$

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Since $1+(\omega q^k)^3 = 1-q^{3k}$ and $1+(\omega^2 q^k)^3 = 1+q^{3k}$, further algebraic manipulation converts (5.3) into

$$\sum_{k=1}^{N-1} \frac{q^k(1+\omega q^{3k})(1-\omega q^k)}{1-q^{6k}} = 0. \tag{5.4}$$

However, we note that

$$\frac{z(1+\omega z^3)(1-\omega z)}{1-z^6} = \frac{1-\omega^2}{3}\Big(\frac{1}{1-z^{-2}} + \frac{1}{1+\omega z^2} - \frac{1}{1-z^{-1}} - \frac{1}{1+\omega z}\Big).$$

Letting $z = q^k$ and $\omega = -\omega^{-2} = -q^{-(2N-1)}$, our summand can be written as

$$\frac{1-\omega^3}{3}\Big(\frac{1}{1-q^{-2k}} + \frac{1}{1-q^{-(2N-1-2k)}} - \frac{1}{1-q^{-k}} - \frac{1}{1-q^{-(2N-1-k)}}\Big).$$

Hence, the claim now becomes

$$\sum_{k=1}^{N-1}\frac{1}{1-q^{-2k}} + \sum_{k=1}^{N-1}\frac{1}{1-q^{-(2(N-k)-1)}} = \sum_{k=1}^{N-1}\frac{1}{1-q^{-k}} + \sum_{k=1}^{N-1}\frac{1}{1-q^{-(2N-1-k)}},$$

which in turn translates to

$$\sum_{k=1}^{N-1}\frac{1}{1-q^{-2k}} + \sum_{k=1}^{N-1}\frac{1}{1-q^{-(2k-1)}} = \sum_{k=1}^{N-1}\frac{1}{1-q^{-k}} + \sum_{k=N}^{2N-2}\frac{1}{1-q^{-k}}.$$

Indeed, equality follows here since both sides of the last equation are equal to $\sum_{k=1}^{2N-2} 1/(1-q^{-k})$. In fact, this is reminiscent of the set-theoretic identity

$$\begin{aligned}&\{k : 1 \le k \le N-1\} \cup \{2N-1-k : 1 \le k \le N-1\}\\ &\quad = \{2k : 1 \le k \le N-1\} \cup \{2N-1-2k : 1 \le k \le N-1\}.\end{aligned}$$

The proof is complete. □

6. Conclusion

For the trigonometric functions enthusiast, the particular equation in (5.4) can be converted to one that involves only these circular functions. To this end, we use the identities

$$\frac{e^{i\theta}}{1-e^{2i\theta}} = \frac{i\csc(\theta)}{2}, \qquad \frac{1}{1+e^{2i\theta}} = \frac{1}{2} - \frac{i\tan(\theta)}{2},$$

and we rewrite $\pi/6 = \pi/2 - (2N-1)x$ followed by replacing tan with cot via $\tan(\pi/2 - t) = \cot(t)$. Here, $x = \pi/(6N-3)$ and the outcome is

$$\begin{aligned}\frac{q^k(1+\omega q^{3k})(1-\omega q^k)}{1-q^{6k}} &= \frac{1-\omega^2}{3}\Big(\frac{q^k}{1-q^{2k}} - \frac{1}{1+\omega q^k} + \frac{1}{1+\omega q^{2k}}\Big)\\ &= \frac{i(1-\omega^2)}{6}\Big(\csc(2kx) + \tan\Big(\frac{\pi}{6}+kx\Big) - \tan\Big(\frac{\pi}{6}+2kx\Big)\Big)\\ &= \frac{i(1-\omega^2)}{6}(\csc(2kx) + \cot((2N-1-k)x) - \cot((2N-1-2k)x).\end{aligned}$$

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Hence, (5.4) reduces to verifying the trigonometric identity

$$\sum_{k=1}^{N-1}(\csc(2kx)+\cot((2N-1-k)x)-\cot((2N-1-2k)x)=0.$$

For the more number-theoretic minded reader, we present below a consequence of the identity in (5.4). We appreciate Terence Tao for allowing us to include his derivation in this paper. For the remainder of this section, specialise to the case where $2N-1$ is coprime to 3.

Introduce the cube root of unity $\epsilon := \omega^2 = e^{2\pi i/3} = q^{2N-1}$, where $q = e^{2\pi i/(6N-3)}$. Expand the numerator in (5.4):

$$\sum_{k=1}^{N-1}\frac{q^k+\epsilon^2 q^{2k}-\epsilon^2 q^{4k}-\epsilon q^{5k}}{1-q^{6k}}=0.$$

From the easily verified 'discrete sawtooth Fourier series' identity, for any k not divisible by $2N-1$,

$$\frac{1}{1-q^{6k}}=-\frac{1}{2N-1}\sum_{j=0}^{2N-2} jq^{6jk}$$

(proven by multiplying out the denominator, cancelling terms and applying the geometric series formula), we can write the preceding identity to be proven as

$$\sum_{j=0}^{2N-2} j\sum_{k=1}^{N-1}(q^{(6j+1)k}+\epsilon^2 q^{(6j+2)k}-\epsilon^2 q^{(6j+4)k}-\epsilon q^{(6j+5)k})=0.$$

Since $\gcd(2N-1,3)=1$, we can write $q=\epsilon^{2N-1}\zeta$ for some primitive $(2N-1)$th root ζ of unity. We then reduce to

$$\begin{aligned}\sum_{j=0}^{2N-2} j\sum_{k=1}^{N-1}&(\epsilon^{(2N-1)k}\zeta^{(6j+1)k}+\epsilon^{2(2N-1)k+2}\zeta^{(6j+2)k})\\ &=\sum_{j=0}^{2N-2} j\sum_{k=1}^{N-1}(\epsilon^{(2N-1)k+2}\zeta^{(6j+4)k}+\epsilon^{2(2N-1)k+1}\zeta^{(6j+5)k}).\end{aligned}$$

From Galois theory, we can see that the net coefficient of ζ^a would have to be independent of a for each primitive residue class a mod $2N-1$. We summarise this discussion in the next declaration.

COROLLARY 6.1. *Let $N>1$ be a natural number such that $2N-1$ is not divisible by 3, let $\chi:\mathbb{Z}\to\mathbb{C}$ be a nonprincipal Dirichlet character of period $2N-1$ and let $\epsilon := e^{2\pi i/3}$. Then, we have the character sum identity*

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$$\Big(\sum_{j=0}^{2N-2} j\cdot\chi(6j+1)\Big)\Big(\sum_{k=1}^{2N-2}\epsilon^{(2N-1)k}\cdot\chi(k)\Big)$$
$$= -\Big(\sum_{j=0}^{2N-2} j\cdot\chi(6j+2)\Big)\Big(\sum_{k=1-N}^{N-1}\epsilon^{2(2N-1)k+2}\cdot\chi(k)\Big). \tag{6.1}$$

We conclude with a problem proposed by Terence Tao.

QUESTION 6.2. Is there a direct proof of the identity (6.1) that does not rely on (5.4)?

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