J. Aust. Math. Soc. **91** (2011), 145–162
doi:10.1017/S1446788711001558

THE GENERALIZED WITT MODULAR LIE SUPERALGEBRA OF CARTAN TYPE

YAN-QIN DONG ✉, YONG-ZHENG ZHANG and ANGELO EBONZO

(Received 28 April 2009; accepted 11 August 2011)

Communicated by J. Du

Abstract

We construct the generalized Witt modular Lie superalgebra $\tilde{W}$ of Cartan type. We give a set of generators for $\tilde{W}$ and show that $\tilde{W}$ is an extension of a subalgebra of $\tilde{W}$ by an ideal $\overline{J}$. Finally, we describe the homogeneous derivations of Z-degree of $\tilde{W}$ and we determine the derivation superalgebra of $\tilde{W}$.

2010 *Mathematics subject classification*: primary 17B50.

Keywords and phrases: modular Lie superalgebras, Z-graded Lie superalgebras, derivation superalgebras.

1. Introduction

The theory of Lie superalgebras over a field of characteristic zero is very well developed (see, for example, [4, 5, 10]). But the same is not true for modular Lie superalgebras. For instance, the classification of finite-dimensional simple modular Lie superalgebras is not yet complete. As far as we know, the $(p, 2p)$-structure for modular Lie superalgebras (analogous to p-mappings for modular Lie algebras) was introduced by Kochetkov and Leites [6]. Later, Petrogradski [9] studied restricted enveloping algebras for modular Lie superalgebras, and Farnsteiner [2] worked on Frobenius extensions and restricted modular Lie superalgebras. In 1997, Zhang [13] constructed four classes of finite-dimensional Cartan type modular Lie superalgebras $X(m, n, \boldsymbol{t})$ and studied their simplicity and restrictiveness, where X is one of the algebras W, S, H or K.

Derivation algebras of Lie algebras play an important role in the study of properties of Lie algebras such as filtrations and automorphism groups. Celousov [1] and Petrogradski [9] investigated derivation algebras of Cartan type modular Lie algebras. Derivation superalgebras of Cartan type modular Lie superalgebras are becoming a

This work was supported by National Science Foundation of China Funded Projects (Nos. 10871057 and 10701019).

subject of interest in the structure theory of Lie superalgebras. Due to the prime characteristic and superstructure of Lie superalgebras, their derivation superalgebras are harder to determine. Despite this, the derivation superalgebras of W, S, H, K, HO and KO have been determined (see [3, 7, 8, 12, 15]).

Our work is motivated by the results and methods for Lie algebras and Lie superalgebras and is based on certain results on modular Lie algebras and Lie superalgebras of Cartan type (see [11, 14, 15]). The paper is organized as follows. In Section 2, we construct the finite-dimensional generalized Witt modular Lie superalgebra. In Section 3, we give a set of generators for $\tilde{W}$ and show that $\tilde{W}$ is not simple; moreover, we show that $\tilde{W}$ is an extension of a subalgebra of $\tilde{W}$ by the ideal $\overline{J}$. In Section 4, we establish some technical lemmas and determine the derivation superalgebra of $\tilde{W}$.

2. Basics and construction

Throughout this paper, F denotes a field of characteristic p, greater than 2, and $Z_2 = \{\bar{0}, \bar{1}\}$ denotes the field of two elements. We use the notation N and N_0 to stand for the sets of positive integers and nonnegative integers, respectively. For $n \in N$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in N_0^n$, we define $|\alpha| = \sum_{i=1}^n \alpha_i$.

Let $\mathcal{O}(n)$ denote the divided power algebra with an F-basis $\{x^{(\alpha)} \mid \alpha \in N_0^n\}$. Put $\boldsymbol{t} = (t_1, t_2, \ldots, t_n) \in N_0^n$ and $\pi_i = p^{t_i} - 1$, where $i = 1, 2, \ldots, n$. Let

$$A(\boldsymbol{t}) := \{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in N_0^n \mid 0 \le \alpha_i \le \pi_i\}.$$

Then

$$\mathcal{O}(n, \boldsymbol{t}) := \mathrm{span}_F\{x^{(\alpha)} \mid \alpha \in A(\boldsymbol{t})\}$$

is a finite-dimensional subalgebra of $\mathcal{O}(n)$.

Let $\Lambda(q)$ denote the Grassmann superalgebra over F in q variables $x_{n+1}, x_{n+2}, \ldots, x_r$, where $r = n + q$. In order to shorten the notation for the Grassmann superalgebra, we put

$$B_k = \{\langle i_1, i_2, \ldots, i_k\rangle \mid n + 1 \le i_1 < i_2 < \cdots < i_k \le r\}$$

and $B(q) = \bigcup_{k=0}^q B_k$, where $B_0 = \emptyset$. When $\mu = \langle i_1, i_2, \ldots, i_k\rangle \in B_k$, we define $|\mu| = k$, $\{\mu\} = \{i_1, i_2, \ldots, i_k\}$ and $x^\mu = x_{i_1} x_{i_2} \cdots x_{i_k}$, where we adopt the conventions that $|\emptyset| := 0$ and $x^\emptyset = 1$. Then the set $\{x^\mu \mid \mu \in B(q)\}$ is an F-basis of $\Lambda(q)$.

We now fix two positive integers m_1 and m_2. We write $m := m_1 + m_2$, $s := r + m_1$ and $s_1 := s + m_2$. Let

$$Q(m) = F[y_{r+1}, \ldots, y_s, y_{s+1}, \ldots, y_{s_1}]$$

be the truncated polynomial algebra such that $y_i^p = 1$ for $i = r + 1, \ldots, s_1$. We let $\Pi = \{0, 1, \ldots, p - 1\}$ denote the prime subfield of F and write $H := \Pi^m$. For every element $\lambda = (\lambda_{r+1}, \ldots, \lambda_{s_1}) \in H$, we define $y^\lambda = \prod_{r+1}^{s_1} y_i^{\lambda_i}$. Then $Q(m)$ has an F-basis $\{y^\lambda \mid \lambda \in H\}$. The tensor product

$$G := \mathcal{O}(n, \boldsymbol{t}) \otimes \Lambda(q) \otimes Q(m)$$

https://doi.org/10.1017/S1446788711001558 Published online by Cambridge University Press

is an associative superalgebra with a Z_2-gradation induced by the standard Z_2-gradation of $\Lambda(q)$. For $f \in \mathcal{O}(n, \boldsymbol{t})$, $g \in \Lambda(q)$ and $h \in Q(m)$, we write fgh for $f \otimes g \otimes h$. Then

$$\{x^{(\alpha)}x^{\mu}y^{\lambda} \mid \alpha \in A(\boldsymbol{t}), \mu \in B(q), \lambda \in H\}$$

is an F-basis for G.

Let $E = \langle n+1, \ldots, r\rangle$ and $\pi = (\pi_1, \ldots, \pi_n)$. Clearly, $E \in B(q)$ and $\pi \in A(\boldsymbol{t})$. For convenience, we write $Y_0 = \{1, \ldots, n\}$, $Y_1 = \{n+1, \ldots, r\}$ and $Y_2 = \{r+1, \ldots, s\}$. Let $Y = Y_0 \cup Y_1$ and $S = Y \cup Y_2$. When $i \in Y_0$ and $\varepsilon_i = (\delta_{i1}, \delta_{i2}, \ldots, \delta_{in})$, we abbreviate $x^{(\varepsilon_i)}$ to x_i. When $i = r+1, \ldots, s_1$ and $\bar{\varepsilon}_i = (\delta_{i(r+1)}, \delta_{i(r+2)}, \ldots, \delta_{is_1})$, we abbreviate $y^{\bar{\varepsilon}_i}$ to y_i.

Let $D_1, D_2, \ldots, D_s$ be the linear transformations of G such that

$$D_i(x^{(\alpha)}x^{\mu}y^{\lambda}) = \begin{cases} x^{(\alpha-\varepsilon_i)}x^{\mu}y^{\lambda} & \text{if } i \in Y_0, \\ x^{(\alpha)}\partial_i(x^{\mu})y^{\lambda} & \text{if } i \in Y_1, \\ \lambda_i x^{(\alpha)}x^{\mu}y^{\lambda} & \text{if } i \in Y_2. \end{cases}$$

Here ∂_i is the derivation of $\Lambda(q)$ such that $\partial_i(x_j) = \delta_{ij}$ for $i, j \in Y_1$. Then D_1, $D_2, \ldots, D_s$ are derivations of the superalgebra G. Let

$$\tilde{W}(n, \boldsymbol{t}, q, m) = \left\{\sum_{i=1}^{s} f_i D_i \;\middle|\; f_i \in G\right\}.$$

For a superalgebra (or a superspace) $L = L_{\bar{0}} \oplus L_{\bar{1}}$, we write $\mathrm{h}(L) = L_{\bar{0}} \cup L_{\bar{1}}$ for the set of all Z_2-homogeneous elements of L and write $|x|$ for the Z_2-degree of a given homogeneous element x. It is clear that $|D_i| = \tilde{\imath}$, where

$$\tilde{\imath} = \begin{cases} \bar{0} & \text{if } i \in Y_0 \cup Y_2, \\ \bar{1} & \text{if } i \in Y_1. \end{cases}$$

Set

$$\tilde{W}_{\gamma} = \mathrm{span}_F\{x^{(\alpha)}x^{\mu}y^{\lambda}D_i \mid \overline{|\mu|} + \tilde{\imath} = \gamma\}$$

for $\gamma \in Z_2$. Then $\tilde{W} = \bigoplus_{\gamma \in Z_2} \tilde{W}_{\gamma}$. The following formula holds in $\tilde{W}(n, \boldsymbol{t}, q, m)$:

$$[fD_i, gD_j] = fD_i(g)D_j - (-1)^{|fD_i||gD_j|}gD_j(f)D_i \tag{2.1}$$

for $f, g \in \mathrm{h}(G)$ and $i, j \in S$. It follows that $\tilde{W}(n, \boldsymbol{t}, q, m)$ is a finite-dimensional Lie superalgebra contained in $\mathrm{der}\, G$. We abbreviate $\tilde{W}(n, \boldsymbol{t}, q, m)$ to $\tilde{W}$ and call $\tilde{W}$ a generalized Witt modular Lie superalgebra of Cartan type.

Let

$$G_i = \mathrm{span}_F\{x^{(\alpha)}x^{\mu}y^{\lambda} \mid |\alpha| + |\mu| = i\}.$$

Then $G = \bigoplus_{i=0}^{\xi} G_i$ is a Z-graded associative superalgebra, where $\xi = \sum_{i=1}^{n} \pi_i + q$. Set

$$\delta(j, Y_2) = \begin{cases} 1 & \text{if } j \in Y_2, \\ 0 & \text{if } j \notin Y_2. \end{cases}$$

https://doi.org/10.1017/S1446788711001558 Published online by Cambridge University Press

Let

$$\tilde{W}_i = \mathrm{span}_F\{x^{(\alpha)}x^{\mu}y^{\lambda}D_j \mid j \in S, |\alpha| + |\mu| + \delta(j, Y_2) - 1 = i\}.$$

Then $\tilde{W} = \bigoplus_{i=-1}^{\xi} \tilde{W}_i$ is a Z-graded Lie superalgebra.

Lemma 2.1. *Let $f \in G$. If $D_i(f) = 0$ for all $i \in Y$, then $f \in G_0$.*

Proof. Without loss of generality, we may assume that f has the form $f = x^{(\alpha)}x^{\mu}y^{\lambda}$. For any $i \in Y_0 \cup Y_1$, our assumption forces $\alpha = 0$ and $\mu = \emptyset$. Thus, $f = y^{\lambda} \in G_0$. □

3. Structure of $\tilde{W}$

Lemma 3.1. *Let*

$$M_1 = \{x^{(k_i\varepsilon_i)}D_j \mid i \in Y_0, j \in Y, 0 \le k_i \le \pi_i\},$$

let

$$M_2 = \{x_l D_t \mid l \in Y_1, t \in Y_2\}$$

and let

$$M = M_1 \cup M_2 \cup \tilde{W}_{-1} \cup \tilde{W}_0.$$

Then $\tilde{W}$ is generated by the set M.

Proof. Let X be the subalgebra of $\tilde{W}$ generated by M. We proceed in several steps to show that $\tilde{W} = X$.

Step 1. We show that $x^{\pi}y^{\lambda}D_1 \in X$. In order to prove the result, we first show that

$$x^{(\pi_1\varepsilon_1+\cdots+\pi_t\varepsilon_t)}D_1 \in X$$

by induction on t, where $t \in Y_0$.

If $t = 1$, then $x^{(\pi_1\varepsilon_1)}D_1 \in M_1 \subseteq X$. Suppose that $x^{(\pi_1\varepsilon_1+\cdots+\pi_{t-1}\varepsilon_{t-1})}D_1 \in X$. We can easily verify that

$$x^{(\pi_t\varepsilon_t)}x_1D_1 = [x^{(\pi_t\varepsilon_t)}D_1, x^{(2\varepsilon_1)}D_1] \in X.$$

Moreover, we get

$$x^{(\pi_1\varepsilon_1+\cdots+\pi_t\varepsilon_t)}D_1 = 1/2[x^{(\pi_1\varepsilon_1+\cdots+\pi_{t-1}\varepsilon_{t-1})}D_1, x^{(\pi_t\varepsilon_t)}x_1D_1] \in X.$$

The induction is completed and $x^{\pi}D_1 \in X$. Since

$$x_1y^{\lambda}D_1 = [y^{\lambda}D_1, x^{(2\varepsilon_1)}D_1] \in [\tilde{W}_{-1}, M_1] \subseteq X,$$

we see that

$$x^{\pi}y^{\lambda}D_1 = 1/2[x^{\pi}D_1, x_1y^{\lambda}D_1] \in X.$$

Step 2. We now show that $x^{\pi}x^{E}y^{\lambda}D_i \in X$ for $i \in S$. We consider three cases below.

https://doi.org/10.1017/S1446788711001558 Published online by Cambridge University Press

Case 1. Suppose that $i \in Y_0$. We first show that $x_{n+1} \cdots x_k D_1 \in X$ by induction on k, where $k \in Y_1$. If $k = n+1$, then $x_{n+1}D_1 \in \tilde{W}_0 \subseteq X$. Suppose that $x_{n+1} \cdots x_{k-1} D_1 \in X$. One can easily verify that

$$x_1 x_k D_1 = [x_k D_1, x^{(2\varepsilon_1)} D_1] \in X.$$

Moreover,

$$x_{n+1} \cdots x_k D_1 = [x_{n+1} \cdots x_{k-1} D_1, x_1 x_k D_1] \in X.$$

The induction is completed and $x^E D_1 \in X$.

Since

$$x^E y^\lambda D_1 = [x^E D_1, x_1 y^\lambda D_1] \in X,$$

we see that

$$x_1 x^E y^\lambda D_i = [x^E y^\lambda D_1, x^{(2\varepsilon_1)} D_i] \in X$$

for any $i \in Y_0$. Moreover,

$$x^\pi x^E y^\lambda D_1 = 1/2[x^\pi D_1, x_1 x^E y^\lambda D_1] \in X$$

and

$$x^\pi x^E y^\lambda D_i = [x^\pi D_1, x_1 x^E y^\lambda D_i] \in X$$

when $i \neq 1$.

Case 2. Suppose that $i \in Y_1$. By Case 1, we deduce that

$$x^\pi x^E y^\lambda D_i = [x^\pi x^E y^\lambda D_1, x_1 D_i] \in X.$$

Case 3. Suppose that $i \in Y_2$. Noting that $x_l D_i \in M_2$ for any $l \in Y_1$, we deduce from Case 2 that

$$x^\pi x^E y^\lambda D_i = [x^\pi x^E y^\lambda D_l, x_l D_i] \in X.$$

Step 3. We shall show that

$$x^{(\alpha)} x^\mu y^\lambda D_i \in X$$

for any $i \in S$ by induction on $(|\pi| + |E|) - (|\alpha| + |\mu|)$, which we call t.

Let $t = 0$. By Step 2, we see that $x^\pi x^E y^\lambda D_i \in X$ for any $i \in S$. Let $t \geq 1$. Suppose that the result is true for $t - 1$. We consider the two cases $|\alpha| < |\pi|$ and $|\alpha| = |\pi|$ separately.

If $|\alpha| < |\pi|$, then there exists $k \in Y_0$ such that $x^{(\alpha+\varepsilon_k)} x^\mu y^\lambda \in G$. By our inductive hypothesis, $x^{(\alpha+\varepsilon_k)} x^\mu y^\lambda D_i \in X$. Moreover,

$$x^{(\alpha)} x^\mu y^\lambda D_i = [D_k, x^{(\alpha+\varepsilon_k)} x^\mu y^\lambda D_i] \in X.$$

If $|\alpha| = |\pi|$, then $|\mu| < |E|$ since $t \geq 1$. Consequently, there exists $k \in Y_1$ such that $x_k x^\mu \neq 0$. By our inductive hypothesis, $x^{(\alpha)} x_k x^\mu y^\lambda D_i \in X$. Moreover,

$$x^{(\alpha)} x^\mu y^\lambda D_i = [D_k, x^{(\alpha)} x_k x^\mu y^\lambda D_i] \in X.$$

Hence, $\tilde{W} = X$ and the proof is completed. □

https://doi.org/10.1017/S1446788711001558 Published online by Cambridge University Press

LEMMA 3.2. *Let $l = |H| = p^m$ and let*

$$\Delta = \Big\{ \sum_{i=1}^{l} a_i y^{\lambda_i} \;\Big|\; \lambda_i \in H, a_i \in F, \sum_{i=1}^{l} a_i = 0 \Big\}.$$

Then Δ is an ideal of $Q(m)$ and $Q(m) = \Delta \oplus F1$.

PROOF. Suppose that $f = \sum_{i=1}^{l} a_i y^{\lambda_i} \in \Delta$ and $g = \sum_{j=1}^{l} b_j y^{\lambda_j} \in Q(m)$, where $a_i, b_j \in F$ and $\lambda_i, \lambda_j \in H$. Then $\sum_{i=1}^{l} a_i = 0$. Write $h := fg = \sum_{k=1}^{l} c_k y^{\lambda_k}$, where $c_k \in F, \lambda_k \in H$. Then

$$\Big(\sum_{i=1}^{l} a_i y^{\lambda_i} \Big)\Big(\sum_{j=1}^{l} b_j y^{\lambda_j} \Big) = \sum_{k=1}^{l} c_k y^{\lambda_k}$$

and we conclude that

$$\sum_{i,j=1}^{l} a_i b_j y^{\lambda_i + \lambda_j} = \sum_{k=1}^{l} c_k y^{\lambda_k}.$$

Since $y^{\lambda_i + \lambda_j} \neq 0$, we see that

$$\sum_{k=1}^{l} c_k = \sum_{i,j=1}^{l} a_i b_j = \Big(\sum_{i=1}^{l} a_i \Big)\Big(\sum_{j=1}^{l} b_j \Big) = 0.$$

Hence, $h \in \Delta$ and Δ is an ideal of $Q(m)$.

Let $f = \sum_{i=1}^{l} a_i y^{\lambda_i}$ be any element of $Q(m)$. Then $f - \sum_{i=1}^{l} a_i \cdot 1 \in \Delta$ and we conclude that $Q(m) = \Delta + F1$. Clearly, $\Delta \cap F1 = \{0\}$. Hence, $Q(m) = \Delta \oplus F1$. □

LEMMA 3.3. *Let*

$$\Gamma = \operatorname{span}_F \{ gh \mid g \in \mathcal{O}(n, \boldsymbol{t}) \otimes \Lambda(q), h \in \Delta \}$$

and let

$$\overline{J} = \Big\{ \sum_{i=1}^{s} f_i D_i \;\Big|\; f_i \in \Gamma, D_{i_k} \cdots D_{i_2} D_{i_1}(f_i) \in \Gamma \ \forall i_k \in S, 1 \leq k \leq s \Big\}.$$

Then Γ is an ideal of G and $\overline{J}$ is an ideal of $\tilde{W}$.

PROOF. Let $f \in G$. Without loss of generality, we may suppose that $f = g'h'$, where $g' \in \mathcal{O}(n, \boldsymbol{t}) \otimes \Lambda(q)$ and $h' \in Q(m)$. Suppose that $gh \in \Gamma$, where $g \in \mathcal{O}(n, \boldsymbol{t}) \otimes \Lambda(q)$ and $h \in \Delta$. Then

$$f(gh) = (g'h')(gh) = (g'g)(h'h) \in \Gamma$$

by Lemma 3.2. Similarly, $(gh)f \in \Gamma$. Thus, Γ is an ideal of G.

Now suppose that $A = \sum_{i=1}^{s} g_i D_i \in \tilde{W}$ and $B = \sum_{j=1}^{s} f_j D_j \in \overline{J}$, where $g_i \in G$, $f_j \in \Gamma$ and $D_{i_k} \cdots D_{i_2} D_{i_1}(f_j) \in \Gamma$. By (2.1), we see that

$$[A, B] = \sum_{i,j=1}^{s} g_i D_i(f_j) D_j - \sum_{i,j=1}^{s} (-1)^{|g_i D_i||f_j D_j|} f_j D_j(g_i) D_i.$$

https://doi.org/10.1017/S1446788711001558 Published online by Cambridge University Press

By our assumption, $D_{i_k}\cdots D_{i_2}D_{i_1}(f_j)\in\Gamma$. Putting $k=1$, we deduce that $D_i(f_j)\in\Gamma$. Consequently, $g_iD_i(f_j)\in\Gamma$ and $f_jD_j(g_i)\in\Gamma$. Moreover, we can easily deduce that

$$D_{i_k}\cdots D_{i_2}D_{i_1}(g_iD_i(f_j))\in\Gamma,\quad D_{i_k}\cdots D_{i_2}D_{i_1}(f_jD_j(g_i))\in\Gamma$$

by induction on k. Hence, $\overline{J}$ is an ideal of $\tilde{W}$. □

Suppose that

$$\overline{X}=\left\{\sum_{i=1}^{s}g_iD_i\,\middle|\,g_i\in G,\exists k\in\{1,\ldots,s\}\text{ such that }i_k\in S\text{ and }D_{i_k}\cdots D_{i_2}D_{i_1}(g_i)\notin\Gamma\right\}.$$

It may be verified that $\overline{X}$ is a subalgebra of $\tilde{W}$. In particular,

$$\left\{\sum_{i=1}^{s}g_iD_i\,\middle|\,g_i\in\mathcal{O}(n,\boldsymbol{t})\otimes\Lambda(q)\right\}$$

is a subalgebra of $\overline{X}$.

Theorem 3.4. *The algebra $\tilde{W}$ is an extension of a subalgebra $\overline{X}$ by the ideal $\overline{J}$.*

Proof. Let $\sum_{i=1}^{s}f_iD_i$ be any element of $\tilde{W}$, where $f_i\in G$. Without loss of generality, we may suppose that $f_i=g_ih_i$, where $g_i\in\mathcal{O}(n,\boldsymbol{t})\otimes\Lambda(q)$ and $h_i\in Q(m)$. It follows by Lemma 3.2 that

$$f_i=g_i(h_i'+a_i1)=g_ih_i'+a_ig_i,$$

where $h_i'\in\Delta$, $a_i\in F$. Thus,

$$\begin{aligned}\sum_{i=1}^{s}f_iD_i&=\sum_{i=1}^{s}(g_ih_i')D_i+\sum_{i=1}^{s}(a_ig_i)D_i\\&=\sum_{D_{i_k}\cdots D_{i_2}D_{i_1}(g_ih_i')\in\Gamma}(g_ih_i')D_i+\sum_{D_{i_k}\cdots D_{i_2}D_{i_1}(g_ih_i')\notin\Gamma}(g_ih_i')D_i+\sum_{i=1}^{s}(a_ig_i)D_i\\&\in\overline{J}+\overline{X}.\end{aligned}$$

It is clear that $\overline{X}\cap\overline{J}=\{0\}$. By Lemma 3.3, $\tilde{W}$ is an extension of $\overline{X}$ by $\overline{J}$. □

4. Derivation superalgebra of $\tilde{W}$

Let $L=\bigoplus_{i\in Z}L_i$ be a Z-graded superalgebra. Let $x\in L$. If there exists $i\in Z$ such that $x\in L_i$, then we call x a Z-homogeneous element and i the Z-degree of x. As usual, the derivation superalgebra of $\tilde{W}$ is a Z-graded Lie superalgebra, that is, $\operatorname{der}\tilde{W}=\bigoplus_{t\in J}\operatorname{der}_t\tilde{W}$, where

$$\operatorname{der}_t\tilde{W}:=\{\varphi\in\operatorname{der}\tilde{W}\mid\varphi(\tilde{W}_i)\subset\tilde{W}_{t+i}\},\quad J=\{-\xi-1,-\xi,\ldots,\xi,\xi+1\}.$$

https://doi.org/10.1017/S1446788711001558 Published online by Cambridge University Press

Set

$$\tau(i) = \begin{cases} \pi_i & \text{if } i \in Y_0, \\ 1 & \text{if } i \in Y_1. \end{cases}$$

Define a linear mapping $\rho_i : G \to G$ such that

$$\rho_i(x^{(\alpha)}x^{\mu}y^{\lambda}) = \begin{cases} x^{(\alpha+\varepsilon_i)}x^{\mu}y^{\lambda} & \text{if } i \in Y_0 \text{ and } \alpha + \varepsilon_i \in A(\boldsymbol{t}), \\ x^{(\alpha)}x_i x^{\mu}y^{\lambda} & \text{if } i \in Y_1, \\ \lambda_i^{-1}x^{(\alpha)}x^{\mu}y^{\lambda} & \text{if } i \in Y_2 \text{ and } \lambda_i \neq 0. \end{cases}$$

We use the convention that $\rho_i(x^{(\alpha)}x^{\mu}y^{\lambda}) = 0$ for $\alpha + \varepsilon_i \notin A(\boldsymbol{t})$ or $\lambda_i = 0$.

Definition 4.1. An element f of G is said to be of D_i-type if $D_i^{\tau(i)}(f) = 0$ for $i \in Y$ and $D_i^{p-1}(f) = f$ for $i \in Y_2$.

Lemma 4.2. *Suppose that $f \in G$.*

(i) *If $i \in Y_2$, then f is of D_i-type if and only if $\lambda_i^{p-1} = 1$.*
(ii) *$D_i(f)$ is of D_i-type for any $i \in S$.*

Proof. Part (i) is obvious.

We now consider part (ii). For $i \in Y$, it is clear that

$$D_i^{\tau(i)}(D_i(f)) = D_i^{\tau(i)+1}(f) = 0.$$

For $i \in Y_2$, we may assume that $f = x^{(\alpha)}x^{\mu}y^{\lambda}$. Since $\lambda_i^p = \lambda_i$, we see that

$$D_i^{p-1}(D_i(f)) = D_i^p(f) = \lambda_i^p x^{(\alpha)}x^{\mu}y^{\lambda} = \lambda_i x^{(\alpha)}x^{\mu}y^{\lambda} = D_i(f).$$

Hence, $D_i(f)$ is of D_i-type. □

Lemma 4.3. *Suppose that $i, j \in S$ and $i \neq j$. Then:*

(i) *if $f \in G$ is of D_i-type, then $D_i\rho_i(f) = f$;*
(ii) *we have the equality*

$$D_i\rho_j = (-1)^{\bar{i}\bar{j}}\rho_j D_i.$$

Proof. To prove (i), suppose that $i \in Y_2$ and $f = x^{(\alpha)}x^{\mu}y^{\lambda}$. Since f is of D_i-type, we deduce from Lemma 4.2(i) that $\lambda_i \neq 0$. Thus,

$$D_i\rho_i(f) = D_i(\lambda_i^{-1}x^{(\alpha)}x^{\mu}y^{\lambda}) = x^{(\alpha)}x^{\mu}y^{\lambda} = f.$$

The remaining cases where $i \in Y_0 \cup Y_1$ are similar.

Part (ii) is obvious. □

Lemma 4.4. *Let $f_{t_1}, f_{t_2}, \ldots, f_{t_k} \in G$, where $t_1, t_2, \ldots, t_k \in S$. If $f_{t_1}, f_{t_2}, \ldots, f_{t_k}$ are of D_i-type and $D_i(f_j) = (-1)^{\bar{i}\bar{j}}D_j(f_i)$ for any $i, j \in \{t_1, t_2, \ldots, t_k\}$, there exists $f \in G$ such that $D_i(f) = f_i$ for all $i = t_1, t_2, \ldots, t_k$.*

Proof. We use induction on k. Let $k = 1$ and $f = \rho_{t_1}(f_{t_1})$. It follows from Lemma 4.3(i) that $D_{t_1}(f) = D_{t_1}\rho_{t_1}(f_{t_1}) = f_{t_1}$.

Suppose that there exists $g \in G$ such that $D_i(g) = f_i$ whenever $i = t_1, t_2, \ldots, t_{k-1}$. Let $f = g + \rho_{t_k}(f_{t_k} - D_{t_k}(g))$. By our inductive hypothesis and Lemma 4.3(ii), we deduce that

$$\begin{aligned} D_i(f) &= D_i(g) + D_i\rho_{t_k}(f_{t_k} - D_{t_k}(g)) \\ &= f_i + (-1)^{\tilde{t}_k\tilde{i}}\rho_{t_k}(D_i(f_{t_k}) - D_iD_{t_k}(g)) \\ &= f_i + (-1)^{\tilde{t}_k\tilde{i}}\rho_{t_k}((-1)^{\tilde{t}_k\tilde{i}}D_{t_k}(f_i) - (-1)^{\tilde{t}_k\tilde{i}}D_{t_k}D_i(g)) \\ &= f_i. \end{aligned}$$

We have to show that $D_{t_k}(f) = f_{t_k}$. By Lemma 4.2(ii), $D_{t_k}(g)$ is of D_{t_k}-type. Consequently, $f_{t_k} - D_{t_k}(g)$ is also of D_{t_k}-type. By Lemma 4.3(i),

$$D_{t_k}(f) = D_{t_k}(g) + D_{t_k}\rho_{t_k}(f_{t_k} - D_{t_k}(g)) = D_{t_k}(g) + (f_{t_k} - D_{t_k}(g)) = f_{t_k}$$

and our result follows. □

Lemma 4.5. *We have $C(\tilde{W}) = 0$, where $C(\tilde{W})$ denotes the center of $\tilde{W}$.*

Proof. Let $D \in C(\tilde{W})$ and write $D = \sum_{k=1}^{s} f_kD_k$, where $f_k \in G$. For any $i \in S$,

$$[D, D_i] = \left[\sum_{k=1}^{s} f_kD_k, D_i\right] = -(-1)^{|f_kD_k|\tilde{i}}\sum_{k=1}^{s} D_i(f_k)D_k = 0.$$

This implies that $D_i(f_k) = 0$ for all $i \in S$.

Moreover, by Lemma 2.1, we see that $f_k \in G_0$ for all $k \in S$. For $j \in Y$ and $t \in Y_2$, $m \in Y_0$, one calculates

$$[D, x_jD_j + y_tD_m] = \left[\sum_{k=1}^{s} f_kD_k, x_jD_j + y_tD_m\right] = f_jD_j + f_ty_tD_m = 0.$$

It follows that $f_j = f_t = 0$ and $D = 0$. □

Lemma 4.6. *Let L be a centerless Lie superalgebra. Let $\varphi \in \mathrm{h}(\mathrm{der}\, L)$, $x \in L_{\bar{0}}$ and $x_1 \in L$. If there exists $k \geq 1$ such that $(\mathrm{ad}\, x)^{p^k} = \mathrm{ad}\, x_1$, then $\varphi(x_1) = (\mathrm{ad}\, x)^{p^k-1}\varphi(x)$.*

Proof. The proof is similar to that of [11, Lemma 8.1, p. 191]. □

Lemma 4.7. *Let $\varphi \in \mathrm{h}(\mathrm{der}_t\, \tilde{W})$, where $t \in J$ and $t \geq 0$. Then there exists $A \in \tilde{W}_t$ such that $\varphi(D_i) = \mathrm{ad}\, A(D_i)$ for all $i \in S$.*

Proof. Let $\varphi(D_i) = \sum_{k=1}^{s} f_{ki}D_k$, where $f_{ki} \in G$. This implies that $|\varphi| + \tilde{i} = |f_{ki}| + \tilde{k}$. Since $[D_i, D_j] = 0$ for any $j \in S$, we see that

$$\left[\sum_{k=1}^{s} f_{ki}D_k, D_j\right] + (-1)^{|\varphi|\tilde{i}}\left[D_i, \sum_{k=1}^{s} f_{kj}D_k\right] = 0.$$

https://doi.org/10.1017/S1446788711001558 Published online by Cambridge University Press

It follows that

$$\sum_{k=1}^{s}[(-1)^{|\varphi|\tilde{\imath}}D_i(f_{kj}) - (-1)^{(|f_{ki}|+\tilde{k})\tilde{\jmath}}D_j(f_{ki})]D_k = 0.$$

Since $|\varphi| + \tilde{\imath} = |f_{ki}| + \tilde{k}$, we see that

$$D_i((-1)^{|\varphi|\tilde{\jmath}} f_{kj}) = (-1)^{\tilde{\imath}\tilde{\jmath}} D_j((-1)^{|\varphi|\tilde{\imath}} f_{ki}). \tag{4.1}$$

For our purposes, it is enough to suppose that f_{ki} is of D_i-type. We treat the three possible cases separately.

Case 1. Suppose that $i \in Y_0$. Since $(\mathrm{ad}\, D_i)^{\pi_i+1} = 0$, we deduce from Lemma 4.6 that $(\mathrm{ad}\, D_i)^{\pi_i}(\varphi(D_i)) = 0$. This implies that $(\mathrm{ad}\, D_i)^{\pi_i}(\sum_{k=1}^{s} f_{ki}D_k) = 0$. It follows that $D_i^{\pi_i}(f_{ki}) = 0$.

Case 2. Suppose that $i \in Y_1$. Putting $j = i$ in (4.1) enables us to deduce that $D_i(f_{ki}) = 0$.

Case 3. Suppose that $i \in Y_2$. Since $\lambda_i^p = \lambda_i$, we see that

$$(\mathrm{ad}\, D_i)^p(x^{(\alpha)}x^u y^\lambda D_j) = \lambda_i^p x^{(\alpha)}x^u y^\lambda D_j = \lambda_i x^{(\alpha)}x^u y^\lambda D_j = \mathrm{ad}\, D_i(x^{(\alpha)}x^u y^\lambda D_j).$$

It follows that

$$(\mathrm{ad}\, D_i)^{p-1}(\varphi(D_i)) = \varphi(D_i)$$

by Lemma 4.6. Consequently,

$$(\mathrm{ad}\, D_i)^{p-1}\Big(\sum_{k=1}^{s} f_{ki}D_k\Big) = \sum_{k=1}^{s} D_i^{p-1}(f_{ki})D_k = \sum_{k=1}^{s} f_{ki}D_k.$$

This implies that $D_i^{p-1}(f_{ki}) = f_{ki}$. Hence, f_{ki} is of D_i-type for all $k, i \in S$.

Equation (4.1) shows that $\{(-1)^{|\varphi|\tilde{\imath}} f_{ki} \mid i \in S\}$ satisfies the conditions of Lemma 4.4. Thus, there exists $g_k \in G$ such that $D_i(g_k) = (-1)^{|\varphi|\tilde{\imath}} f_{ki}$. This implies that $\tilde{\imath} + |g_k| = |f_{ki}|$. Note that $|\varphi| = |g_k| + \tilde{k}$. Write

$$B := -\sum_{k=1}^{s} g_k D_k \in \tilde{W}.$$

One deduces that

$$[B, D_i] = \sum_{k=1}^{s}(-1)^{(|g_k|+\tilde{k})\tilde{\imath}}D_i(g_k)D_k = \sum_{k=1}^{s}(-1)^{|\varphi|\tilde{\imath}}D_i(g_k)D_k = \sum_{k=1}^{s} f_{ki}D_k = \varphi(D_i).$$

Since $\tilde{W}$ is Z-graded, we may suppose that $B = \sum_{l=-1}^{\xi} B_l$, where $B_l \in \tilde{W}_l$. It follows that $\varphi(D_i) = [B_t, D_i]$. Thereby, we find $A = B_t \in \tilde{W}_t$ such that $\varphi(D_i) = \mathrm{ad}\, A(D_i)$ for $i \in S$. □

https://doi.org/10.1017/S1446788711001558 Published online by Cambridge University Press

Write

$$\Theta := Q(m)^{m_2} = Q(m) \times Q(m) \times \cdots \times Q(m).$$

For

$$\theta = (h_{s+1}(y), h_{s+2}(y), \ldots, h_{s_1}(y)) \in \Theta,$$

define

$$\tilde{\theta} : H \longrightarrow Q(m), \quad \lambda \longmapsto \sum_{j=s+1}^{s_1} \lambda_j h_j(y).$$

For $\lambda, \eta \in H$, we are able to verify that

$$\tilde{\theta}(\lambda + \eta) = \tilde{\theta}(\lambda) + \tilde{\theta}(\eta).$$

For $\theta \in \Theta$, define a linear mapping $D_\theta : \tilde{W} \longrightarrow \tilde{W}$ such that

$$D_\theta(x^{(\alpha)} x^\mu y^\lambda D_i) = \tilde{\theta}(\lambda) x^{(\alpha)} x^\mu y^\lambda D_i \quad \forall i \in S.$$

LEMMA 4.8. *For any $\theta \in \Theta$, we have $D_\theta \in \mathrm{der}_{\bar{0}}(\tilde{W})$.*

PROOF. For $i \in Y_0$ and $k \in Y_2$, a direct computation shows that

$$[x^{(\alpha)} x^\mu y^\lambda D_i, x^{(\beta)} x^\nu y^\eta D_k] = x^{(\alpha)} x^\mu x^\nu y^\eta [y^\lambda D_i, x^{(\beta)} D_k].$$

Consequently,

$$\begin{aligned}
D_\theta&[x^{(\alpha)} x^\mu y^\lambda D_i, x^{(\beta)} x^\nu y^\eta D_k] \\
&= D_\theta(x^{(\alpha)} x^\mu x^\nu y^\eta y^\lambda D_i(x^{(\beta)}) D_k - x^{(\alpha)} x^\mu x^\nu y^\eta x^{(\beta)} D_k(y^\lambda) D_i) \\
&= D_\theta(x^{(\alpha)} x^\mu x^\nu y^{\lambda+\eta} D_i(x^{(\beta)}) D_k) - D_\theta(\lambda_k x^{(\alpha)} x^{(\beta)} x^\mu x^\nu y^{\lambda+\eta} D_i) \\
&= \tilde{\theta}(\lambda + \eta) x^{(\alpha)} x^\mu x^\nu y^\eta (y^\lambda D_i(x^{(\beta)}) D_k - x^{(\beta)} D_k(y^\lambda) D_i) \\
&= (\tilde{\theta}(\lambda) + \tilde{\theta}(\eta)) x^{(\alpha)} x^\mu x^\nu y^\eta [y^\lambda D_i, x^{(\beta)} D_k] \\
&= (\tilde{\theta}(\lambda) + \tilde{\theta}(\eta)) [x^{(\alpha)} x^\mu y^\lambda D_i, x^{(\beta)} x^\nu y^\eta D_k] \\
&= [D_\theta(x^{(\alpha)} x^\mu y^\lambda D_i), x^{(\beta)} x^\nu y^\eta D_k] + [x^{(\alpha)} x^\mu y^\lambda D_i, D_\theta(x^{(\beta)} x^\nu y^\eta D_k)].
\end{aligned}$$

Hence, we conclude that $D_\theta \in \mathrm{der}_{\bar{0}}(\tilde{W})$. The argument for the remaining cases is similar. □

LEMMA 4.9. *Let $\varphi \in \mathrm{h}(\mathrm{der}\, \tilde{W})$. If $\varphi(D_j) = 0$ for all $j \in S$, then there exists $\theta \in \Theta$ such that $\varphi(y^\lambda D_i) = D_\theta(y^\lambda D_i)$ for any $\lambda \in H$ and $i \in Y$.*

PROOF. We proceed in several steps.

Step 1. Let $\varphi(y^\lambda D_i) = \sum_{k=1}^{s} g_{ki\lambda} D_k$, where $g_{ki\lambda} \in G$. Since $[D_j, y^\lambda D_i] = 0$ for $j \in Y$, we see that

$$[\varphi(D_j), y^\lambda D_i] + (-1)^{|\varphi|\bar{j}} [D_j, \varphi(y^\lambda D_i)] = 0.$$

Consequently, it follows by our assumption that $\varphi(D_j) = 0$ that

$$[D_j, \varphi(y^\lambda D_i)] = \left[D_j, \sum_{k=1}^{s} g_{ki\lambda} D_k\right] = \sum_{k=1}^{s} D_j(g_{ki\lambda}) D_k = 0.$$

We now deduce from Lemma 2.1 that $g_{ki\lambda} \in G_0$ for all $k \in S$.

https://doi.org/10.1017/S1446788711001558 Published online by Cambridge University Press

Step 2. Let $\varphi(x_iD_i) = \sum_{k=1}^{s} a_kD_k$, where $a_k \in G$. Since $[D_i, x_iD_i] = D_i$, we see that

$$\Big[D_i, \sum_{k=1}^{s} a_kD_k\Big] = \sum_{k=1}^{s} D_i(a_k)D_k = 0.$$

This means that $a_k \in G_0$ by Lemma 2.1.

Since $[y^\lambda D_i, x_iD_i] = y^\lambda D_i$, we deduce that

$$\Big[\sum_{k=1}^{s} g_{ki\lambda}D_k, x_iD_i\Big] + (-1)^{|\varphi|\bar{\imath}}\Big[y^\lambda D_i, \sum_{k=1}^{s} a_kD_k\Big] = \sum_{k=1}^{s} g_{ki\lambda}D_k.$$

This implies that

$$g_{ii\lambda}D_i - \sum_{k\in Y_2}(-1)^{|\varphi|\bar{\imath}} a_kD_k(y^\lambda)D_i = \sum_{k=1}^{s} g_{ki\lambda}D_k.$$

It follows that $g_{ki\lambda} = 0$ for all $k \in S \setminus \{i\}$ and $\varphi(y^\lambda D_i) = g_{ii\lambda}D_i$. We abbreviate $g_{ii\lambda}$ to $g_{i\lambda}$. Set $h_{i\lambda}(y) = g_{i\lambda}y^{-\lambda}$. Then

$$\varphi(y^\lambda D_i) = g_{i\lambda}D_i = h_{i\lambda}(y)y^\lambda D_i.$$

Step 3. We claim that

$$h_{i\lambda}(y) + h_{j\eta}(y) = h_{j(\lambda+\eta)}(y)$$

for any $\lambda, \eta \in H$ and $i, j \in Y$.

Suppose that $\varphi(x_iy^\eta D_j) = \sum_{k=1}^{s} f_kD_k$, where $f_k \in G$. Since $[D_i, x_iy^\eta D_j] = y^\eta D_j$, we deduce that

$$(-1)^{|\varphi|\bar{\imath}}\Big[D_i, \sum_{k=1}^{s} f_kD_k\Big] = (-1)^{|\varphi|\bar{\imath}} \sum_{k=1}^{s} D_i(f_k)D_k = h_{j\eta}(y)y^\eta D_j.$$

This implies that $D_i(f_k) = 0$ for all $k \in S \setminus \{j\}$ and $D_i(f_j) = (-1)^{|\varphi|\bar{\imath}}h_{j\eta}(y)y^\eta$. Therefore, we may assume that $f_j = (-1)^{|\varphi|\bar{\imath}}h_{j\eta}(y)y^\eta x_i + g_j$, where $g_j \in G$ and $D_i(g_j) = 0$. Since $[y^\lambda D_i, x_iy^\eta D_j] = y^{\lambda+\eta}D_j$, we deduce that

$$\begin{aligned}
&[h_{i\lambda}(y)y^\lambda D_i, x_iy^\eta D_j] + (-1)^{|\varphi|\bar{\imath}}\Big[y^\lambda D_i, \sum_{k=1}^{s} f_kD_k\Big] \\
&\quad = [h_{i\lambda}(y)y^\lambda D_i, x_iy^\eta D_j] + (-1)^{|\varphi|\bar{\imath}}\Big[y^\lambda D_i, (-1)^{|\varphi|\bar{\imath}}h_{j\eta}(y)y^\eta x_iD_j + g_jD_j + \sum_{k\neq j} f_kD_k\Big] \\
&\quad = h_{i\lambda}(y)y^{\lambda+\eta}D_j + h_{j\eta}(y)y^{\lambda+\eta}D_j - \sum_{k\in Y_2}(-1)^{(|\varphi|+|f_kD_k|)\bar{\imath}} f_kD_k(y^\lambda)D_i \\
&\quad = h_{j(\lambda+\eta)}(y)y^{(\lambda+\eta)}D_j.
\end{aligned}$$

https://doi.org/10.1017/S1446788711001558 Published online by Cambridge University Press

In the following, we consider the two cases where $i \neq j$ and $i = j$ separately. If $i \neq j$, then the assertion is obvious. Moreover, we deduce that

$$\sum_{k \in Y_2} (-1)^{(|\varphi|+|f_k D_k|)\tilde{\iota}} f_k D_k(y^\lambda) = 0.$$

Hence, if $i = j$, then the equality $h_{i\lambda}(y) + h_{j\eta}(y) = h_{j(\lambda+\eta)}(y)$ also holds. We have established our claim.

Step 4. Since λ, η, i, j have been chosen randomly,

$$h_{i\lambda}(y) + h_{j\lambda}(y) = h_{j(2\lambda)}(y) = h_{j\lambda}(y) + h_{j\lambda}(y).$$

We deduce that $h_{i\lambda}(y) = h_{j\lambda}(y)$. We write $h_{i\lambda}(y)$ for $h_\lambda(y)$ for any $i \in Y$. Then $\varphi(y^\lambda D_i) = h_\lambda(y) y^\lambda D_i$. By Step 3, $h_\lambda(y) + h_\eta(y) = h_{\lambda+\eta}(y)$. In particular,

$$h_{\bar{\varepsilon}_k}(y) + h_{\bar{\varepsilon}_k}(y) = h_{2\bar{\varepsilon}_k}(y) = 2h_{\bar{\varepsilon}_k}(y), h_{2\bar{\varepsilon}_k}(y) + h_{\bar{\varepsilon}_k}(y) = h_{3\bar{\varepsilon}_k}(y) = 3h_{\bar{\varepsilon}_k}(y).$$

Moreover, we see that $h_{c\bar{\varepsilon}_k}(y) = ch_{\bar{\varepsilon}_k}(y)$ for any $c \in \Pi$ and $k = r+1, \ldots, s_1$. We abbreviate $h_{\bar{\varepsilon}_k}(y)$ by $h_k(y)$.

Step 5. We now complete the proof. Set

$$H_1 = \{\lambda \in H \mid \lambda_k = 0 \ \forall k = s+1, s+2, \ldots, s_1\}$$

and

$$H_2 = \{\lambda \in H \mid \lambda_k = 0 \ \forall k \in Y_2\}.$$

For any $\lambda \in H$, we can find $\lambda' \in H_1$ and $\lambda'' \in H_2$ such that $\lambda = \lambda' + \lambda''$.

Suppose that λ_t is the first number of λ'' which is not equal to 0, where t is one of $s+1, \ldots, s_1$. Then

$$\begin{aligned} h_\lambda(y) &= h_{\lambda'+\lambda''}(y) = h_{\lambda'}(y) + h_{\lambda''}(y) \\ &= h_{\lambda'}(y) + h_{\lambda_t\bar{\varepsilon}_t+\cdots+\lambda_{s_1}\bar{\varepsilon}_{s_1}}(y) \\ &= h_{\lambda'}(y) + \lambda_t h_t(y) + \cdots + \lambda_{s_1} h_{s_1}(y) \\ &= \lambda_{s+1} h_{s+1}(y) + \cdots + \lambda_t(\lambda_t^{-1} h_{\lambda'}(y) + h_t(y)) + \cdots + \lambda_{s_1} h_{s_1}(y). \end{aligned}$$

Set

$$\theta = (h_{s+1}(y), \ldots, \lambda_t^{-1} h_{\lambda'}(y) + h_t(y), h_{t+1}(y), \ldots, h_{s_1}(y)).$$

Then $\theta \in \Theta$ and

$$\varphi(y^\lambda D_i) = h_\lambda(y) y^\lambda D_i = \tilde{\theta}(\lambda) y^\lambda D_i = D_\theta(y^\lambda D_i).$$

This completes the proof. □

LEMMA 4.10. *Let $A \in \tilde{W}$. If $[D_i, A] = [y_j D_t, A] = 0$ for all $i \in Y$, $t \in Y_1$ and $j \in Y_2$, then $A \in \tilde{W}_{-1}$.*

https://doi.org/10.1017/S1446788711001558 Published online by Cambridge University Press

PROOF. Suppose that $A = \sum_{k=1}^{s} f_k D_k$, where $f_k \in G$. Then

$$[D_i, A] = \left[D_i, \sum_{k=1}^{s} f_k D_k\right] = \sum_{k=1}^{s} D_i(f_k) D_k = 0$$

and we conclude that $D_i(f_k) = 0$. By Lemma 2.1, this shows that $f_k \in G_0$ for all $k \in S$. Since $[y_j D_t, A] = [y_j D_t, \sum_{k=1}^{s} f_k D_k] = 0$, it follows that $f_j y_j D_t = 0$. This shows that $f_j = 0$ for all $j \in Y_2$, whence $A = \sum_{k=1}^{r} f_k D_k \in \tilde{W}_{-1}$. □

LEMMA 4.11. *Let $\varphi \in \mathrm{h}(\mathrm{der}_t \tilde{W})$, where $t \in J$. Suppose that $k \geq -1$ and $\varphi(\tilde{W}_j) = 0$, where $j = -1, 0, \ldots, k$. If $k + t \geq -1$, then $\varphi = 0$.*

PROOF. We let $l \geq k$ and show that $\varphi(\tilde{W}_l) = 0$ by induction on l. By our assumption that $\varphi(\tilde{W}_j) = 0$, it will then follow that $\varphi(\tilde{W}_k) = 0$.

Suppose that $l > k$ and $\varphi(\tilde{W}_{l-1}) = 0$. Lemma 4.10 allows us to deduce that

$$\varphi(A) \in \tilde{W}_{-1} \cap \tilde{W}_{l+t} = 0,$$

since $[D_i, A] \in \tilde{W}_{l-1}$ for any $A \in \tilde{W}_l$ and $i \in Y$ and $[y_h D_\nu, A] \in \tilde{W}_{l-1}$ for any $h \in Y_2$ and $\nu \in Y_1$, while $\varphi(D_i) = \varphi(y_h D_\nu) = 0$. Hence, $\varphi(\tilde{W}_l) = 0$ and we may conclude that $\varphi = 0$. □

PROPOSITION 4.12. *Let $\varphi \in \mathrm{h}(\mathrm{der}_t \tilde{W})$, where $t \in J$ and $t \geq 0$. Then there exist $A \in \tilde{W}_t$ and $\theta \in \Theta$ such that $\varphi = \mathrm{ad}\, A + D_\theta$.*

PROOF. By Lemma 4.7, there exists $A \in \tilde{W}_t$ such that $\varphi(D_i) = \mathrm{ad}\, A(D_i)$ for all $i \in S$. Thus, we may find $\theta \in \Theta$ such that $(\varphi - \mathrm{ad}\, A - D_\theta)(y^\lambda D_j) = 0$ for any $\lambda \in H$ and $j \in Y$ by Lemma 4.9. This allows us to deduce that $(\varphi - \mathrm{ad}\, A - D_\theta)(\tilde{W}_{-1}) = 0$ and $\varphi = \mathrm{ad}\, A + D_\theta$ by Lemma 4.11. □

REMARK 4.13. It is possible to add the following conclusions to Proposition 4.12. If $\varphi \in (\mathrm{der}_0 \tilde{W})_{\bar{0}}$, then there exist $A \in \tilde{W}_t$ and $\theta \in \Theta$ such that $\varphi = \mathrm{ad}\, A + D_\theta$. Otherwise there exists $A \in \tilde{W}_t$ such that $\varphi = \mathrm{ad}\, A$.

PROPOSITION 4.14. *Let $\Omega = \{D_\theta \mid \theta \in \Theta\}$. Then the following statements hold.*

(i) *The space Ω is a subspace of* $\mathrm{der}\, \tilde{W}$.
(ii) *The intersection* $\mathrm{ad}\, \tilde{W} \cap \Omega = \{0\}$.

PROOF. We first prove (i). Since $Q(m)$ is a linear space over F, we see that $\Theta = Q(m)^{m_2}$ is also a linear space over F. Suppose that

$$\theta = (h_{s+1}(y), \ldots, h_{s_1}(y)), \qquad \eta = (g_{s+1}(y), \ldots, g_{s_1}(y))$$

for any $\theta, \eta \in \Theta$. Then

$$\theta + \eta = (h_{s+1}(y) + g_{s+1}(y), \ldots, h_{s_1}(y) + g_{s_1}(y)).$$

https://doi.org/10.1017/S1446788711001558 Published online by Cambridge University Press

For $\lambda \in H$,

$$\begin{aligned}\tilde{\theta}(\lambda) + \tilde{\eta}(\lambda) &= \sum_{j=s+1}^{s_1} \lambda_j h_j(y) + \sum_{j=s+1}^{s_1} \lambda_j g_j(y) \\ &= \sum_{j=s+1}^{s_1} \lambda_j (h_j(y) + g_j(y)) = (\theta + \eta)\tilde{}(\lambda).\end{aligned}$$

We deduce that

$$\begin{aligned}(D_\theta + D_\eta)(x^{(\alpha)} x^\mu y^\lambda D_i) &= \tilde{\theta}(\lambda) x^{(\alpha)} x^\mu y^\lambda D_i + \tilde{\eta}(\lambda) x^{(\alpha)} x^\mu y^\lambda D_i \\ &= (\theta + \eta)\tilde{}(\lambda) x^{(\alpha)} x^\mu y^\lambda D_i \\ &= D_{\theta+\eta}(x^{(\alpha)} x^\mu y^\lambda D_i)\end{aligned}$$

and we conclude that $D_\theta + D_\eta = D_{\theta+\eta} \in \Omega$. Similarly, $kD_\theta = D_{k\theta} \in \Omega$ for any $k \in F$. Thus, Ω is a subspace of der $\tilde{W}$.

To prove (ii), let X be an arbitrary element of ad $\tilde{W} \cap \Omega$. Then there exist $B = \sum_{k=1}^{s} f_k D_k \in \tilde{W}$ and $\theta \in \Theta$ such that $X = \mathrm{ad}\, B = D_\theta$. Consequently,

$$\mathrm{ad}\, B(D_j) = \Big[\sum_{k=1}^{s} f_k D_k, D_j\Big] = \sum_{k=1}^{s} (-1)^{\bar{j}|f_k D_k|} D_j(f_k) D_k = D_\theta(D_j) = 0$$

for all $j \in Y$. Lemma 2.1 shows that $f_k \in G_0$ for all $k \in S$. Since $B \in \tilde{W}_{\bar{0}}$ by Lemma 4.8, we may assume that

$$B = \sum_{k=1}^{n} f_k D_k + \sum_{k'=r+1}^{s} f_{k'} D_{k'}.$$

Thus,

$$\mathrm{ad}\, B(x_i D_i + y_t D_j) = f_i D_i + f_t y_t D_j = D_\theta(x_i D_i + y_t D_j) = 0$$

for any $i \in Y_0$, $j \in Y_1$, $t \in Y_2$. This implies that $f_i = f_t = 0$, whence $X = \mathrm{ad}\, B = 0$. The proof is now complete. □

Proposition 4.15. *We have the equality of sets* $\mathrm{der}_{-1} \tilde{W} = \mathrm{ad}\, \tilde{W}_{-1}$.

Proof. Let $\varphi \in \mathrm{h}(\mathrm{der}_{-1} \tilde{W})$. We see that

$$\tilde{W}_0 = \mathrm{span}_F\{x_i D_j, x_i D_i, x_i y^\lambda D_j, x_i y^\lambda D_i, y^\lambda D_l \mid \lambda \in H, i, j \in Y, i \neq j, l \in Y_2\}.$$

Clearly, ad $\tilde{W}_{-1} \subseteq \mathrm{der}_{-1} \tilde{W}$. It remains to show that ad $\tilde{W}_{-1} \supseteq \mathrm{der}_{-1} \tilde{W}$. We proceed in several steps.

Step 1. Let $\varphi(x_i D_j) = \sum_{k=1}^{r} a_k D_k$ and $\varphi(x_h D_l) = \sum_{k=1}^{r} b_k D_k$ for any $h, l \in Y \setminus \{i, j\}$, where $a_k, b_k \in G_0$. Since $[x_i D_j, x_h D_l] = 0$, we see that

$$\Big[\sum_{k=1}^{r} a_k D_k, x_h D_l\Big] + (-1)^{|\varphi|(\bar{i}+\bar{j})}\Big[x_i D_j, \sum_{k=1}^{r} b_k D_k\Big] = 0.$$

https://doi.org/10.1017/S1446788711001558 Published online by Cambridge University Press

It follows that $a_h D_l - (-1)^{(\bar{i}+\bar{j})(|\varphi|+\bar{i})} b_i D_j = 0$. This means that $a_h = 0$ for every $h \in Y \setminus \{i, j\}$. Hence, $\varphi(x_i D_j) = a_i D_i + a_j D_j$.

Moreover, we may suppose that $\varphi(x_i D_h) = c_i D_i + c_h D_h$ and $\varphi(x_h D_j) = d_h D_h + d_j D_j$, where $c_i, c_h, d_h, d_j \in G_0$. Since $[x_i D_h, x_h D_j] = x_i D_j$, we see that

$$[c_i D_i + c_h D_h, x_h D_j] + (-1)^{|\varphi|(\bar{i}+\bar{h})}[x_i D_h, d_h D_h + d_j D_j] = a_i D_i + a_j D_j.$$

It follows that $a_i = 0$ and $\varphi(x_i D_j) = a_j D_j$.

In particular, suppose that $\varphi(x_i D_{i+1}) = h_i D_{i+1}$ for $i = 1, \ldots, r-1$ and $\varphi(x_r D_1) = h_r D_1$, where $h_k \in G_0$ for $k = 1, \ldots, r$. Let $\psi = \varphi - \sum_{k=1}^{r} \operatorname{ad}(h_k D_k)$. Then

$$\psi(x_i D_{i+1}) = \varphi(x_i D_{i+1}) - \sum_{k=1}^{r} \operatorname{ad}(h_k D_k)(x_i D_{i+1}) = h_i D_{i+1} - h_i D_{i+1} = 0$$

and $\psi(x_r D_1) = 0$. In the following steps, we shall prove that $\psi(\tilde{W}_0) = 0$.

Step 2. We claim that $\psi(x_i D_j) = 0$. Indeed, if $i < j$, then by Step 1 we have

$$\psi(x_i D_{i+2}) = \psi([x_i D_{i+1}, x_{i+1} D_{i+2}]) = 0$$

and it follows that $\psi(x_i D_j) = 0$. If $i > j$, then

$$\psi(x_{r-1} D_1) = \psi([x_{r-1} D_r, x_r D_1]) = 0.$$

It follows that $\psi(x_i D_1) = 0$. Consequently, $\psi(x_i D_2) = \psi([x_i D_1, x_1 D_2]) = 0$ and it follows that $\psi(x_i D_j) = 0$, establishing our claim.

Step 3. We claim that $\psi(x_i D_i) = 0$. Suppose that $\psi(x_i D_i) = \sum_{k=1}^{r} e_k D_k$, where $e_k \in G_0$. Since $[x_i D_i, x_j D_{j+1}] = 0$ for any $j \in Y \setminus \{i-1, i, r\}$, we see that

$$\left[\sum_{k=1}^{r} e_k D_k, x_j D_{j+1}\right] = e_j D_{j+1} = 0.$$

This implies that $e_j = 0$. It follows that

$$\psi(x_i D_i) = e_{i-1} D_{i-1} + e_i D_i + e_r D_r.$$

Let $i \in Y \setminus \{1, r\}$. By applying ψ to

$$[x_i D_i, x_i D_{i+1}] = x_i D_{i+1}, \quad [x_i D_i, x_{i-1} D_i] = -x_{i-1} D_i, \quad [x_i D_i, x_r D_1] = 0,$$

we deduce that $e_i = e_{i-1} = e_r = 0$. Hence, $\psi(x_i D_i) = 0$ for any $i \in Y \setminus \{i, r\}$. We can similarly verify that $\psi(x_1 D_1) = \psi(x_r D_r) = 0$ and we have established our claim.

Step 4. We claim that $\psi(x_i y^\lambda D_j) = 0$. Suppose that $\psi(x_i y^\lambda D_j) = \sum_{k=1}^r f_k D_k$, where $f_k \in G_0$. Now Steps 2 and 3 imply that $\psi(x_h D_l) = 0$. Since also $[x_i y^\lambda D_j, x_h D_l] = 0$ for $h, l \in Y$ with $h \neq j$ and $l \neq i$, we deduce that

$$\Big[\sum_{k=1}^{r} f_k D_k, x_h D_l\Big] = f_h D_l = 0.$$

It follows that $f_h = 0$ and $\psi(x_i y^\lambda D_j) = f_j D_j$. Since $[x_i D_i, x_i y^\lambda D_j] = x_i y^\lambda D_j$, we see that $0 = [x_i D_i, f_j D_j] = f_j D_j$ by Step 3. It follows that $f_j = 0$ and $\psi(x_i y^\lambda D_j) = 0$, establishing our claim.

Step 5. We claim that $\psi(x_i y^\lambda D_i) = 0$. Suppose that $\psi(x_i y^\lambda D_i) = \sum_{k=1}^r g_k D_k$, where $g_k \in G_0$. Since $[x_i y^\lambda D_i, x_j D_j] = 0$ for any $j \in Y \setminus \{i\}$, we see that

$$\Big[\sum_{k=1}^{r} g_k D_k, x_j D_j\Big] = g_j D_j = 0.$$

It follows that $g_j = 0$ and $\psi(x_i y^\lambda D_i) = g_i D_i$. Since

$$[x_i y^\lambda D_i, x_i D_j] = x_i y^\lambda D_j,$$

we deduce that

$$[g_i D_i, x_i D_j] = g_i D_j = 0.$$

It follows that $g_i = 0$ and $\psi(x_i y^\lambda D_i) = 0$, establishing our claim.

Step 6. To complete the proof, we first show that $\psi(y^\lambda D_l) = 0$. Let $\psi(y^\lambda D_l) = \sum_{k=1}^r a'_k D_k$, where $a'_k \in G_0$. Since $\psi(x_i y^{-\lambda} D_i) = 0$ for any $i \in Y$ by Step 5, we may apply ψ to

$$[y^\lambda D_l, x_i y^{-\lambda} D_i] = -\lambda_l x_i D_i$$

to deduce that $[\sum_{k=1}^r a'_k D_k, x_i y^{-\lambda} D_i] = 0$. It follows that $a'_i = 0$ and $\psi(y^\lambda D_l) = 0$. From the discussion above, we conclude that $\psi(\tilde{W}_0) = 0$. Thus, $\psi = 0$ by Lemma 4.11 and $\mathrm{der}_{-1}\, \tilde{W} = \mathrm{ad}\, \tilde{W}_{-1}$. □

We can use a similar method to that used to prove [15, Propositions 3 and 4] to deduce the following proposition.

PROPOSITION 4.16. *Let $t \in J$ and $t > 1$. If there is no $k \in N$ such that $t = p^k$, then $\mathrm{der}_{-t}\, \tilde{W} = 0$. If there exists $k \in N$ such that $t = p^k$, then*

$$\mathrm{der}_{-t}\, \tilde{W} = \mathrm{Span}_{G_0}\{\mathrm{ad}\, D_i^t \mid i \in Y_0\}.$$

THEOREM 4.17. *We have the equality*

$$\mathrm{der}\, \tilde{W} = \mathrm{ad}\, \tilde{W} \oplus \Omega \oplus \mathrm{Span}_{G_0}\{(\mathrm{ad}\, D_i)^{p^{k_i}} \mid i \in Y_0,\ 1 \le k_i < t_i\}.$$

PROOF. This is a direct consequence of Propositions 4.12, 4.14, 4.15 and 4.16. □

https://doi.org/10.1017/S1446788711001558 Published online by Cambridge University Press

Acknowledgement

The authors thank the referee for helpful suggestions.

References

[1] M. J. Celousov, 'Derivations of Lie algebras of Cartan-type', *Izv. Vyssh. Uchebn. Zaved. Mat.* **98** (1970), 126–134 (in Russian).
[2] R. Farnsteiner, 'Note on Frobenius extensions and restricted Lie superalgebras', *J. Pure Appl. Algebra* **108** (1996), 241–256.
[3] J.-Y. Fu, Q.-C. Zhang and C.-B. Jiang, 'The Cartan type modular Lie superalgebras KO', *Comm. Algebra* **34** (2006), 129–142.
[4] V. G. Kac, 'Lie superalgebras', *Adv. Math.* **98** (1977), 8–96.
[5] V. G. Kac, 'Classification of infinite-dimensional simple linearly compact Lie superalgebras', *Adv. Math.* **139** (1998), 1–55.
[6] Yu. Kochetkov and D. Leites, 'Simple Lie algebras in characteristic 2 recovered from superalgebras and on the notion of a simple finite group', *Contemp. Math.* **131** (1992), 59–67.
[7] W.-D. Liu, Y.-Z. Zhang and X.-L. Wang, 'The derivation algebra of the Cartan type Lie superalgebras HO', *J. Algebra* **273** (2004), 176–205.
[8] F.-M. Ma and Q.-C. Zhang, 'Derivation algebra of modular Lie superalgebra K of Cartan type', *J. Math. (Wuhan)* **20** (2000), 431–435.
[9] V. M. Petrogradski, 'Identities in the enveloping algebras for modular Lie superalgebras', *J. Algebra* **145** (1992), 1–21.
[10] M. Scheunert, *Theory of Lie Superalgebras*, Lecture Notes in Mathematics, 716 (Springer, New York, 1979).
[11] H. Strade and R. Farnsteiner, *Modular Lie Algebras and Their Representations*, Monographs and Textbooks in Pure and Applied Mathematics, 116 (Marcel Dekker, New York, 1988).
[12] Y. Wang and Y.-Z. Zhang, 'Derivation algebra Der(H) and central extensions of Lie superalgebra', *Comm. Algebra* **32** (2004), 4117–4131.
[13] Y.-Z. Zhang, 'Finite-dimensional Lie superalgebras of Cartan-type over fields of prime characteristic', *Chinese Sci. Bull.* **42** (1997), 720–724.
[14] Y.-Z. Zhang and W.-D. Liu, *Modular Lie Superalgebras* (Science Press, Beijing, 2004), (in Chinese).
[15] Q.-C. Zhang and Y.-Z. Zhang, 'Derivation algebra of modular Lie superalgebras W and S of Cartan type', *Acta Math. Sci.* **20** (2000), 137–144.

YAN-QIN DONG, School of Mathematics and Statistics,
Northeast Normal University, Changchun 130024, Jilin, PR China
and
Aviation University of Air Force, Changchun 130022, Jilin, PR China
e-mail: dongyq384@nenu.edu.cn

YONG-ZHENG ZHANG, School of Mathematics and Statistics,
Northeast Normal University, Changchun 130024, Jilin, PR China
e-mail: zhyz@nenu.edu.cn

ANGELO EBONZO, Transportation Management College,
Dalian Maritime University, Dalian 116026, PR China
e-mail: angedan2000@yahoo.fr