CENTRAL SIDON AND CENTRAL Λ_p SETS

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Introduction

Central Sidon sets and central Λ_p sets are defined and equivalent characterizations are given. It is shown that a central Sidon set with an upper bound on the degrees of its elements is a Λ_p set $(1 < p < \infty)$. The bound on the degrees is shown to be necessary by an example.

Two sufficient conditions are given which insure that a set is central Sidon. The first uses Riesz polynomials and provides a method for constructing infinite central Sidon sets under appropriate conditions. The second generalizes the notion of independent sets and shows that the set of nontrivial projections in the product of subgroups of unitary groups is a central Sidon set.

1. Preliminaries

We assume the notation, definitions and theorems of [1] and [2]. Throughout G will be a compact group with dual object Σ. For $\sigma \in \Sigma$, U^σ will be a continuous irreducible unitary representation of G with degree d_σ and trace χ_σ. I_σ will denote the identity operator on the Hilbert space upon which U^σ acts.

If A is one of the algebras $\boldsymbol{M}(G)$, $\boldsymbol{L}_p(G)$ $(1 \leqq p \leqq \infty)$, $\boldsymbol{C}(G)$, $\boldsymbol{T}(G)$, or $\boldsymbol{K}(G)$, then A^z denotes the center of this algebra under convolution multiplication. If A is one of the $\mathfrak{E}_p(\Sigma)$ algebras [2, (28.34)], A^z is the center of this algebra under composition multiplication.

If $f \in \boldsymbol{L}_1^z(G)$, the Fourier transform of f is given by

$$\hat{f}(\sigma) = d_\sigma^{-1} a_\sigma I_\sigma$$

and the Fourier series of f is

$$f \sim \sum_{\sigma \in \Sigma} a_\sigma \chi_\sigma$$

where $a_\sigma = \int_G f \overline{\chi_\sigma}\, d\lambda$. Such an f has absolutely convergent Fourier series ($f \in \boldsymbol{K}(G)$) if

$$\| \hat{f} \|_1 \equiv \sum_{\sigma \in \Sigma} d_\sigma |a_\sigma| < \infty .$$

LEMMA 1.1. *Let G be a compact group and A be one of the algebras $L_p(G)$ $(1 \leqq p < \infty)$ or $C(G)$. Then $A^{z*} = A^{*z}$.*

PROOF. To simplify notation we will at times consider $L_p(G)$ $(1 \leqq p \leqq \infty)$ as a subset of $M(G)$; i.e., $\mu \in L_p(G)$ if $d\mu = g d\lambda$ where $g \in L_p(G)$ and λ is Haar measure on G.

Let A be as in the statement of the theorem and let $\mu \in A^{*z}$. Then μ restricted to A^z remains a bounded linear functional. To show $A^{*z} \subset A^{z*}$ we need only show the restriction mapping is one-to-one. It suffices to show that if $\mu \in A^{*z}$ and $\mu \not\equiv 0$, then $\mu | A^z \not\equiv 0$. This, however, follows from the fact that $\mu \to \hat{\mu}$ is an isomorphism and $\operatorname{tr} \hat{\mu}(\sigma) = \int_G \overline{\chi_\sigma} d\mu$.

Now suppose $S \in A^{z*}$. By the Hahn-Banach theorem, S can be extended to a linear functional, S', on A without increasing the norm. For $f \in A$, let $f_a^a(x) = f(a^{-1}xa)$. Since $a \to f_a^a$ is continuous, $a \to S'(f_a^a)$ is a continuous function on G. Set

$$M(f) = \int_G S'(f_a^a) d\lambda(a).$$

Then M is a linear functional on A, $M(f) = S(f)$ for all $f \in A^z$, and $\| M \| = \| S \| = \| S' \|$. Hence, there is a measure $\nu \in A^*$ such that $M(f) = \int_G f d\nu$ for all f in A. Since $\nu * f = f * \nu$ for each $f \in C(G)$, $\nu \in A^{*z}$. It follows that the mapping $\mu \to \mu | A^z$ carries A^{*z} onto A^{z*}. An examination of the details shows that this mapping is norm preserving and the lemma is proved.

2. Central Sidon sets and central Λ_p sets

In this section central Sidon sets and central Λ_p sets are defined and equivalent characterizations of these sets are given. Theorems in this section should be compared with those in [2, §37]. In particular the proofs of theorems 2.1, 2.2 and 2.3 are for the most part a carry over into our setting of the proofs given by Hewitt and Ross [2, (37.2), (37.7) and (37.9)]. For this reason, instead of giving complete proofs for these theorems, we will only indicate the ways in which our proofs differ from the proofs given in [2].

For $P \subset \Sigma$, $C_P^z(G)$ will denote the set of elements in the center of $C(G)$ (under convolution as multiplication) whose Fourier transforms are 0 off P. Call P a *central Sidon set* if

$$C_P^z(G) \subset K(G).$$

It is immediate that every Sidon set in Σ is a central Sidon set [2, (37.1)].

THEOREM 2.1. *Let $P \subset \Sigma$. The following assertions are equivalent:*

(i) *P is a central Sidon set;*

(ii) *given E in* $\mathfrak{E}^z_\infty(P)$, *there is a measure* $\mu \in \boldsymbol{M}^z(G)$ *such that* $\hat{\mu}(\sigma) = E_\sigma$ *for all* $\sigma \in P$;

(iii) *given E in* $\mathfrak{E}^z_0(P)$, *there is a function* $f \in \boldsymbol{L}^z_1(G)$ *such that* $\hat{f}(\sigma) = E_\sigma$ *for all* $\sigma \in P$;

(iv) $\boldsymbol{L}^z_{\infty P}(G) \subset \boldsymbol{K}(G)$;

(v) *there exists a constant k such that* $\|\hat{f}\|_1 \leqq k\|f\|_\infty$ *for all* $f \in \boldsymbol{L}^z_{\infty P}(G)$;

(vi) *there exists a constant k such that* $\|\hat{f}\|_1 \leqq k\|f\|_u$ *for all f in* $\boldsymbol{C}^z_P(G)$;

(vii) *there exists a constant k such that* $\|f\|_1 \leqq k\|f\|_u$ *for all f in* $\boldsymbol{T}^z_P(G)$;

(viii) *for each* $W \in \Pi_{\sigma \in P}\varepsilon_\sigma I_\sigma$, $\varepsilon_\sigma \in \{-1, 1\}$, *there is a measure* $\mu \in \boldsymbol{M}^z(G)$ *such that*

$$\sup\{\|W_\sigma - \hat{\mu}(\sigma)\|_{\phi_\infty} : \sigma \in P\} < 1;$$

(ix) *for each* $W \in \Pi_{\sigma \in P}\varepsilon_\sigma I_\sigma$, $\varepsilon_\sigma \in \{-1, 1\}$, *there is a measure* $\mu \in \boldsymbol{M}(G)$ *such that*

$$\sup\{\|W_\sigma - \hat{\mu}(\sigma)\|_{\phi_\infty} : \sigma \in P\} < 1.$$

PROOF. The proof of our theorem is easily obtained by modifying the proof of equivalent properties for Sidon sets [2, (37.2)]. This is done by replacing in each instance (except in (ix) implies (i)) all functions and algebras by central functions and central algebras. We mention several steps at which this modification may not be transparent. In the proof that (vii) implies (iv) we note that the approximate identity used is contained in $\boldsymbol{T}^z(G)$. In the proof that (i) implies (ii) we use the fact established in Lemma 1.1 that the dual of $\boldsymbol{C}^z(G)$ is $\boldsymbol{M}^z(G)$. Finally, in the proof that (ii) implies (iii) we note that by the factorization theorem [2, (32.22)],

$$\boldsymbol{L}^z_1(G)^\wedge \circ \mathfrak{E}^z_0(\Sigma) = \mathfrak{E}^z_0(\Sigma).$$

REMARK. Dunkl and Ramirez [3] have shown that the dual object of an infinite compact group is not a central Sidon set.

We next define and obtain equivalent properties for central Λ_p sets. Let P be a subset of Σ, and let $1 < p \leqq \infty$. Then P is said to be of *type central* Λ_p or a *central* Λ_p *set* if every function in $\boldsymbol{L}^z_{1P}(G)$ belongs to $\boldsymbol{L}^z_{pP}(G)$, i.e., if

$$\boldsymbol{L}^z_{1P}(G) = \boldsymbol{L}^z_{pP}(G).$$

Note that every Λ_p set is also a central Λ_p set [2, (37.6)].

THEOREM 2.2. *Suppose that* $P \subset \Sigma$ and $1 < q < p < \infty$. *The following assertions are equivalent:*

(i) *P is of type central* Λ_p;

(ii) $\boldsymbol{L}^z_{pP}(G) = \boldsymbol{L}^z_{qP}(G)$;

(iii) *there is a constant k such that* $\|f\|_p \leqq k\|f\|_q$ *for all* $f \in \boldsymbol{T}^z_P(G)$;

(iv) *there is a constant k such that* $\|f\|_p \leqq k\|f\|_1$ *for all* $f \in \boldsymbol{T}^z_P(G)$;

(v) $\boldsymbol{M}^z_P(G) = \boldsymbol{L}^z_{pP}(G)$.

PROOF. Again the proof follows easily from the proof for equivalent properties for Λ_p sets [2, (37.7)]. We need only mention that in the proof that (iv) implies (v) we need the fact that for $f \in L_p^z(G)$,

$$\|f\|_p = \sup\{\|f * h\|_p : h \in T^z(G), \|h\|_1 \leqq 1\}; \tag{2.2.1}$$

see [4, (35.11)].

REMARK. If $1 < q < p \leqq \infty$ and P is a central Λ_p set, then P is also a central Λ_q set. This is obvious from the definition and the inclusion

$$L_p^z(G) \subset L_q^z(G).$$

For $1 < p < \infty$, p' denotes the number for which $1/p + 1/p' = 1$.

THEOREM 2.3. *Suppose that $P \subset \Sigma$ and $1 < p < \infty$. The following assertions are equivalent*:

(i) *P is of type central Λ_p*;

(ii) *for each $g \in L_{p'}^z(G)$, there is an $h \in L_\infty^z(G)$ such that $\hat{g}(\sigma) = \hat{h}(\sigma)$ for all $\sigma \in P$*;

(iii) *for each $g \in L_{p'}^z(G)$, there is an $h \in C^z(G)$ such that $\hat{g}(\sigma) = \hat{h}(\sigma)$ for all $\sigma \in P$.*

If $p > 2$, the above are equivalent to:

(iv) *for $g \in L_{p'}^z(G)$, we have*

$$\sum_{\sigma \in P} d_\sigma \|\hat{g}(\sigma)\|_{\phi_2}^2 < \infty.$$

PROOF. Again the proof follows easily from the proof of equivalent properties for Λ_p sets [2, (37.9)]. In (i) implies (ii) we need the fact that the dual of $L_p^z(G)$ is $L_{p'}^z(G)$; see Lemma 1.1. In (ii) implies (iii) we apply the factorization theorem [2, (32.22)] to obtain

$$L_{p'}^z(G) = L_1^z(G) * L_{p'}^z(G).$$

Finally to obtain (iv) implies (i), we use formula (2.2.1).

We now indicate a relationship between central Sidon sets and central Λ_p sets.

THEOREM 2.4. *Let G be a compact group with dual object Σ and let P be a central Sidon subset of Σ such that $N = \sup\{d_\sigma : \sigma \in P\} < \infty$. Then P is a Λ_p set and hence a central Λ_p set for all p in $]1, \infty[$.*

REMARK. The corresponding theorems for Sidon and Λ_p sets holds even if $N = \infty$. It is due to Figà-Talamanca and Rider [4, Corollary 9]; an alternative proof is given by Hewitt and Ross [2, (37.10)]. In section 4 below we give an example which shows that the hypothesis $N < \infty$ is necessary. It should be noted that our conclusion states that P is a Λ_p set and not simply a central Λ_p set.

PROOF. Let s be an arbitrary integer such that $s > \max\{2, 1/2p\}$. By the properties of Λ_p sets [2, (37.7) and (37.8)], it suffices to show that there is a constant k' (depending on s) such that

(2.4.1) $$\|f\|_{2s} \leqq k'\|f\|_2 \text{ for all } f \in \boldsymbol{T}_P(G).$$

The mapping $\mu \to (\hat{\mu}(\sigma))_{\sigma \in P}$ is plainly a bounded linear mapping of $M^z(G)$ into $\mathfrak{E}_\infty(P)$. Since P is a central Sidon set, (2.1.ii) shows this mapping carries $M^z(G)$ onto $\mathfrak{E}^z_\infty(P)$. By a corollary to the open mapping theorem [2, (E.2)], there is a constant k such that for each $E \in \mathfrak{E}^z_\infty(P)$, there exists a $\mu \in M^z(G)$ such that

(2.4.2) $$\|\mu\| \leqq k\|E\|_\infty \text{ and } \hat{\mu}(\sigma) = E_\sigma \text{ for all } \sigma \in P.$$

Now consider f in $\boldsymbol{T}_P(G)$, and write

$$f(x) = \sum_{\sigma \in P} d_\sigma \operatorname{tr}(A_\sigma U^\sigma_x).$$

Let $\boldsymbol{G}$ be the group $\prod_{\sigma \in P} T_\sigma$ where $T_\sigma = T$, the circle group. For $\boldsymbol{t} \in \boldsymbol{G}$, write $\boldsymbol{t} = (t_\sigma)$ where $|t_\sigma| = 1$. Define

$$F(\boldsymbol{t},x) = \sum_{\sigma \in P} d_\sigma t_\sigma \operatorname{tr}(A_\sigma U^\sigma_x),$$

for $(\boldsymbol{t},x) \in \boldsymbol{G} \times G$. It is easily checked that F is continuous on $\boldsymbol{G} \times G$. Using (2.4.2), we obtain for each $\boldsymbol{t} \in \boldsymbol{G}$ a measure $\mu_{\boldsymbol{t}} \in M_z(G)$ such that $\|\mu_{\boldsymbol{t}}\| \leqq k$ and $\hat{\mu}_{\boldsymbol{t}}(\sigma) = t_\sigma I_\sigma$ for all $\sigma \in P$. Let $f_{\boldsymbol{t}}(x) = F(\boldsymbol{t},x)$ for all $x \in G$. It is easily checked that $\hat{f}(\sigma) = \hat{\mu}_{\boldsymbol{t}}(\sigma)\hat{f}_{\boldsymbol{t}}(\sigma)$ for all $\sigma \in \Sigma$. The uniqueness of Fourier-Stieltjes transforms implies that $f = \mu_{\boldsymbol{t}} * f_{\boldsymbol{t}}$ for all $\boldsymbol{t} \in \boldsymbol{G}$. A simple calculation yields

(2.4.3) $$\|f\|_{2s}^{2s} \leqq k^{2s}\|f_{\boldsymbol{t}}\|_{2s}^{2s}.$$

It is easily checked that $t \to \|f_{\boldsymbol{t}}\|_{2s}^{2s}$ is a continuous function on $\boldsymbol{G}$. Hence, we can integrate the inequality (2.4.3) over $\boldsymbol{G}$. We use the fact [2, (36.2.ii)] that

$$\int_{\boldsymbol{G}} |F(\boldsymbol{t},x)|^{2s} d\boldsymbol{t} \leqq \left(\int_{\boldsymbol{G}} |F(\boldsymbol{t},x)|^2 d\boldsymbol{t}\right)^s \cdot 4^s s!$$

and Fubini's theorem to obtain

(2.4.4) $$\|f\|_{2s}^{2s} \leqq k^{2s}\int_G 4^s s!\left(\int_{\boldsymbol{G}} |F(\boldsymbol{t},x)|^2 d\boldsymbol{t}\right)^s dx.$$

We now compute the inner integral in (2.4.4). For fixed $x \in G$, we have, using Plancherel's theorem applied to the group $\boldsymbol{G}$ and [2, (D.39.ii)],

$$\begin{aligned}\int_G |F(t,x)|^2 dt &= \int_G \Big|\sum_{\sigma\in P} d_\sigma t_\sigma \mathrm{tr}(A_\sigma U_x^\sigma)\Big|^2 dt \\ &= \sum_{\sigma\in P} d_\sigma^2 |\mathrm{tr}(A_\sigma U_x^\sigma)|^2 \\ &\leqq \sum_{\sigma\in P} d_\sigma^2 \| A_\sigma\|_{\phi_2}^2 \| U_x^\sigma\|_{\phi_2}^2 \\ &= \sum_{\sigma\in P} d_\sigma^3 \| A_\sigma\|_{\phi_2}^2 \leqq N^2 \sum_{\sigma\in P} d_\sigma \| A_\sigma\|_{\phi_2}^2 = N^2 \|f\|_2^2 .\end{aligned}$$

Thus (2.4.4) may be recast as

$$\| f\|_{2s}^{2s} \leqq k^{2s}4^s s!\, N^{2s} \| f\|_2^{2s}.$$

This implies (2.4.1) and the theorem is proved.

3. Riesz polynomials

In this section we show the existence of central Sidon sets by the method of Riesz polynomials following the proof given by Rider [5]. In particular we show that if there is an integer P such that $\{\sigma\in\Sigma \mid d_\sigma < P\}$ is infinite, then Σ contains an infinite central Sidon set. For other results along this line see Hewitt and Zuckerman [6] and Rider [5].

Let $E = \{\sigma_1,\sigma_2,\cdots\}$ be a countable subset of Σ. For $\sigma\in\Sigma$ and for positive integers m, s and N let $R_s(E,\sigma,N,m)$ be the number of subsets $\{\tau_1,\cdots,\tau_s\}$ of Σ satisfying

$$\tau_k = \sigma_{n_k} \text{ or } \overline{\sigma_{n_k}} \text{ for some } \sigma_{n_k} \text{ in } E,$$

$(k = 1,2,\cdots,s)$, $n_1 < n_2 < \cdots < n_s \leqq N$, and

$$\int_G \chi_{\tau_1}\cdots\chi_{\tau_s}\bar{\chi}_\sigma d\lambda = m.$$

Let

$$R_s(E,\sigma,N) = \sum_{n=1}^{\infty} m\cdot R_s(E,\sigma,N,m),$$

and

$$R_s(E,\sigma) = \lim_{N\to\infty} R_s(E,\sigma,N).$$

Let 1 denote the representation which is identically one.

LEMMA 3.1. *Suppose P and B are positive integers with $B \geqq P$ satisfying:*
(i) *$d_\sigma \leqq P$ for all $\sigma\in E$,*
(ii) $1\notin E$,
(iii) *If $\sigma\in E$ and $\sigma\neq\bar\sigma$, then $\bar\sigma\notin E$, and*
(iv) $R_s(E,1)\leqq B^s$, $(s = 1,2,\cdots)$.

Then

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(v) $\sum_{s=1}^{\infty} (2B)^{-s} R_s(E,\sigma) \leqq 2P$ *for all* $\sigma \in \Sigma$.

In particular, $R_s(E,\sigma) \leqq 2P(2B)^s \leqq (4BP)^s$.

PROOF. Let $\beta = 1/(2B)$ and let

$$f_k(x) = \begin{cases} 1 - \beta\chi_\sigma(x) + \overline{\beta\chi_\sigma(x)} & \text{if } \sigma_k \neq \overline{\sigma_k}, \\ 1 + \beta\chi_\sigma(x) & \text{if } \sigma_k = \sigma_k. \end{cases}$$

Form the Riesz products,

$$P_N(x) = \prod_{k=1}^{N} f_k(x).$$

Since a product of irreducible characters decomposes into a sum of irreducible characters, we can write

$$P_N(x) = 1 + \sum_{\sigma \in \Sigma} C_N(\sigma)\chi_\sigma(x).$$

A straightforward calculation shows

$$C_N(\sigma) = \sum_{s=1}^{N} R_s(E,\sigma,N)\beta^s.$$

In particular,

$$C_N(1) \leqq \sum_{s=1}^{\infty} \beta^s R_s(E,1) \leqq \sum_{s=1}^{\infty} \beta^s B^s = \sum_{s=1}^{\infty} 2^{-s} = 1.$$

Since $|\beta\chi_\sigma| \leqq 1/2$ and P_N is real valued, P_N is non-negative. Thus

$$\| P_N \|_1 = \int P_N d\lambda = \hat{P}_N(1) = 1 + C_N(1) \leqq 2.$$

A computation of the Fourier transform of P_N yields

$$d_\sigma \| \hat{P}_N(\sigma) \|_{\phi\infty} = C_N(\sigma)$$

for $\sigma \neq 1$. Also for $\sigma \neq 1$, it is easily seen that

$$\lim_{N \to \infty} C_N(\sigma) = \sum_{s=1}^{\infty} R_s(E,\sigma)\beta^s.$$

Since

$$\| \hat{P}_N(\sigma) \|_{\phi_\infty} \leqq \| P_N \|_1, \sum_{s=1}^{\infty} (2B)^{-s} R_s(E,\sigma) \leqq 2P.$$

THEOREM 3.2. *Let E be an infinite subset of Σ and B and P be positive integers such that*

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(i) $d_\sigma \leqq P$ *for all* $\sigma \in E$, *and*

(ii) $R_s(E,1) \leqq B^s$ $(s = 1,2,\cdots)$.

Then $E \cup \bar{E}$ *is a central Sidon set, and so* E *is a central Sidon set.*

PROOF. Without loss of generality $B \geqq P$ and by the last lemma we may assume $R_s(E,\sigma) \leqq B^s$ for all $\sigma \in \Sigma$. Also we may assume E is countable since E is central Sidon if and only if every countable subset of E is central Sidon. Finally we assume that if $\sigma \in E$ and $\sigma \neq \bar{\sigma}$ then $\bar{\sigma} \notin E$ (this does not change $E \cup \bar{E}$) and for the time being we assume $1 \notin E$.

We will show $E \cup \bar{E}$ satisfies (2.1.viii), that is for each

$$W \in \prod_{\sigma \in E \cup \bar{E}} \varepsilon_\sigma I_\sigma, \ \varepsilon_\sigma \in \{-1,1\},$$

there is a measure $\mu \in M^z(G)$ such that

$$\sup\{\| W_\sigma - \hat{\mu}(\sigma)\|_{\phi_\infty} : \sigma \in E \cup \bar{E}\} < 1. \tag{3.2.1}$$

Now let $W \in \prod_{\sigma \in E \cup \bar{E}} \varepsilon_\sigma I_\sigma$ be fixed. Let $0 < \delta < 1/P^2$. Write

$$E_1 = \{\sigma \in E : \varepsilon_\sigma = \varepsilon_{\bar{\sigma}}\}, \text{ and}$$

$$E_2 = \{\sigma \in E : \varepsilon_\sigma = -\varepsilon_{\bar{\sigma}}\}.$$

Then E is the disjoint union of E_1 and E_2,

$$E_2 \cap \bar{E} = \varnothing \text{ and } E_1 \cap \bar{E} = \{\sigma \in E : \sigma = \bar{\sigma}\}.$$

Choose $k \geqq 2$ such that $2/(k-1) < \delta$ and let $\beta = 1/(kB^2)$. Write $E_1 = \{\sigma_1, \sigma_2, \cdots\}$ and let

$$\beta_k = d_{\sigma_k} \beta \varepsilon_{\sigma_k} P^{-2}.$$

Clearly $|\beta_k \chi_{\sigma_k}| \leqq \beta \leqq 1/2$ $(k = 1,2,\cdots)$. Let

$$f_k(x) = \begin{cases} 1 + \beta_k \chi_{\sigma_k}(x) + \overline{\beta_k \chi_{\sigma_k}(x)} & \text{if } \sigma_k \neq \overline{\sigma_k}, \\ 1 + \beta_k \chi_{\sigma_k}(x) & \text{if } \sigma_k = \overline{\sigma_k}. \end{cases}$$

Again we form the Riesz product

$$P_N(x) = \prod_{k=1}^{N} f_k(x).$$

Then

$$P_N = 1 + \sum_{k=1}^{N} \beta_k \chi_{\sigma_k} + \sum_{\substack{k=1 \\ \sigma_k \neq \bar{\sigma}_k}}^{N} \overline{\beta_k \chi_{\sigma_k}} + \sum_{\sigma \in \Sigma} D_N(\sigma) \chi_\sigma,$$

where

$$|D_N(\sigma)| \leqq \sum_{s=2}^{N} R_s(E,\sigma,N)\beta^s \leqq \sum_{s=2}^{\infty} R_s(E,\sigma)\beta^s \leqq \sum_{s=2}^{\infty} B^s \beta^s$$

$$= 1/(kB(kB-1)) \leqq 1/(k(k-1)B^2) < \beta\delta/2.$$

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Since $P_N \geqq 0$, we have

$$\| P_N \|_1 = \| \hat{P}_N(1) \|_{\phi_\infty} = 1 + D_N(1) \leqq 1 + \beta\delta/2.$$

Since $P_N \in L_1^z(G)$ for all N and $M^z(G) = C^z(G)^*$ (Lemma 1.1), Alaoglu's theorem implies that

$$\{\mu \in M^z(G): \| \mu \| \leqq 1 + \beta\delta/2\}$$

is weak-* compact. Hence there is a subnet $\{P_\alpha\}$ of $\{P_N\}_{N=1}^\infty$ and a $\mu_1 \in M^z(G)$ such that

$$\| \mu_1 \| \leqq 1 + \beta\delta/2$$

and $P_\alpha \to \mu_1$ in the weak-* topology. Routine calculations show that if $\sigma \in E_1 \cup \bar{E}_1$,

$$\| \hat{\mu}_1(\sigma) - \beta P^{-2} \varepsilon_\sigma I_\sigma \|_{\phi_\infty} < \beta\delta/2,$$

and that if $1 \neq \sigma \notin E_1 \cup \bar{E}_1$, then

$$\| \hat{\mu}(\sigma) \|_{\phi_\infty} < \beta\delta/2.$$

In a similar manner by constructing Riesz polynomials using E_2,

$$\beta_k = i\varepsilon_{\sigma_k} d_{\sigma_k} \beta P^{-2}, \text{ and } f_k(x) = 1 + \beta_k \chi_{\sigma_k} + \overline{\beta_k \chi_{\sigma_k}},$$

we get a measure $\mu_2 \in M^z(G)$ satisfying

$$\| \mu_2 \| \leqq 1 + \beta\delta/2,$$

$$\| \hat{\mu}_2(\sigma) - i\beta P^{-2} \varepsilon_\sigma I_\sigma \|_{\phi_\infty} < \beta\delta/2 \text{ for all } \sigma \in E_2 \cup \bar{E}_2,$$

and

$$\| \hat{\mu}_2(\sigma) \|_{\phi_\infty} < \beta\delta/2 \text{ if } 1 \neq \sigma \notin E_2 \cup \bar{E}_2.$$

Now let $\mu = P^2 \beta^{-1}(\mu_1 - i\mu_2)$. If $\sigma \in E \cup \bar{E}$,

$$\| \hat{\mu}(\sigma) - \varepsilon_\sigma I_\sigma \| < P^2\delta < 1,$$

and if $1 \neq \sigma \notin E \cup \bar{E}$,

$$\| \hat{\mu}(\sigma) \|_{\phi_\infty} < P^2\delta < 1.$$

Finally we take care of 1. If $1 \in E$, select a multiple α of the trigonometric polynomial 1 so that

$$(\mu + \alpha 1)\hat{}\,(1) = \varepsilon_1.$$

If $1 \notin E$, select a multiple α of 1 so that

$$(\mu + \alpha 1)\hat{}\,(1) = 0.$$

Replacing μ by $\mu + \alpha 1$, we now have

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$$\| \hat{\mu}(\sigma) - \varepsilon_\sigma I_\sigma \|_{\phi_\infty} < P^2\delta \text{ for all } \sigma \in E \cup \bar{E} \text{ and}$$

$$\| \hat{\mu}(\sigma) \|_{\phi_\infty} < P^2\delta \text{ for all } \sigma \in \Sigma \backslash (E \cup \bar{E}).$$

Since $\delta < P^{-2}$, μ satisfies (3.2.1) and $E \cup \bar{E}$ is a central Sidon set.

COROLLARY 3.3. *If E satisfies the hypothesis of theorem 3.2 and F is a central Sidon set, then $E \cup F$ is a central Sidon set. In particular any finite union of sets satisfying the hypothesis of theorem 3.2 is a central Sidon set.*

PROOF. Repeat Rider's proof for Abelian G [5, (1.6)].

COROLLARY 3.4. *If $E \subset \Sigma$ is infinite and*

(i) $\sup\{d_\sigma : \sigma \in E\} < \infty$,

then E contains an infinite central Sidon set.

PROOF. Since (i) holds, it is possible to construct by induction a sequence of subsets $\{F_n\}$ of E such that for each n the cardinality of F_n is n, $F_n \subset F_{n+1}$ and $R_s(F_n, 1) = 0$, $(s = 1, 2, \cdots)$. Let $F = \bigcup_{n=1}^{\infty} F_n$. Then $F \subset E$ and satisfies the hypothesis of theorem 3.2.

4. I-sets

In this section we generalize the notion of independence as defined for Abelian groups. It will be shown that subsets of Σ which satisfy this generalized independence property are central Sidon sets. An interesting example shows there are central Sidon sets which are not Sidon sets.

Let I, II, III, and IV be the usual quadrants of the complex plane. The axes are considered to lie in their adjacent quadrants and the origin to lie in all four quadrants. Note that if $\alpha \in II$ and $\beta \in III$, then $\alpha\beta \in I \cup IV$.

Let E be a subset of Σ. We say that E is an *I-set* if

$$1 \notin E, \tag{4.0.1}$$

and if for every finite subset Φ of E and for every ordered partition $\Phi = \Phi_1 \cup \Phi_2 \cup \Phi_3$ $(\Phi_1 \neq \varnothing)$ there is an $x \in G$ such that

$$\chi_\sigma(x) = d_\sigma \quad \text{if } \sigma \in \Phi_1, \tag{4.0.2}$$

$$\chi_\sigma(x) \in II \quad \text{if } \sigma \in \Phi_2, \text{ and} \tag{4.0.3}$$

$$\chi_\sigma(x) \in III \quad \text{if } \sigma \in \Phi_3. \tag{4.0.4}$$

REMARK. By using the compactness of G and the continuity of characters (4.0.2)–(4.0.4) can be replaced by:

For every $\varepsilon > 0$, there is an $x \in G$ such that (4.0.3) and (4.0.4) hold and

$$|\chi_\sigma(x) - d_\sigma| < \varepsilon. \tag{4.0.2'}$$

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Using this and a result on independent characters [7, page 98], it can be shown that independent subsets of the dual group of a compact Abelian group which do not contain 1 are I-sets. This justifies our calling I-sets generalized independent sets.

The main theorem of this section shows that I-sets are central Sidon sets.

THEOREM 4.1. *Let G be a compact group and let P be an I-set in Σ. Then P is a central Sidon set.*

PROOF. Let $f = \sum_{\sigma \in F} \alpha_\sigma \chi_\sigma \in T_P^z(G)$ where $\alpha_\sigma \neq 0$ for $\sigma \in F$ and F is a finite subset of P. Note that

$$\| \hat{f} \|_1 = \textstyle\sum_{\sigma \in F} d_\sigma |\alpha_\sigma| .$$

By elementary consideration it is possible to choose a complex number θ such such that $\theta^4 = 1$ and a subset S of F such that

$$S = \{\sigma \in F : \operatorname{Re}(\alpha_\sigma \theta) \geqq 0\} \text{ and}$$

$$\operatorname{Re}\Big(\sum_{\sigma \in S} d_\sigma \theta \alpha_\sigma \Big) \geqq 1/4 \sum_{\sigma \in F} d_\sigma |\alpha_\sigma| = 1/4 \| \hat{f} \|_1 .$$

Let

$$S_3 = \{\sigma \in F \backslash S : \operatorname{Im}(\alpha_\sigma \theta) \geqq 0\} \text{ and } S_2 = \{\sigma \in F \backslash S : \operatorname{Im}(\alpha_\sigma \theta) < 0\} .$$

Then $S \cup S_2 \cup S_3$ is an ordered partition of F. By the definition of I-set, there exists an x in G such that

$$\begin{aligned} &\chi_\sigma(x) = d_\sigma && \text{if } \sigma \in S, \\ &\chi_\sigma(x) \in II && \text{if } \sigma \in S_2, \text{ and} \\ &\chi^\sigma(x) \in III && \text{if } \sigma \in S_3 . \end{aligned}$$

Since $\theta\alpha_\sigma \in III$ if $\sigma \in S_2$ and $\theta\alpha_\sigma \in II$ if $\sigma \in S_3$, we see that $\operatorname{Re}(\theta\alpha_\sigma\chi_\sigma(x)) \geqq 0$ for $\sigma \in S_2 \cup S_3$. Also $\operatorname{Re}(\theta\alpha_\sigma\chi_\sigma(x)) = \operatorname{Re}(\theta\alpha_\sigma)d_\sigma \geqq 0$ if $\sigma \in S$. Thus

$$\begin{aligned} \| f \|_u \geqq |f(x)| \geqq \operatorname{Re}(\theta f(x)) &= \operatorname{Re}\Big(\sum_F \theta\alpha_\sigma\chi_\sigma(x)\Big) \\ \geqq \operatorname{Re}\Big(\sum_S \theta\alpha_\sigma\chi_\sigma(x)\Big) &= \operatorname{Re}\Big(\sum_S \theta\alpha_\sigma d_\sigma\Big) \\ \geqq 1/4 \sum_F d_\sigma |\alpha_\sigma| &= 1/4 \| \hat{f} \|_1 . \end{aligned}$$

Since f was an arbitrary element of $T_P^z(G)$, P is a central Sidon set by (2.1.vii).

COROLLARY 4.2. *Let $\{G_\alpha\}_{\alpha \in A}$ be set of compact groups and for each $\alpha \in A$ let U_α be a non-one continuous irreducible unitary representation of G_α. Let $G = \prod_{\alpha \in A} G_\alpha$ and let π_α be the projection of G onto G_α. Let σ_α be the equivalence class of $U_\alpha \circ \pi_\alpha$ for each $\alpha \in A$. Then $P = \{\sigma_\alpha : \alpha \in A\}$ is a central Sidon set.*

PROOF. It suffices to show P is an I-set. Let Φ be a finite subset of P. Let $\Phi = \Phi_1 \cup \Phi_2 \cup \Phi_3$ be any ordered partition of Φ. Let χ_α be the character of U_α for $\alpha \in A$. Then the character of σ_α is $\chi_\alpha \circ \pi_\alpha$. Define the element $\boldsymbol{x} = (x_\alpha) \in G$ as follows:

x_α is the identity of G_α if $\sigma_\alpha \in \Phi_1$ or of σ_α is not in Φ;

x_α is such that $\chi_\alpha(x_\alpha) = 0$ if $\sigma_\tau \in \Phi_2 \cup \Phi_3$ and $d_{\sigma_\alpha} \neq 1$ (this is possible by a theorem of Gallagher [8]);

x_α is such that $\chi_\alpha(x_\alpha) \in II$ if $\sigma_\alpha \in \Phi_2$, $d_{\sigma_\alpha} = 1$, and σ_α is not identically 1;

x_α is such that $\chi_\alpha(x_\alpha) \in III$ if $\sigma_\alpha \in \Phi_3$, $d_{\sigma_\alpha} = 1$, and σ_α is not identically 1.

Then $\boldsymbol{x}$ is well defined and

$$\begin{array}{lll} \chi_\sigma(\boldsymbol{x}) = d_\sigma & \text{if } \sigma \in \Phi_1, & \\ \chi_\sigma(\boldsymbol{x}) \in II & \text{if } \sigma \in \Phi_2, & \text{and} \\ \chi_\sigma(\boldsymbol{x}) \in III & \text{if } \sigma \in \Phi_3. & \end{array}$$

Since Φ was an arbitrary finite subset of P, P is an I-set and hence a central Sidon set.

EXAMPLE 4.3. (See the remarks concerning Theorem 2.4.) For each $n = 1, 2, \cdots$ let $G_n = SU(2)$; see [2, §29]. Let

$$G = \prod_{n=1}^{\infty} G_n.$$

For each $n = 1, 2, \cdots$, let U_n be a continuous irreducible unitary representation of G_n of dimension n. Let χ_n be the character of U_n. Let σ_n be the projection of G onto G_n. Let σ_n be the equivalence class of $U_n \circ \pi_n$.

The character χ_{σ_n} of σ_n is $\chi_n \circ \pi_n$. Thus for positive r,

$$\int_G |\chi_{\sigma_n}|^r \, d\lambda = \int_{G_n} |\chi_n|^r d\lambda.$$

It follows [4, (37.21.b)] that $\| \chi_{\sigma_n} \|_2 = 1$ and $\| \chi_{\sigma_n} \|_4 = n^{1/4}$. Since $\chi_{\sigma_n} \in \boldsymbol{T}_E(G)$, $E = \{\sigma_1, \sigma_2, \cdots\}$ is not a central Λ_4 set (2.2.iii), a Λ_4 set, nor a Sidon set [2, (36.10)]. However, E is a central Sidon set by (4.2).

5. Open questions

The results of this paper by no means answer all the questions which can be raised with regard to central Sidon and central Λ_p sets. The following questions might be pursued to advantage.

5.1. Are there any compact groups whose dual objects contain no infinite central Sidon set? It is known that there is a group whose dual object contains no infinite central Λ_4 sets, namely, $SU(2)$. However, it is not known whether the dual object of $SU(2)$ contains an infinite central Sidon set.

Note added in proof: Since the submission of this paper, question 5.1 has been answered. Many groups have dual objects which lack infinite central Sidon sets see Rogozin [9].

5.2. Given that a subset P of Σ is a central Sidon set (or a central Λ_p set), is there some additonal condition which, if imposed on P, would imply that P is a Sidon set (a Λ_p set)? A natural condition to try would be

(i) $\sup\{d_\sigma : \sigma \in P\} = N < \infty$.

Of course, if $N = 1$ or P is finite the question has an obvious answer. A closely related question is: Must an infinite set P satisfying (i) contain an infinite Sidon subset?

5.3. Are there any non Abelian, non finite compact groups such that every central Sidon set is a Sidon set?

5.4. The definition of I-sets is not entirely satisfactory. Can a better definition be devised? Also, in the Abelian case, what is the relation of I-sets to other independence notions such as dissociate sets?

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References

[1] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis*, I (Springer-Verlag, Berlin, 1963).
[2] E. Hewitt, and K. A. Ross, *Abstract Harmonic Analysis*, II (Springer-Verlag, Berlin, 1970).
[3] C. Dunkl and D. Ramirez, 'Sidon Sets on Compact Groups', *Manatsheft für Math.* 75 (1971), 111–117.
[4] A. Figa-Talamancà and D. Rider, 'A theorem of Littlewood and lacunary series for compact groups', *Pacific J. Math.* 16 (1966), 505–514.
[5] D. Rider, 'Gap series on circles and spheres', *Canad. J. Math.* 18 (1966), 389–398.
[6] E. Hewitt, and H. S. Zuckerman, 'Some theorems on lacunary Fourier series, with extensions to compact groups', *Trans. Amer. Math. Soc.* 93 (1959), 1–19.
[7] W. Rudin, *Fourier Analysis on Groups* (Interscience Publishers, New York, 1962).
[8] P. Gallagher, 'Zeros of group characters', *Math. Z.* 87 (1965), 363–364.
[9] D. Ragozin, '*Central measures on compact simple Lie groups*', (*to appear*).

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