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Unbounded Fredholm Operators and Spectral Flow

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Abstract. We study the gap (= "projection norm" = "graph distance") topology of the space of all (not necessarily bounded) self-adjoint Fredholm operators in a separable Hilbert space by the Cayley transform and direct methods. In particular, we show the surprising result that this space is connected in contrast to the bounded case. Moreover, we present a rigorous definition of spectral flow of a path of such operators (actually alternative but mutually equivalent definitions) and prove the homotopy invariance. As an example, we discuss operator curves on manifolds with boundary.

Introduction

The main purpose of this paper is to study the topology of the space of all (generally unbounded) self-adjoint Fredholm operators, and to put the notion of spectral flow for continuous paths of such operators on a firm mathematical footing with clear concise definitions and proofs.

The natural topology on the space of all such operators, denoted by $\mathcal{CF}^{\mathrm{sa}}$, (for a fixed separable Hilbert space, H) is given by the graph distance topology. That is, we consider the topology induced by the metric: $\delta(T_1, T_2) = \|P_1 - P_2\|$ where P_i is the projection onto the graph of T_i in the space $H \times H$ for $i = 1, 2$. This metric is called the gap metric. The space of unbounded Fredholm operators has been studied systematically in the seminal paper by Cordes and Labrousse [6].

Many users of the notion of spectral flow feel that the definition and basic properties are already well-understood. However, there are some difficulties with the currently available definitions which this paper aims to remedy. A feature of our approach is the use of the Cayley Transform,

$$T \mapsto \boldsymbol{\kappa}(T) = (T - i)(T + i)^{-1}.$$

We show that the image $\boldsymbol{\kappa}(\mathcal{CF}^{\mathrm{sa}})$ is precisely the set

$$\{U \in \mathcal{U}(H) \mid (U + I) \text{ is Fredholm and } (U - I) \text{ is injective}\} =: {}_{\mathcal{F}}\mathcal{U}_{\mathrm{inj}},$$

and that the map $\boldsymbol{\kappa}$ induces an equivalent metric, $\widetilde{\delta}$, on $\mathcal{CF}^{\mathrm{sa}}$ via

$$\widetilde{\delta}(T_1, T_2) = \|\boldsymbol{\kappa}(T_1) - \boldsymbol{\kappa}(T_2)\|.$$

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This Cayley picture of $\mathcal{CF}^{\mathrm{sa}}$ leads us to a more careful study of the metric space $\mathcal{CF}^{\mathrm{sa}}$ by studying its image ${}_{\mathcal{F}}\mathcal{U}_{\mathrm{inj}} = \boldsymbol{\kappa}(\mathcal{CF}^{\mathrm{sa}})$. In contrast to the space of bounded self-adjoint Fredholm operators, we prove the surprising result that $\mathcal{CF}^{\mathrm{sa}}$ is (path-)connected. In particular, the operator I can be connected to $-I$ in $\mathcal{CF}^{\mathrm{sa}}$.

Furthermore, using the Cayley picture of $\mathcal{CF}^{\mathrm{sa}}$, we are able to give two different (but equivalent) definitions of the spectral flow of a continuous path in $\mathcal{CF}^{\mathrm{sa}}$ and to show that these definitions are invariant under homotopy. We use neither Kato's Selection Theorem nor any differentiability or regularity assumptions. Thus, spectral flow induces a surjective homomorphism SF, from the fundamental group $\pi_1(\mathcal{CF}^{\mathrm{sa}})$ to $\mathbb{Z}$.

On the other hand, the space $\mathcal{F}^{\mathrm{sa}}$ of bounded operators in $\mathcal{CF}^{\mathrm{sa}}$ inherits its usual (norm) topology with the gap metric δ and $\mathcal{F}^{\mathrm{sa}}$ has three connected components by a result of Atiyah and Singer. To add to the confusion, $\mathcal{F}^{\mathrm{sa}}$ is also dense in $\mathcal{CF}^{\mathrm{sa}}$. Unfortunately, we have been unable to decide whether $\mathrm{SF}\colon \pi_1(\mathcal{CF}^{\mathrm{sa}}) \to \mathbb{Z}$ is injective or whether $\mathcal{CF}^{\mathrm{sa}}$ is a classifying space for the K^1-functor (*cf.* Remark 1.11 below).

Finally, we consider a fixed compact Riemannian manifold M with boundary Σ, a family $\{D_s\}$ of symmetric elliptic differential operators of first order and of Dirac type on M acting on sections of a fixed Hermitian vector bundle E with coefficients depending continuously on a parameter s, and a norm-continuous family $\{P_t\}$ of orthogonal projections of $L^2(\Sigma; E|_\Sigma)$ defining well-posed boundary problems. Here "Dirac type" means that each operator D_s can be written in product form near any hypersurface (for details of the definition see Assumption 3.1 (1), Equation (3.1) below).

With a view to applications in low-dimensional topology and gauge theories (see *e.g.,* [13]), we do not assume that the metric structures of M and E are of product form near Σ. Consequently the tangential symmetric and skew-symmetric operator components may depend on the normal variable near Σ. Solely exploiting elliptic regularity and the *unique continuation property* of operators of Dirac type, we show that the induced two-parameter family

$$(s,t) \mapsto (D_s)_{P_t}$$

of self-adjoint $L^2(M;E)$-extensions with compact resolvent is continuous in

$$\mathcal{CF}^{\mathrm{sa}}(L^2(M;E))$$

in the gap metric without any further assumptions or restrictions.

The results of this paper have been announced in [4].

Notations 0.1 Let H be a separable complex Hilbert space. First let us introduce some notation for various spaces of operators in H:

$\mathcal{C}(H)$ closed densely defined operators in H,
$\mathcal{B}(H)$ bounded linear operators $H \to H$,
$\mathcal{U}(H)$ unitary operators $H \to H$,
$\mathcal{K}(H)$ compact linear operators $H \to H$,

$\mathcal{F}(H)$ bounded Fredholm operators $H \to H$,
$\mathcal{CF}(H)$ closed densely defined Fredholm operators in H.

If no confusion is possible we will omit "(H)" and write $\mathcal{C}, \mathcal{B}, \mathcal{K}$, etc. By $\mathcal{C}^{\mathrm{sa}}, \mathcal{B}^{\mathrm{sa}}$, etc. we denote the set of self-adjoint elements in $\mathcal{C}, \mathcal{B}$, etc.

1 The Space of Unbounded Self-Adjoint Fredholm Operators

1.1 The Topology of $\mathcal{C}^{\mathrm{sa}}(H)$

We present a few facts about the so-called gap topology on $\mathcal{C}^{\mathrm{sa}}$, *cf.* [6, 12, 15]. As explained, *e.g.,* in [15, Section 1], there are two natural metrics on $\mathcal{C}^{\mathrm{sa}}$: the Riesz metric and the gap metric. The *Riesz metric* is the metric such that the bijection

$$F\colon \mathcal{C}^{\mathrm{sa}} \to \left\{ S \in \mathcal{B}^{\mathrm{sa}} \;\middle|\; \|S\| \le 1 \text{ and } S \pm I \text{ both injective} \right\}, \qquad T \mapsto F_T := T(I+T^2)^{-1/2} \tag{1.1}$$

is an isometry. That is, given $T_1, T_2 \in \mathcal{C}^{\mathrm{sa}}$ then their Riesz distance $\varphi(T_1, T_2)$ is defined to be $\|F_{T_1} - F_{T_2}\|$. Note that the image of F is neither open nor closed in the closed unit ball of $\mathcal{B}^{\mathrm{sa}}$. Note also that F maps the space $\mathcal{CF}^{\mathrm{sa}}$ of (generally unbounded) self-adjoint Fredholm operators onto the intersection of the space $\mathcal{F}^{\mathrm{sa}}$ of bounded self-adjoint Fredholm operators with $F(\mathcal{C}^{\mathrm{sa}})$, see also Subsection 1.2. It is clear that F is injective. We postpone the proof that F as defined in (1.1) is surjective (see Proposition 1.5 below).

The *gap metric* $\delta(T_1, T_2)$ is given as follows: let P_j denote the orthogonal projections onto the graphs of T_j in $H \times H$. Then $\delta(T_1, T_2) := \|P_1 - P_2\|$. It is shown in [15, Section 1] that the Riesz topology is finer than the gap topology. By an example due to Fuglede [15, Section 1], (see also Example 2.14 below), the Riesz topology is not equal to the gap topology and hence the Riesz topology is strictly finer than the gap topology. This means in particular that the Riesz map F is not continuous on $(\mathcal{C}^{\mathrm{sa}}, \delta)$. This was also noted in [3, Section 4.2].

The next result shows that, as for the Riesz topology, the gap topology can be obtained from a map into the bounded linear operators.

Recall that two metrics for the same set are *topologically equivalent* if and only if they define the same topology and *uniformly equivalent* if and only if they can be estimated mutually in a uniform way. In the latter case the maps $\mathrm{id}\colon (X, \delta_1) \to (X, \delta_2)$ and $\mathrm{id}\colon (X, \delta_2) \to (X, \delta_1)$ are Lipschitz continuous and thus uniformly continuous.

Theorem 1.1

(a) *On $\mathcal{C}^{\mathrm{sa}}$ the gap metric is uniformly equivalent to the metric γ given by*

$$\gamma(T_1, T_2) = \|(T_1 + i)^{-1} - (T_2 + i)^{-1}\|.$$

(b) *Let $\kappa\colon \mathbb{R} \to S^1 \setminus \{1\}, x \mapsto \frac{x-i}{x+i}$ denote the Cayley transform. Then κ induces a homeomorphism*

$$\boldsymbol{\kappa}\colon \mathcal{C}^{\mathrm{sa}}(H) \to \left\{ U \in \mathcal{U}(H) \;\middle|\; U - I \text{ is injective} \right\}, \qquad T \mapsto \boldsymbol{\kappa}(T) = (T-i)(T+i)^{-1}. \tag{1.2}$$

More precisely, the gap metric is (uniformly) equivalent to the metric $\widetilde{\delta}$ *defined by* $\widetilde{\delta}(T_1, T_2) = \|\boldsymbol{\kappa}(T_1) - \boldsymbol{\kappa}(T_2)\| = \frac{1}{2}\gamma(T_1, T_2)$.

Proof First we recall that for $T \in \mathcal{C}^{\mathrm{sa}}$ the orthogonal projection P_T onto the graph of T is given by

$$\begin{pmatrix} R_T & TR_T \\ TR_T & T^2 R_T \end{pmatrix}, \quad R_T := (I + T^2)^{-1}. \tag{1.3}$$

Hence, the gap metric δ is (uniformly) equivalent to

$$\delta_1(T_1, T_2) = \|R_{T_1} - R_{T_2}\| + \|T_1 R_{T_1} - T_2 R_{T_2}\|, \tag{1.4}$$

(see also [6, Lemma 3.10]). The identities

$$(T - i)^{-1} = (T + i)(T^2 + I)^{-1} = TR_T + iR_T,$$
$$(T + i)^{-1} = (T - i)(T^2 + I)^{-1} = TR_T - iR_T$$

yield

$$R_T = \frac{1}{2i}\left((T - i)^{-1} - (T + i)^{-1}\right), \qquad T \in \mathcal{C}^{\mathrm{sa}}, \tag{1.5}$$
$$TR_T = \frac{1}{2}\left((T - i)^{-1} + (T + i)^{-1}\right),$$

and we infer that the metric δ_1 is (uniformly) equivalent to the metric γ given by

$$\begin{aligned} \gamma(T_1, T_2) &= \frac{1}{2}\left(\|(T_1 + i)^{-1} - (T_2 + i)^{-1}\| + \|(T_1 - i)^{-1} - (T_2 - i)^{-1}\|\right) \\ &= \|(T_1 + i)^{-1} - (T_2 + i)^{-1}\|. \end{aligned} \tag{1.6}$$

In the last equality we have used that for any $A \in \mathcal{B}(H)$ one has $\|A\| = \|A^*\|$. This proves (a).

To prove (b) we note for $T \in \mathcal{C}^{\mathrm{sa}}$ the identities $\operatorname{range}(T + i) = H$ and

$$\boldsymbol{\kappa}(T) = I - 2i(T + i)^{-1}. \tag{1.7}$$

These imply

$$\|(T_1 + i)^{-1} - (T_2 + i)^{-1}\| = \frac{1}{2}\left\|\boldsymbol{\kappa}(T_1) - \boldsymbol{\kappa}(T_2)\right\|. \tag{1.8}$$

This shows that the gap metric and the metric $\widetilde{\delta}$ are (uniformly) equivalent. This equivalence implies that the Cayley transform is a homeomorphism onto its image. It remains to identify the image of the Cayley transform.

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Given $T \in \mathcal{C}^{\mathrm{sa}}$, its Cayley transform $\boldsymbol{\kappa}(T)$ is certainly a unitary operator. To show that $\boldsymbol{\kappa}(T) - I$ is injective consider $x \in H$ such that $\boldsymbol{\kappa}(T)x = x$. In view of (1.7) this implies

$$x = \boldsymbol{\kappa}(T)x = x - 2i(T+i)^{-1}x;$$

thus $(T+i)^{-1}x = 0$ and hence $x = 0$.

Conversely, let U be a unitary operator such that $U - I$ is injective. From the following proposition and corollary, we obtain the existence of a $T \in \mathcal{C}^{\mathrm{sa}}$ such that $\boldsymbol{\kappa}(T) = U$. The theorem is proved. ■

Proposition 1.2 *If U is unitary and $U - I$ is injective, then $T := i(I+U)(I-U)^{-1}$ is self-adjoint on* domain$(T) :=$ range$(I-U)$. *Moreover, $T = i(I-U)^{-1}(I+U)$.*

A similar result is proved in [19, Theorem 13.19]. Our argument seems to be shorter and more appropriate in our context.

Proof $\overline{\operatorname{range}(I-U)} = \ker(I-U^*)^{\perp} = \ker(I-U)^{\perp} = \{0\}^{\perp} = H$ since U normal implies $\ker(I-U^*) = \ker(I-U)$. Thus, domain T is dense in H. Now,

$$\begin{aligned}(I+U)(I-U)^{-1} &= (I-U)^{-1}(I-U)(I+U)(I-U)^{-1}\\ &= (I-U)^{-1}(I+U)|_{\operatorname{range}(I-U)} \subseteq (I-U)^{-1}(I+U).\end{aligned}$$

On the other hand, if $x \in \operatorname{domain}\big((I-U)^{-1}(I+U)\big)$ then

$$(I+U)x \in \operatorname{domain}\big((I-U)^{-1}\big) = \operatorname{range}(I-U),$$

and so there exists a $y \in H$ with $(I+U)x = (I-U)y$. Solving,

$$x = (I-U)y + (I-U)x - x$$

and so $x = (I-U)\frac{1}{2}(x+y) \in \operatorname{domain}\big[(I+U)(I-U)^{-1}\big]$. Thus,

$$T = i(I+U)(I-U)^{-1} = i(I-U)^{-1}(I+U).$$

It is an elementary calculation that T is symmetric and so

$$T \subseteq T^* = -i(I-U^*)^{-1}(I+U^*).$$

(We have the "$=$" since $I+U$ is bounded and on the left in the formula for T, see *e.g.,* [17, p. 299].) By the same argument as for T we get

$$T^* = -i(I-U^*)^{-1}(I+U^*) = -i(I+U^*)(I-U^*)^{-1}$$

and T^* is symmetric, so that

$$T^* \subseteq T^{**} = i(I-U)^{-1}(I+U) = T.$$

Hence, $T = T^*$. ■

Corollary 1.3 *With U and T as above, $\boldsymbol{\kappa}(T) = U$.*

Proof

$$\begin{aligned}(T + iI) &= i(I - U)^{-1}(I + U) + i(I - U)^{-1}(I - U)\\ &= i(I - U)^{-1} \cdot 2 = 2i(I - U)^{-1},\end{aligned}$$

so that
$$(T + iI)^{-1} = \frac{1}{2i}(I - U).$$

By a similar calculation,

$$(T - iI) = 2i(I - U)^{-1}U = 2iU(I - U)^{-1}$$

so that,
$$\boldsymbol{\kappa}(T) = (T - iI)(T + iI)^{-1} = U. \qquad \blacksquare$$

Remark 1.4 (a) In the definition of the metric γ in (1.6) we may replace i by $-i$ or, more generally, by any $-\lambda$ with $\lambda \in \varrho(T_1) \cap \varrho(T_2)$, $\varrho(T_j) := \mathbb{C} \setminus \operatorname{spec} T_j$ denoting the resolvent set. All these metrics are (uniformly) equivalent with the gap metric.

(b) We recall the basic spectral argument for Cayley transforms, namely that the identity $\lambda I - T = (\lambda + i)(\kappa(\lambda) - \boldsymbol{\kappa}(T))(I - \boldsymbol{\kappa}(T))^{-1}$ implies

$$\lambda \in \operatorname{spec} T \iff \kappa(\lambda) \in \operatorname{spec} \boldsymbol{\kappa}(T), \tag{1.9}$$

$$\lambda \in \operatorname{spec}_{\mathrm{discr}} T \iff \kappa(\lambda) \in \operatorname{spec}_{\mathrm{discr}} \boldsymbol{\kappa}(T). \tag{1.10}$$

Here $\operatorname{spec}_{\mathrm{disc}}$ denotes the discrete spectrum, *cf.* subsection 1.2 below.

Following the same pattern as the preceding proof of Proposition 1.2 we show:

Proposition 1.5 *If S is a bounded self-adjoint operator with $\|S\| \leq 1$ and $S \pm I$ injective, then $T := S(I - S^2)^{-\frac{1}{2}}$ is densely defined and self-adjoint. Moreover,*

$$T = (I - S^2)^{-\frac{1}{2}}S \quad \textit{and} \quad S = T(I + T^2)^{-\frac{1}{2}}.$$

Proof Since $I - S^2$ is injective, it has dense range and so $(I - S^2)^{-1}$ and $(I - S^2)^{-\frac{1}{2}}$ are densely defined and self-adjoint. Since S commutes with $(I - S^2)^{\frac{1}{2}}$, we have that $S(I - S^2)^{-\frac{1}{2}} \subseteq (I - S^2)^{-\frac{1}{2}}S$ by an argument in the proof of Proposition 1.2. On the other hand, for $x \in \operatorname{domain}((I - S^2)^{-\frac{1}{2}}S)$ we have $Sx \in \operatorname{domain}((I - S^2)^{-\frac{1}{2}}) = \operatorname{range}((I - S^2)^{\frac{1}{2}})$ so that
$$Sx = (I - S^2)^{\frac{1}{2}}y$$
for some y. Hence, $S^2x = S(I-S^2)^{\frac{1}{2}}y = (I-S^2)^{\frac{1}{2}}Sy$. Or, $(I-S^2)x = x-(I-S^2)^{\frac{1}{2}}Sy$. That is,

$$x = (I - S^2)x + (I - S^2)^{\frac{1}{2}}Sy = (I - S^2)^{\frac{1}{2}}\big((I - S^2)^{\frac{1}{2}}x + Sy\big)$$

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is in the range of $(I-S^2)^{\frac{1}{2}}$, which is domain$((I-S^2)^{-\frac{1}{2}}) =$ domain$(S(I-S^2)^{-\frac{1}{2}})$. That is, $(I-S^2)^{-\frac{1}{2}}S = S(I-S^2)^{-\frac{1}{2}}$. By an argument in the proof of Proposition 1.2, this implies that $T := (I-S^2)^{-\frac{1}{2}}S$ is self-adjoint.

Now, since S commutes with $(I-S^2)^{-\frac{1}{2}}$, one calculates

$$(I+T^2) = I + (I-S^2)^{-1}S^2 = (I-S^2)^{-1}\big((I-S^2)+S^2\big) = (I-S^2)^{-1}.$$

From this we easily calculate $T(I+T^2)^{-\frac{1}{2}} = S$. ∎

It was proved in [6, Addendum] that the topology induced by the gap metric on the set of bounded operators is the same as the topology induced by the natural metric $s(T_1,T_2) = \|T_1 - T_2\|$. However, the reader should be warned that the metric s is not (uniformly) equivalent to the gap metric. In other words, the uniform structures induced by the gap metric and by the operator norm on the space of bounded linear operators are different. This follows from the fact that the metric s is complete while the gap metric on the set of bounded operators is not complete. The latter follows from the following result.

Proposition 1.6 *With respect to the gap metric, the set $\mathcal{B}^{\mathrm{sa}}(H)$ is dense in $\mathcal{C}^{\mathrm{sa}}(H)$.*

Proof Let $T \in \mathcal{C}^{\mathrm{sa}}$ and denote by $(E_\lambda)_{\lambda\in\mathbb{R}}$ the spectral resolution of T. Put

$$T_n := \int_{[-n,n]} \lambda\, dE_\lambda + \int_{|\lambda|>n} n(\operatorname{sgn}\lambda)\, dE_\lambda. \tag{1.11}$$

Then T_n is a bounded self-adjoint operator and

$$\begin{aligned}\gamma(T,T_n) &= \|(T+i)^{-1} - (T_n+i)^{-1}\| \\ &= \Big\| \int_{|\lambda|>n} (\lambda+i)^{-1} - (n(\operatorname{sgn}\lambda)+i)^{-1}\, dE_\lambda \Big\| \le \frac{2}{n}.\end{aligned} \tag{1.12}$$

Hence $T_n \to T$ in the γ-metric. In view of Theorem 1.1(a), this proves the assertion. ∎

1.2 The Connectedness of $\mathcal{CF}^{\mathrm{sa}}$

We determine the image under the Cayley transform of the space $\mathcal{CF}^{\mathrm{sa}}$ of (not necessarily bounded) self-adjoint Fredholm operators. Moreover, we will show that this space is path connected. For the general theory of unbounded Fredholm operators we refer to [12, Section IV.5].

We recall that for a closed operator T in a Hilbert space the essential spectrum, $\operatorname{spec}_{\mathrm{ess}} T$, consists of those $\lambda \in \mathbb{C}$ for which $T-\lambda$ is not a Fredholm operator. Then $\operatorname{spec}_{\mathrm{ess}} T$ is a closed subset of $\operatorname{spec} T$. The discrete spectrum, $\operatorname{spec}_{\mathrm{discr}} T$, consists of those isolated points of $\operatorname{spec} T$ which are not in $\operatorname{spec}_{\mathrm{ess}} T$.

It is well known that if T is self-adjoint then λ is an isolated point of $\operatorname{spec} T$ if and only if range$(T-\lambda)$ is closed ([8, Definition XIII.6.1 and Theorem XIII.6.5];

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note that loc. cit. define the essential spectrum differently). Consequently, for a self-adjoint operator T we have

$$\begin{aligned}\operatorname{spec}_{\mathrm{discr}} T &= \operatorname{spec} T \setminus \operatorname{spec}_{\mathrm{ess}} T \\ &= \Big\{ \lambda \in \mathbb{C} \mid \lambda \text{ is an isolated point of } \operatorname{spec} T \text{ which is} \\ &\qquad\qquad \text{an eigenvalue of } \textit{finite} \text{ multiplicity of } T \Big\} \\ &= \{ \lambda \in \mathbb{C} \mid 0 < \dim\ker(T-\lambda) < \infty \text{ and } \operatorname{range}(T-\lambda) \text{ closed} \}.\end{aligned}$$

We note an immediate consequence of the Cayley picture:

Proposition 1.7 *For $\lambda \in \mathbb{R}$ the sets*

$$\big\{ T \in \mathcal{C}^{\mathrm{sa}}(H) \bigm| \lambda \notin \operatorname{spec} T \big\} \quad \textit{and} \quad \big\{ T \in \mathcal{C}^{\mathrm{sa}}(H) \bigm| \lambda \notin \operatorname{spec}_{\mathrm{ess}} T \big\}$$

are open in the gap topology.

Proof By Theorem 1.1 (see also Remark 1.4(b)) we have

$$\big\{ T \in \mathcal{C}^{\mathrm{sa}}(H) \bigm| \lambda \notin \operatorname{spec} T \big\} = \boldsymbol{\kappa}^{-1} \big\{ U \in \mathcal{U}(H) \bigm| \kappa(\lambda) \notin \operatorname{spec} U \big\},$$
$$\big\{ T \in \mathcal{C}^{\mathrm{sa}}(H) \bigm| \lambda \notin \operatorname{spec}_{\mathrm{ess}} T \big\} = \boldsymbol{\kappa}^{-1} \big\{ U \in \mathcal{U}(H) \bigm| \kappa(\lambda) \notin \operatorname{spec}_{\mathrm{ess}} U \big\},$$

where the spaces of unitary operators on the right side are open in the range of $\boldsymbol{\kappa}$ by the openness of the spaces of bounded invertible resp., bounded Fredholm operators. Now the assertion follows. ■

Corollary 1.8 *The set $\mathcal{CF}^{\mathrm{sa}} = \big\{ T \in \mathcal{C}^{\mathrm{sa}} \bigm| 0 \notin \operatorname{spec}_{\mathrm{ess}} T \big\} = \boldsymbol{\kappa}^{-1}({}_{\mathcal{F}}\mathcal{U})$, ${}_{\mathcal{F}}\mathcal{U} := \big\{ U \in \mathcal{U} \bigm| -1 \notin \operatorname{spec}_{\mathrm{ess}} U \big\}$, of (not necessarily bounded) self-adjoint Fredholm operators is open in $\mathcal{C}^{\mathrm{sa}}$.*

Remark 1.9 By Proposition 1.6, the preceding corollary implies that the set $\mathcal{F}^{\mathrm{sa}}$ is dense in $\mathcal{CF}^{\mathrm{sa}}$ with respect to the gap metric.

In contrast to the bounded case and somewhat surprisingly, the space of unbounded self-adjoint Fredholm operators is connected. More precisely we have:

Theorem 1.10

(a) *The set $\mathcal{CF}^{\mathrm{sa}}$ is path-connected with respect to the gap metric.*
(b) *Moreover, its Cayley image*

$${}_{\mathcal{F}}\mathcal{U}_{\mathrm{inj}} := \big\{ U \in \mathcal{U} \bigm| U + I \textit{ Fredholm and } U - I \textit{ injective} \big\} = \boldsymbol{\kappa}(\mathcal{CF}^{\mathrm{sa}})$$

is dense in ${}_{\mathcal{F}}\mathcal{U}$.

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Proof (a) Once again we look at the Cayley transform picture. We shall use the following notation:

$$\mathcal{U}_{\mathrm{inj}} := \left\{ U \in \mathcal{U} \;\middle|\; U - I \text{ injective} \right\} = \boldsymbol{\kappa}(\mathcal{C}^{\mathrm{sa}}).$$

Note that ${}_{\mathcal{F}}\mathcal{U}_{\mathrm{inj}} = {}_{\mathcal{F}}\mathcal{U} \cap \mathcal{U}_{\mathrm{inj}}$. We consider a fixed $U \in {}_{\mathcal{F}}\mathcal{U}_{\mathrm{inj}}$. Then H is the direct sum of the spectral subspaces $H_{\pm}$ of U corresponding to $[0,\pi)$ and $[\pi, 2\pi]$ respectively and we may decompose $U = U_{+} \oplus U_{-}$. More precisely, we have

$$\operatorname{spec}(U_{+}) \subset \left\{ e^{it} \;\middle|\; t \in [0,\pi) \right\} \quad \text{and} \quad \operatorname{spec}(U_{-}) \subset \left\{ e^{it} \;\middle|\; t \in [\pi, 2\pi] \right\}.$$

Note that there is no intersection between the spectral spaces in the endpoints: if -1 belongs to $\operatorname{spec}(U)$, it is an isolated eigenvalue by our assumption and hence belongs only to $\operatorname{spec}(U_{-})$; if 1 belongs to $\operatorname{spec}(U)$, it can belong both to $\operatorname{spec}(U_{+})$ and $\operatorname{spec}(U_{-})$, but in any case, it does not contribute to the decomposition of U since, by our assumption, 1 is not an eigenvalue at all.

By spectral deformation (squeezing the spectrum down to $+i$ and $-i$) we contract U_{+} to iI_{+} and U_{-} to $-iI_{-}$, where $I_{\pm}$ denotes the identity on $H_{\pm}$. We do this on the upper half arc and the lower half arc, respectively, in such a way that 1 does not become an eigenvalue under the course of the deformation: actually it will no longer belong to the spectrum; neither will -1 belong to the spectrum. That is, we have connected U and $iI_{+} \oplus -iI_{-}$ within $\boldsymbol{\kappa}(\mathcal{CF}^{\mathrm{sa}})$.

We distinguish two cases: If H_{-} is finite-dimensional, we now rotate $-iI_{-}$ up through -1 into iI_{-}. More precisely, we consider $\{iI_{+} \oplus e^{i(\pi/2+(1-t)\pi)} I_{-}\}_{t\in[0,1]}$. This proves that we can connect U with $iI_{+} \oplus iI_{-} = iI$ within $\boldsymbol{\kappa}(\mathcal{CF}^{\mathrm{sa}})$ in this first case.

If H_{-} is infinite-dimensional, we "un-contract" $-iI_{-}$ in such a way that no eigenvalues remain. To do this, we identify H_{-} with $L^2([0,1])$. Now multiplication by $-i$ on $L^2([0,1])$ can be connected to multiplication by a function whose values are a short arc centred on $-i$ and so that the resulting operator V_{-} on H_{-} has no eigenvalues. This will at no time introduce spectrum near $+1$ or -1. We then rotate this arc up through $+1$ (which keeps us in the right space) until it is centred on $+i$. Then we contract the spectrum on H_{-} to be $+i$. That is, also in this case we have connected our original operator U to $+iI$. To sum up this second case (see also Figure 1):

$$\begin{aligned} U &\sim iI_{+} \oplus -iI_{-} \sim iI_{+} \oplus V_{-} \sim iI_{+} \oplus e^{it\pi} V_{-} \quad \text{for } t \in [0,1] \\ &\sim iI_{+} \oplus -V_{-} \sim iI_{+} \oplus -(-iI_{-}) \sim iI. \end{aligned}$$

To prove (b), we just decompose any $V \in {}_{\mathcal{F}}\mathcal{U}$ into $V = U \oplus I_1$ where $U \in {}_{\mathcal{F}}\mathcal{U}_{\mathrm{inj}}(H_0)$ and I_1 denotes the identity on the 1-eigenspace $H_1 = \ker(V - I)$ of V with $H = H_0 \oplus H_1$ an orthogonal decomposition. Then for $\varepsilon > 0$, $U \oplus e^{i\varepsilon} I_1 \in {}_{\mathcal{F}}\mathcal{U}_{\mathrm{inj}}$ approaches U for $\varepsilon \to 0$. ∎

Remark 1.11 Recall that $\mathcal{F}^{\mathrm{sa}}$ has three connected components

$$\mathcal{F}^{\mathrm{sa}}_{\pm} = \left\{ T \in \mathcal{F}^{\mathrm{sa}} \;\middle|\; \operatorname{spec}_{\mathrm{ess}}(T) \subset \mathbb{R}_{\pm} \right\},$$

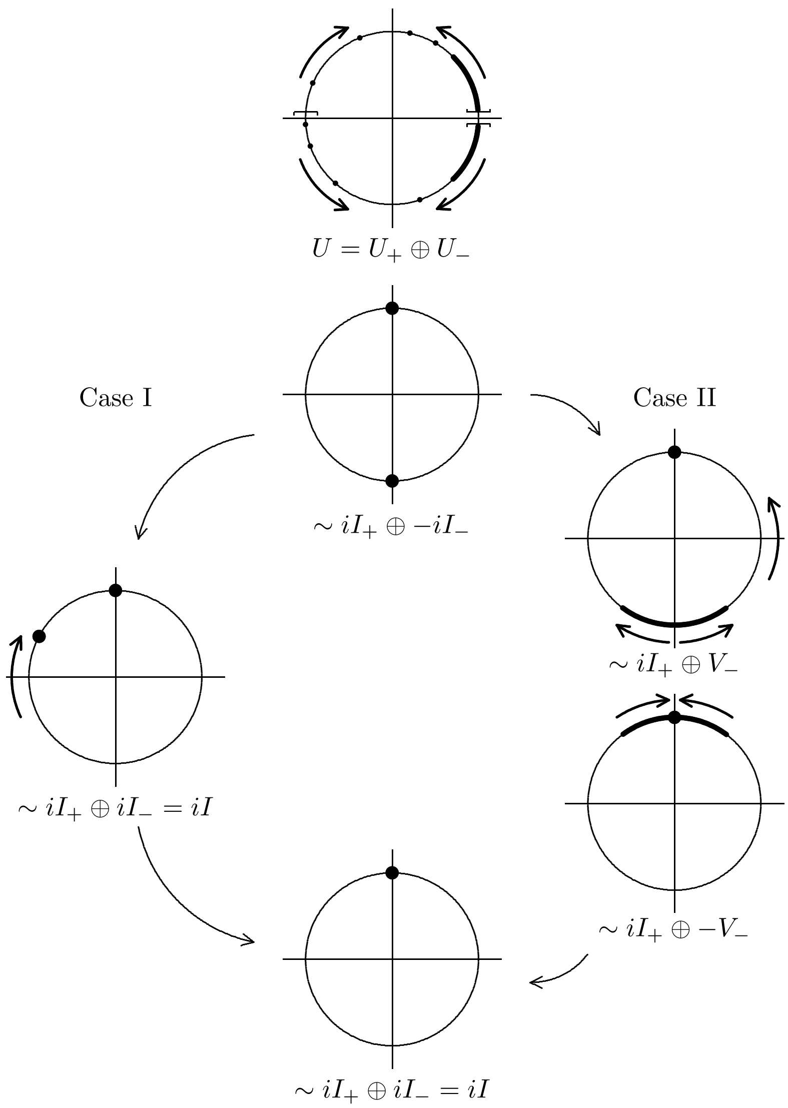

Figure 1: Connecting a fixed U in ${}_{\mathcal{F}}\mathcal{U}_{\text{inj}}$ to iI. Case I (finite rank U_-) and Case II (infinite rank U_-)

and $\mathcal{F}^{\mathrm{sa}}_* = \mathcal{F}^{\mathrm{sa}} \setminus \left(\mathcal{F}^{\mathrm{sa}}_+ \cup \mathcal{F}^{\mathrm{sa}}_-\right)$. Now $\mathcal{F}^{\mathrm{sa}}_\pm$ are contractible and $\mathcal{F}^{\mathrm{sa}}_*$ is a classifying space for the K^1-functor [1]. Recall that K-theory is a generalized cohomology theory; a classifying space for K^1 is a topological space Z such that $K^1(X)$ is naturally isomorphic to the homotopy classes, $[X, Z]$, of maps $X \to Z$.

The preceding proof shows also that the two subsets of $\mathcal{CF}^{\mathrm{sa}}$

$$\mathcal{CF}^{\mathrm{sa}}_\pm = \left\{ T \in \mathcal{CF}^{\mathrm{sa}} \;\middle|\; \operatorname{spec}_{\mathrm{ess}}(T) \subset \mathbb{R}_\pm \right\},$$

the spaces of all essentially positive, resp., all essentially negative, self-adjoint Fredholm operators, are no longer open. The third of the three complementary subsets

$$\mathcal{CF}^{\mathrm{sa}}_* = \mathcal{CF}^{\mathrm{sa}} \setminus \left(\mathcal{CF}^{\mathrm{sa}}_+ \cup \mathcal{CF}^{\mathrm{sa}}_-\right) \tag{1.13}$$

is also not open. We do not know whether the two "trivial" components are contractible as in the bounded case nor whether the whole space is a classifying space for K^1 as is the non-trivial component in the bounded case.

Independently of the Fuglede example, the connectedness of $\mathcal{CF}^{\mathrm{sa}}$ and the non-connectedness of $\mathcal{F}^{\mathrm{sa}}$ show that the Riesz map is not continuous on $\mathcal{CF}^{\mathrm{sa}}$ in the gap topology.

2 Spectral Flow for Unbounded Self-Adjoint Operators

2.1 First Approach, Via Cayley Transform and Winding Number

In [13, Section 6] it was shown that the natural inclusion

$$\mathcal{U}_{\mathcal{K}}(H) := \{U \in \mathcal{U} \mid U - I \text{ is compact}\} \hookrightarrow {}_{\mathcal{F}}\mathcal{U}(H) := \{U \in \mathcal{U} \mid -1 \notin \operatorname{spec}_{\mathrm{ess}} U\}$$

is a homotopy equivalence. As a consequence the classical winding number extends to an isomorphism

$$\operatorname{wind} : \pi_1({}_{\mathcal{F}}\mathcal{U}, I) \to \mathbb{Z}, \tag{2.1}$$

see also [9, Appendix] for a different proof (*cf.* also Proposition 2.5 below).

Furthermore, in [13, l.c.] it was shown that to any continuous (not necessarily closed) curve $f\colon [0,1] \to {}_{\mathcal{F}}\mathcal{U}$ one can assign an integer $\operatorname{wind}(f)$ in such a way that the mapping wind is

(1) *Path additive*: Let $f_1, f_2\colon [0,1] \to {}_{\mathcal{F}}\mathcal{U}(H)$ be continuous paths with $f_2(0) = f_1(1)$. Then $\operatorname{wind}(f_1 * f_2) = \operatorname{wind}(f_1) + \operatorname{wind}(f_2)$.

(2) *Homotopy invariant*: Let f_1, f_2 be continuous paths in ${}_{\mathcal{F}}\mathcal{U}$. Assume that there is a homotopy $H\colon [0,1] \times [0,1] \to {}_{\mathcal{F}}\mathcal{U}$ such that $H(0,t) = f_1(t), H(1,t) = f_2(t)$ and such that $\dim\ker(H(s,0)+I)$, $\dim\ker(H(s,1)+I)$ are independent of s. Then $\operatorname{wind}(f_1) = \operatorname{wind}(f_2)$. In particular, wind is invariant under homotopies leaving the endpoints fixed.

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Roughly speaking, the mapping wind is the "spectral flow" across -1; that is, wind counts the net number of eigenvalues of $f(t)$ which cross -1 from the upper half-plane into the lower half-plane. One has to choose a convention for those cases in which $-1 \in \operatorname{spec} f(0)$ or $-1 \in \operatorname{spec} f(1)$. Contrary to the convention which was chosen in [13], our convention is chosen as follows: choose $\varepsilon > 0$ so small that $-1 \notin \operatorname{spec}(f(j)e^{i\varphi})$, $j = 0, 1$ for all $0 < |\varphi| \le \varepsilon$. Then put $\operatorname{wind}(f) := \operatorname{wind}(fe^{i\varepsilon})$. This means that an eigenvalue running from the lower half-plane into -1 is not counted while an eigenvalue running from the upper half-plane into -1 contributes 1 to the winding number.

In analogy to [16] we can give an explicit description of $\operatorname{wind}(f)$. Alternatively, it can be used as a definition of wind:

Proposition 2.1 *Let $f\colon [0,1] \to {}_{\mathcal{F}}\mathcal{U}$ be a continuous path.*

(a) *There is a partition $\{0 = t_0 < t_1 < \cdots < t_n = 1\}$ of the interval and positive real numbers $0 < \varepsilon_j < \pi$, $j = 1, \dots, n$, such that $\ker(f(t) - e^{i(\pi \pm \varepsilon_j)}) = \{0\}$ for $t_{j-1} \le t \le t_j$.*

(b) *Then*

$$\operatorname{wind}(f) = \sum_{j=1}^{n} k(t_j, \varepsilon_j) - k(t_{j-1}, \varepsilon_j), \tag{2.2}$$

where

$$k(t, \varepsilon_j) := \sum_{0 \le \theta < \varepsilon_j} \dim \ker(f(t) - e^{i(\pi+\theta)}).$$

(c) *In particular, this calculation of $\operatorname{wind}(f)$ is independent of the choice of the segmentation of the interval and of the choice of the barriers, ε_j.*

Proof In (a) we use that $f(t) \in {}_{\mathcal{F}}\mathcal{U}$ and f continuous. (b) follows from the path additivity of wind. (c) is immediate from (b). ∎

This idea of a *spectral flow across* -1 was introduced first in [3, Section 1.3], where it was used to give a definition of the Maslov index in an infinite dimensional context.

After these explanations the definition of spectral flow for paths in $\mathcal{CF}^{\mathrm{sa}}$ is straightforward:

Definition 2.2 Let $f\colon [0,1] \to \mathcal{CF}^{\mathrm{sa}}(H)$ be a continuous path. Then the *spectral flow* of f, $\operatorname{SF}(f)$ is defined by

$$\operatorname{SF}(f) := \operatorname{wind}(\boldsymbol{\kappa} \circ f).$$

From the properties of $\boldsymbol{\kappa}$ and of the winding number we infer immediately:

Proposition 2.3 SF *is path additive and homotopy invariant in the following sense: let $f_1, f_2\colon [0,1] \to \mathcal{CF}^{\mathrm{sa}}$ be continuous paths and let*

$$H : [0,1] \times [0,1] \to \mathcal{CF}^{\mathrm{sa}}$$

be a homotopy such that $H(0,t) = f_1(t), H(1,t) = f_2(t)$ *and such that* $\dim\ker H(s,0)$, $\dim\ker H(s,1)$ *are independent of* s. *Then* $\mathrm{SF}(f_1) = \mathrm{SF}(f_2)$. *In particular,* SF *is invariant under homotopies leaving the endpoints fixed.*

In [14] it is shown that SF is the only integer-valued homotopy invariant of paths in $\mathcal{CF}^{\mathrm{sa}}$ satisfying standard properties.

From Proposition 2.1 we get:

Proposition 2.4 *For a continuous path* $f\colon [0,1] \to \mathcal{F}^{\mathrm{sa}}$ *our definition of spectral flow coincides with the definition in* [16].

Note that also the conventions coincide for $0 \in \operatorname{spec} f(0)$ or $0 \in \operatorname{spec} f(1)$.

Returning to the Cayley picture, we have that the mapping wind induces a surjection of $\pi_1({}_{\mathcal{F}}\mathcal{U}_{\mathrm{inj}})$ onto $\mathbb{Z}$. Because $\mathbb{Z}$ is free, there is a right inverse of wind and a normal subgroup G of $\pi_1({}_{\mathcal{F}}\mathcal{U}_{\mathrm{inj}})$ such that we have a split short exact sequence

$$0 \to G \to \pi_1({}_{\mathcal{F}}\mathcal{U}_{\mathrm{inj}}) \to \mathbb{Z} \to 0. \tag{2.3}$$

For now, an open question is whether G is trivial: does the winding number distinguish the homotopy classes? That is, the question is whether each loop with winding number 0 can be contracted to a constant point, or, equivalently, whether two continuous paths in $\mathcal{CF}^{\mathrm{sa}}$ with same endpoints and with same spectral flow can be deformed into each other? Or is $\pi_1({}_{\mathcal{F}}\mathcal{U}_{\mathrm{inj}}) \cong \mathbb{Z} \rtimes G$ the semi-direct product of a non-trivial factor G with $\mathbb{Z}$? We know a little more than (2.3):

Proposition 2.5 *There exists a continuous map* ${}_{\mathcal{F}}\mathcal{U} \to \mathcal{U}_\infty$ *which induces an isomorphism* $\pi_1({}_{\mathcal{F}}\mathcal{U}) \to \pi_1(\mathcal{U}_\infty) = \mathbb{Z}$. *Moreover, the restriction of this map to* ${}_{\mathcal{F}}\mathcal{U}_{\mathrm{inj}}$ *induces a map such that the following diagram commutes:*

$$\begin{tikzcd} \pi_1({}_{\mathcal{F}}\mathcal{U}_{\mathrm{inj}}) \arrow[r] \arrow[dr, "\mathrm{wind}"'] & \pi_1(\mathcal{U}_\infty) \arrow[d, "\mathrm{wind}", "\cong"'] \\ & \mathbb{Z} \end{tikzcd} \tag{2.4}$$

Proof Let $U_0 \in {}_{\mathcal{F}}\mathcal{U}$. Then there exists a neighbourhood N_{ε_0} of U_0 in ${}_{\mathcal{F}}\mathcal{U}$ and $\varepsilon_0 > 0$ such that for each $U \in N_{\varepsilon_0}$ the projection $\chi_{\varepsilon_0}(U)$ has finite rank where χ_{ε_0} denotes the characteristic function of the arc $\{e^{it} \mid t \in [\pi - \varepsilon_0, \pi + \varepsilon_0]\}$ of the unit circle $\mathbb{T}$. Now, there is a continuous function $f_{\varepsilon_0} : \mathbb{T} \to \mathbb{T}$ such that:

$$f_{\varepsilon_0}(z) = \begin{cases} z & \text{for } z \in \{e^{it} \mid t \in [\pi - \frac{\varepsilon_0}{2}, \pi + \frac{\varepsilon_0}{2}]\} \\ 1 & \text{for } z \in \{e^{it} \mid t \in [0, \pi - \varepsilon_0] \cup [\pi + \varepsilon_0, 2\pi]\} \end{cases}$$

with

$$f_{\varepsilon_0} : \begin{cases} \{e^{it} \mid t \in [\pi - \varepsilon_0, \pi - \frac{\varepsilon_0}{2}]\} \to \{e^{it} \mid t \in [0, \pi - \varepsilon_0]\} \text{ is injective} \\ \{e^{it} \mid t \in [\pi + \frac{\varepsilon_0}{2}, \pi + \varepsilon_0]\} \to \{e^{it} \mid t \in [\pi + \varepsilon_0, 2\pi]\} \text{ is injective.} \end{cases}$$

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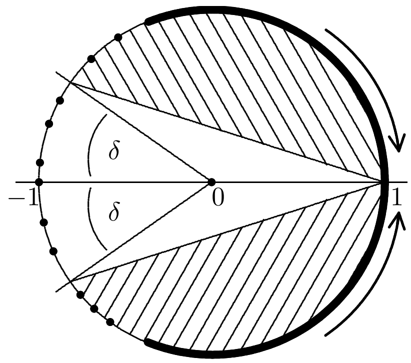

Figure 2: Convex regions of finite linear combinations

Then, actually, $U \mapsto f_{\varepsilon_0}(U) : N_{\varepsilon_0} \to \mathcal{U}_\infty$!

Since ${}_{\mathcal{F}}\mathcal{U}$ is metric, it is paracompact and so the open cover $\{N_{\varepsilon_0}(U)\}$ has an open locally finite refinement, say $\{N_\alpha\}$, and each N_α carries a function $f_\alpha \colon N_\alpha \to \mathcal{U}_\infty$ given by a function $f_\alpha \colon \mathbb{T} \to \mathbb{T}$ corresponding to a positive ε_0. We let $\{p_\alpha\}$ be a partition of unity subordinate to the cover. Then $f \colon {}_{\mathcal{F}}\mathcal{U} \to \mathcal{B}(H)$ is continuous where $f(U) := \sum_\alpha p_\alpha(U) f_\alpha(U)$. We claim that $f(U)$ is normal and invertible so that $g(U) = f(U)|f(U)|^{-1}$ is unitary. To see this, we observe that for each single U we have $f(U) = \sum_{i=1}^n \lambda_i f_{\alpha_i}(U)$ with the f_{α_i} as above. Moreover, if we let δ denote the minimum of the corresponding $\{\frac{1}{2}\varepsilon_{\alpha_i}\}$ then $h = \sum_{i=1}^n \lambda_i f_{\alpha_i}$ satisfies

$$h(z) = z \quad \text{for all } z \in \{e^{it} \mid t \in [\pi - \delta, \pi + \delta]\}, \tag{2.5}$$

$$h(z) = 1 \quad \text{for all } z \in \{e^{it} \mid t \in [0, \Delta] \cup [2\pi - \Delta, 2\pi]\}, \tag{2.6}$$

where $\Delta = \max \varepsilon_{\alpha_i} > 0$ and $\chi_{[\Delta, 2\pi - \Delta]}(U)$ is of finite rank; $h(z)$ lies in one of the shaded convex regions of Figure 2 for all other z on the circle.

Thus, $f(U) = h(U)$ is normal and invertible. Moreover, since each

$$f_{\alpha_i}(U) \in \mathcal{U}_\infty \subset \{I + \text{ finite rank operators}\},$$

$f(U)$ is in $\{I +$ finite rank operators$\}$ so that $g(U) = f(U)|f(U)|^{-1}$ is in $\mathcal{U}_\infty$. Moreover, clearly $\chi_\delta(U) = \chi_\delta(g(U))$ and so we get the commuting diagram (as the covering is neighbourhood-finite we get $\chi_\delta(V) = \chi_\delta(g(V))$ for V in a neighbourhood of U. ■

Summing up, it remains an open problem to determine the fundamental group of the space $\mathcal{CF}^{\mathrm{sa}}(H)$ or, even more, to determine whether, as in the bounded case, it is a classifying space for K^1.

Robbin and Salamon [18] introduced the spectral flow for a family of unbounded self-adjoint operators under the assumption that the domain is fixed and that each operator of the family has a compact resolvent. Along the lines of their method one can prove the following generalization of [18, Theorem 4.25]:

Proposition 2.6 *Let $f\colon [0,1] \to \mathcal{CF}^{\mathrm{sa}}(H)$ be a closed continuous path. Then there is a continuous path of self-adjoint matrices $g\colon [0,1] \to \mathrm{Mat}(n,\mathbb{C})$ such that $f \oplus g$ is homotopic to a closed continuous path of* invertible *operators*

$$h\colon [0,1] \to \mathcal{CF}^{\mathrm{sa}}(H \oplus \mathbb{C}^n).$$

If h were a family of bounded invertible operators then it would be clear that it is homotopic to a constant path. Unfortunately, this is not clear for a path of unbounded operators. If we could conclude that h is homotopic to a constant path, then we would know at least that the "stable" fundamental group of $\mathcal{CF}^{\mathrm{sa}}(H)$ is isomorphic to $\mathbb{Z}$.

2.2 Second Approach, after [16]

There is another way of looking at continuous curves of self-adjoint Fredholm operators which more closely resembles what is done in the bounded self-adjoint setting. The fact that one can (continuously) isolate the spectra of the unbounded Fredholm operators in an open interval about 0 is quite appealing from an operator algebra point of view: it is surprising that this can be done without the Riesz map being continuous. Therefore both approaches are included in this note.

In [16] the third author introduced a new method to define spectral flow of a continuous family of bounded operators. The interesting new feature of his approach was that it works directly for any continuous family without first changing the family to a generic situation (see also Proposition 2.1 above).

In this subsection we adapt the method of [16] to unbounded operators.

Lemma 2.7 *Let $K \subset \mathbb{C}$ be a compact set. Then $\{ T \in \mathcal{C}^{\mathrm{sa}} \mid K \subset \varrho(T) \}$ is open in the gap topology. Here, $\varrho(T) := \mathbb{C} \setminus \operatorname{spec} T$ denotes the resolvent set of T.*

Similarly, $\{ T \in \mathcal{C}^{\mathrm{sa}} \mid K \subset \varrho_{\mathrm{ess}}(T) \}$, $\varrho_{\mathrm{ess}}(T) := \mathbb{C} \setminus \operatorname{spec}_{\mathrm{ess}}(T)$, is open in the gap topology.

Proof In view of Theorem 1.1 we find

$$\begin{aligned} \{ T \in \mathcal{C}^{\mathrm{sa}} \mid K \subset \varrho(T) \} &= \{ T \in \mathcal{C}^{\mathrm{sa}} \mid \operatorname{spec} T \subset K^c \cap \mathbb{R} \} \\ &= \{ T \in \mathcal{C}^{\mathrm{sa}} \mid \operatorname{spec} \boldsymbol{\kappa}(T) \subset \kappa(K^c \cap \mathbb{R}) \cup \{1\} \} \\ &= \boldsymbol{\kappa}^{-1} \{ U \in \mathcal{U} \mid \operatorname{spec} U \subset \kappa(K^c \cap \mathbb{R}) \cup \{1\} \}. \end{aligned} \tag{2.7}$$

Since K is compact the set $\kappa(K^c \cap \mathbb{R}) \cup \{1\}$ is open. Consequently

$$\{ U \in \mathcal{U} \mid \operatorname{spec} U \subset \kappa(K^c \cap \mathbb{R}) \cup \{1\} \}$$

is open and since $\boldsymbol{\kappa}$ is a homeomorphism we reach the first conclusion.

The proof for $\varrho_{\mathrm{ess}}(T)$ instead of $\varrho(T)$ proceeds along the same lines. ∎

Lemma 2.8 *Let $K \subset \mathbb{C}$ be a compact set and let $\Omega := \left\{ T \in \mathcal{C}^{\mathrm{sa}} \;\middle|\; K \subset \varrho(T) \right\}$ be equipped with the gap topology. Then the map $R\colon K \times \Omega \to \mathcal{B}, (\lambda, T) \mapsto (T - \lambda)^{-1}$ is continuous.*

Proof For $(\lambda, T) \in K \times \Omega$ we have

$$
\begin{aligned}
R(\lambda, T) = (T-\lambda)^{-1} &= (I - (i+\lambda)(T+i)^{-1})^{-1}(T+i)^{-1} \\
&=: F(\lambda, (T+i)^{-1}) =: F \circ G(\lambda, T).
\end{aligned} \tag{2.8}
$$

In view of Theorem 1.1 the map

$$
\begin{aligned}
G\colon K \times \Omega \to K \times \left\{ S \in \mathcal{B}^{\mathrm{sa}} \;\middle|\; (K+i)^{-1} \subset \varrho(S) \right\}, \\
(\lambda, T) \mapsto (\lambda, (T+i)^{-1})
\end{aligned} \tag{2.9}
$$

is continuous. Furthermore, the map

$$
\begin{aligned}
F\colon K \times \left\{ S \in \mathcal{B} \;\middle|\; (K+i)^{-1} \subset \rho(S) \right\} \to \mathcal{B}, \\
(\lambda, S) \mapsto (I - (i+\lambda)S)^{-1} S
\end{aligned} \tag{2.10}
$$

is continuous. This proves the assertion. ∎

Lemma 2.9 *Let $a < b$ be real numbers. Then the set*

$$\Omega_{a,b} := \left\{ T \in \mathcal{C}^{\mathrm{sa}} \;\middle|\; a, b \notin \operatorname{spec} T \right\}$$

is open in the gap topology and the map

$$\Omega_{a,b} \to \mathcal{B}, \quad T \mapsto 1_{[a,b]}(T)$$

is continuous.

Proof That $\Omega_{a,b}$ is open follows from Proposition 1.7. Next, denote by Γ the circle of radius $(b-a)/2$ and centre $(a+b)/2$. Then

$$1_{[a,b]}(T) = \frac{1}{2\pi i} \int_\Gamma (\lambda - T)^{-1}\, d\lambda. \tag{2.11}$$

The assertion now follows from Lemma 2.8. ∎

We collect what we have so far:

Proposition 2.10 *Fix $T_0 \in \mathcal{CF}^{\mathrm{sa}}$.*

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(a) *Then there is a positive number a and an open neighbourhood* $\mathcal{N} \subset \mathcal{CF}^{sa}$ *of* T_0 *in the gap topology such that the map*

$$\mathcal{N} \to \mathcal{B}, \quad T \mapsto 1_{[-a,a]}(T)$$

is continuous and finite-rank projection-valued, and hence $T \mapsto T1_{[-a,a]}(T)$ *is also continuous. (We may as well assume the rank to be constant).*

(b) *If* $-a \le c < d \le a$ *are points such that* $c, d \notin \operatorname{spec}(T)$ *for all* $T \in \mathcal{N}$*, then the map* $T \mapsto 1_{[c,d]}(T)$ *is continuous on* $\mathcal{N}$ *and has finite rank on* $\mathcal{N}$*. Of course, on any connected subset of* $\mathcal{N}$ *this rank is constant.*

Proof $T_0 \in \mathcal{CF}^{sa}$ is equivalent to $0 \notin \operatorname{spec}_{ess}(T_0)$. Thus either $0 \notin \operatorname{spec} T_0$ or 0 is an isolated point of $\operatorname{spec} T_0$ and an eigenvalue of finite multiplicity. Hence there is an $a > 0$ such that $\operatorname{spec} T_0 \cap [-a, a] \subset \{0\}$. By Lemma 2.7 the set

$$\mathcal{N} := \left\{ T \in \mathcal{C}^{sa} \;\middle|\; [-a, a] \subset \varrho_{ess}(T), \text{ and } \pm a \notin \operatorname{spec}(T) \right\} \tag{2.12}$$

is open in the gap topology and the map $T \mapsto 1_{[-a,a]}(T)$ is continuous by Lemma 2.9. Moreover, $\mathcal{N} \subset \mathcal{CF}^{sa}$ and $1_{[-a,a]}(T)$ is of finite rank. This follows from the fact that $[-a, a] \subset \varrho_{ess}(T)$. This proves (a). Now (b) follows from Lemma 2.9. ■

Remark 2.11 The preceding proposition is a precise copy of the corresponding result for norm-continuous curves of bounded self-adjoint Fredholm operators. It explains why, after all, spectral flow of gap-topology continuous curves of (possibly unbounded) self-adjoint Fredholm operators can be defined in precisely the same way as in the bounded case and with the same properties. In substance, the proposition was announced in [5, p. 140] without proof but with reference to [12, IV.3.5] (the continuity of a finite system of eigenvalues).

Now we proceed exactly as in [16, p. 462]. We strive for almost literal repetition to emphasize the analogy (and the differences wherever they occur) between the bounded and the unbounded case.

First a notation: if E is a finite-rank spectral projection for a self-adjoint operator T, let $E^{\geq}$ denote the projection on the subspace of $E(H)$ spanned by those eigenvectors for T in $E(H)$ having non-negative eigenvalues.

Definition 2.12 Let $f\colon [0, 1] \to \mathcal{CF}^{sa}(H)$ be a continuous path. By compactness and the previous proposition, choose a partition, $\{0 = t_0 < t_1 < \cdots < t_n = 1\}$ of the interval and positive real numbers ε_j, $j = 1, \ldots, n$ such that for each $j = 1, 2, \ldots, n$ the function $t \mapsto E_j(t) := 1_{[-\varepsilon_j,\varepsilon_j]}(f(t))$ is continuous and of finite rank on $[t_{j-1}, t_j]$. We redefine the *spectral flow* of f, $\operatorname{SF}(f)$ to be

$$\sum_{j=1}^{n} \left(\dim\left(E_j^{\geq}(t_j)\right) - \dim\left(E_j^{\geq}(t_{j-1})\right) \right).$$

By definition, spectral flow is path additive when defined this way, and we obtain in exactly the same way as in [16]:

Proposition 2.13 *Spectral flow is well defined; that is, it depends only on the continuous mapping* $f\colon [0,1] \to \mathcal{CF}^{\mathrm{sa}}$.

Propositions 2.10 and 2.13 show that pathological examples like piecewise linear curves of self-adjoint unbounded Fredholm operators with infinitely fast oscillating spectrum and hence without well-defined spectral flow are excluded; more precisely, they can not be continuous in the gap topology.

Example 2.14 Let H be a separable Hilbert space and $\{e_k\}_{k\in\mathbb{N}}$ be a complete orthonormal system. Consider the multiplication operator which is defined by

$$T_0\colon \operatorname{domain}(T_0) \to H, \quad \sum_k a_k e_k \mapsto \sum_k k a_k e_k$$

with $\operatorname{domain}(T_0) = \left\{ \sum_k a_k e_k \;\middle|\; \sum_k k^2 |a_k|^2 < +\infty \right\}$. Then T_0 is self-adjoint and invertible and so $T_0 \in \mathcal{CF}^{\mathrm{sa}}$. Set

$$P_n\colon H \to H, \quad e_k \mapsto \begin{cases} k e_k & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

Then the sequence of unbounded self-adjoint Fredholm operators $\{T_n := T_0 - 2P_n\}_{n\in\mathbb{N}}$ converges to T_0 for $n \to \infty$ in the gap topology. To see this, we apply Theorem 1.1(a) and get

$$(2.13)\quad \gamma(T_n, T_0) = \left\| (T_n + iI)^{-1} - (T_0 + iI)^{-1} \right\| = \left| \frac{1}{i-n} - \frac{1}{i+n} \right| \to 0 \text{ for } n \to \infty.$$

For the Riesz transformation we note, however, that

$$\|F_{T_n} e_n - F_{T_0} e_n\| = \left| \frac{2n}{\sqrt{1+n^2}} \right| \to 2 \text{ for } n \to \infty.$$

This is the aforementioned Fuglede example. Clearly the full spectrum (*i.e.*, the parts which are increasingly remote from 0) does not change continuously for $n \to \infty$. The corresponding linear interpolations $(1-t)T_n + tT_{n+1}$ all belong to $\mathcal{CF}^{\mathrm{sa}}$ and have rapidly oscillating spectrum also near 0, hence the piecewise linear curve may not be continuous in the gap topology by the previous proposition; and it is not, as clearly seen by Theorem 1.1(a). We find, *e.g.*,

$$\begin{aligned} \gamma\Big(\frac{1}{2}T_n + \frac{1}{2}T_{n+1}, T_0\Big) &\geq \left\| \left(\left(\frac{1}{2}T_n + \frac{1}{2}T_{n+1} + i \right)^{-1} - \left(T_0 + i\right)^{-1} \right) e_n \right\| \\ &= \left| \frac{1}{i} - \frac{1}{i+n} \right| \to 1 \text{ for } n \to \infty. \end{aligned}$$

The example also shows that it is unlikely that the Cayley image ${}_{\mathcal{F}}\mathcal{U}_{\mathrm{inj}}$ of $\mathcal{CF}^{\mathrm{sa}}$ can be retracted to the subspace where 1 does not belong to the spectrum at all (that is, the image of $\mathcal{F}^{\mathrm{sa}}$ in ${}_{\mathcal{F}}\mathcal{U}_{\mathrm{inj}}$). Differently put, it shows that the eigenvalues of the Cayley transforms flip around +1 in the same way that the eigenvalues of the operators in $\mathcal{CF}^{sa}$ flip around $\pm\infty$. More precisely, consider the sequence of Cayley transforms $U_n := \boldsymbol{\kappa}(T_n) \in {}_{\mathcal{F}}\mathcal{U}_{\mathrm{inj}}$. The spectrum of U_n consists of discrete eigenvalues which all lie in the lower half-plane except one in the upper-half plane with a corresponding hole in the lower half-plane sequence, plus the accumulation point 1 where $U_n - I$ is injective, but not invertible. The same is true for $U_0 := \boldsymbol{\kappa}(T_0)$, but now having all eigenvalues in the lower half plane. By (2.13) the sequence $\{U_n\}_{n\in\mathbb{N}}$ converges to U_0 in ${}_{\mathcal{F}}\mathcal{U}_{\mathrm{inj}}$. We see that the eigenvalues of the sequence flip from the upper half-plane to the lower half-plane close to +1 without actually crossing +1. It seems, however, unlikely that there is a continuous path from U_1 to U_0 which avoids any crossing.

Note that the linear path from T_0 to T_1 is continuous and has SF equal to -1. The corresponding curve from U_0 to U_1 has one crossing at -1 from the lower half-plane to the upper one.

Remark 2.15 So far we have established that spectral flow based on the approach in [16], *i.e.,* Definition 2.12, is well defined for gap continuous paths of self-adjoint Fredholm operators. To do this we have repeatedly used the local continuity proposition (Proposition 2.10) for continuous families in the gap topology. The surprising fact is that this same local continuity proposition suffices to prove the homotopy invariance. Initially, this may sound a little counter-intuitive since we admit varying domains for our operators and therefore might not expect nice parametrizations of the spectrum for these perturbations.

Of course it would suffice to show that Definition 2.12 coincides with the previous definition based on the Cayley transform and the winding number (Definition 2.2). Then, the homotopy invariance of Definition 2.12 would follow from Proposition 2.4 which is based on general topological arguments. We prefer, however, to emphasize the existence of a self-contained proof based only on Definition 2.2 and Proposition 2.10.

Proposition 2.16 *Spectral flow as defined in Definition* 2.12 *is homotopy invariant.*

Proof As in [16]. ■

As a direct consequence of Proposition 2.1 we obtain:

Proposition 2.17 *Spectral flows as defined in Definitions* 2.2 *and* 2.12 *coincide.*

Remark 2.18 In spite of the density of $\mathcal{F}^{sa}$ in $\mathcal{CF}^{sa}$ (Remark 1.9) not any gap continuous path in $\mathcal{CF}^{sa}$ with endpoints in $\mathcal{F}^{sa}$ can be continuously deformed into an operator norm continuous path in $\mathcal{F}^{sa}$. One reason is that the one space is connected, but not the other by Theorem 1.10(a).

3 Operator Curves on Manifolds With Boundary

In low-dimensional topology and quantum field theory, various examples of operator curves appear which take their departure in a symmetric elliptic differential operator of first order (usually an operator of Dirac type) on a fixed compact Riemannian smooth manifold M with boundary Σ. Posing a suitable well-posed boundary value problem provides for a nicely spaced discrete spectrum near 0. Then, varying the coefficients of the differential operator and the imposed boundary condition suggests the use of the powerful topological concept of spectral flow. In this Section we show under which conditions the curves of the induced self-adjoint L^2-extensions become continuous curves in $\mathcal{CF}^{\mathrm{sa}}(L^2(M;E))$ in the gap topology such that their spectral flow is well defined and truly homotopy invariant.

3.1 Notation and Basic Facts

We fix the notation and recall basic facts, partially following [5, 10].

Let $D\colon C^\infty(M;E) \to C^\infty(M;E)$ be an elliptic symmetric (*i.e.,* formally self-adjoint) first order differential operator on M acting on sections of a Hermitian vector bundle E. Different from the case of closed manifolds, now D is no longer essentially self-adjoint and $\ker D$ is infinite dimensional and varies with the regularity of the underlying Sobolev space. Among the many extensions of D to a closed operator in $L^2(M;E)$, we recall first the definition of the *minimal* and the *maximal* closed extension with

$$\operatorname{domain}(D^{\min}) = \overline{\left\{ u \in C^\infty(M;E) \;\middle|\; \operatorname{supp} u \subset M \setminus \Sigma \right\}}^{H^1(M;E)} \quad \text{and}$$
$$\operatorname{domain}(D^{\max}) = \left\{ u \in L^2(M;E) \;\middle|\; Du \in L^2(M;E) \right\}.$$

Now we make three basic (mutually related) assumptions:

Assumptions 3.1

(1) The operator D takes the form

$$D|_U = \sigma(y,\tau)\Big(\frac{\partial}{\partial\tau} + A_\tau + B_\tau\Big) \tag{3.1}$$

in a bi-collar $U = \Xi \times [-\varepsilon,\varepsilon]$ of any hypersurface $\Xi \subset M \setminus \Sigma$, and a similar form in a collar of Σ, where

$$\sigma(\,\cdot\,,\tau), A_\tau, B_\tau : C^\infty(\Xi_\tau; E|_{\Xi_\tau}) \to C^\infty(\Xi_\tau; E|_{\Xi_\tau}) \tag{3.2}$$

are a unitary bundle morphism, a symmetric elliptic differential operator of first order, and a skew-symmetric bundle morphism, respectively, with

$$\sigma(\,\cdot\,,\tau)^2 = -I, \quad \sigma(\,\cdot\,,\tau)A_\tau = -A_\tau\sigma(\,\cdot\,,\tau), \quad \text{and} \quad \sigma(\,\cdot\,,\tau)B_\tau = B_\tau\sigma(\,\cdot\,,\tau). \tag{3.3}$$

Here τ denotes the normal variable and Ξ_τ a hypersurface parallel to Ξ in a distance τ.

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(2) The operator D satisfies the (weak) Unique Continuation Property

$$\ker D^{\max} \cap \operatorname{domain}(D^{\min}) = \{0\}. \tag{3.4}$$

(3) The operator D can be continued to an *invertible* elliptic differential operator $\widetilde{D}$ on a closed smooth Riemannian manifold $\widetilde{M}$ which contains M, and acting on sections in a smooth Hermitian bundle $\widetilde{E}$ which is a smooth continuation of E over the whole of $\widetilde{M}$; in particular, $\widetilde{M}$ is partitioned by Σ so that we have $\widetilde{M} = M_- \cup_\Sigma M_+$ with $M_+ = M$, $M_- \cap M_+ = \partial M_\pm = \Sigma$.

Remark 3.2 All (compatible) Dirac operators satisfy Assumption 3.1(1) (see e.g. [2] or [10]). Then Assumptions 3.1(2) and the sharper 3.1(3) follow by [5, Chapters 8, 9].

Let $\widetilde{\varrho}, \varrho^\pm$ denote the trace maps from $C^\infty(\widetilde{M}; E), C^\infty(M_\pm; E)$ to $C^\infty(\Sigma; E|_\Sigma)$. (We write E also for $\widetilde{E}$ and $\widetilde{E}|_{M_-}$). Furthermore, $r^\pm$ denotes restriction to $M_\pm$ and $e^\pm$ denotes extension by 0 from $M_\pm$ to $\widetilde{M}$.

Under the fundamental Assumption 3.1(3), it is well known that the *Poisson operator K* is given by

$$K := r^+ \widetilde{D}^{-1} \widetilde{\varrho}^* \sigma. \tag{3.5}$$

The Poisson operator K extends to a bounded mapping of $H^s(E|_\Sigma)$ onto

$$Z^{s+1/2} = \left\{ u \in H^{s+1/2}(M_+; E) \;\middle|\; Du = 0 \text{ in the interior of } M_+ \right\}$$

and provides a left inverse for $\varrho^+ \colon Z^{s+1/2} \to H^s(E|_\Sigma)$. Note that by the ellipticity of D, the trace map ϱ^+ can be extended to $Z^{s+1/2}$ for all real s (*cf.* [5, Theorem 12.4]).

The Calderón projector is then given by

$$P_+ = \varrho^+ K. \tag{3.6}$$

It is a pseudodifferential projection (idempotent). By definition, its extension to $H^s(E_{|\Sigma})$ has the *Cauchy data space* $\varrho^+(Z^{s+1/2})$ as its range. Without loss of generality we can assume that the extension of P_+ to $L^2(E|_\Sigma)$ is orthogonal (see [5, Lemma 12.8]).

3.2 Well-Posed Boundary Problems

To obtain self-adjoint Fredholm extensions of D in $L^2(M_+; E)$ we must impose suitable boundary conditions.

Definition 3.3 The self-adjoint Fredholm Grassmannian of D is defined by

$$\begin{aligned}\operatorname{Gr}^{\mathrm{sa}}(D) := \big\{ P \text{ pseudodifferential projection} \mid P^* = P,\ P = \sigma_0(I-P)\sigma_0^*, \\ \text{and } PP_+ \colon \text{ range } P_+ \to \text{range } P \text{ is Fredholm} \big\},\end{aligned}$$

where $\sigma_0 \colon E|_\Sigma \to E|_\Sigma$ denotes the unitary bundle morphism over the boundary according to Assumption 3.1(1). The topology is given by the operator norm.

It is well known (see *e.g.*, [7, Appendix B]) that $\mathrm{Gr}^{\mathrm{sa}}(D)$ is connected with the higher homotopy groups given by Bott periodicity.

Remark 3.4 At Σ, the "tangential operator" A_0 defines a spectral projection $\Pi_{\geq}$ of $L^2(\Sigma; E|_\Sigma)$ onto the subspace spanned by the eigensections of A_0 for non-negative eigenvalues, the *Atiyah–Patodi–Singer projection*. If A_0 is invertible, then $\Pi_{\geq} = \Pi_{>}$ belongs to $\mathrm{Gr}^{\mathrm{sa}}(D)$. If A_0 is not invertible, then one adds to $\Pi_{>}$ a projection onto a Lagrangian subspace of $\ker A_0$ (relative to σ_0) to obtain an element in $\mathrm{Gr}^{\mathrm{sa}}(D)$.

We recall the main result of the analysis of well-posed boundary problems (see *e.g.*, [5, Corollary 19.2, Theorem 19.5, and Proposition 20.3]):

Theorem 3.5

(a) *Each* $P \in \mathrm{Gr}^{\mathrm{sa}}(D)$ *defines a self-adjoint extension* D_P *in* $L^2(M; E)$ *with compact resolvent by*

$$\mathrm{domain}(D_P) := \left\{ u \in H^1(M; E) \;\middle|\; P(u|_\Sigma) = 0 \right\}.$$

(b) *The Calderón extension* D_{P_+} *is invertible. In fact, the inverse of* D_{P_+} *can be expressed in terms of* $\widetilde{D}^{-1}$ *and the Poisson operator:*

$$D_{P_+}^{-1} = r^+ \widetilde{D}^{-1} e^+ - K P_+ \widetilde{\varrho} \widetilde{D}^{-1} e^+. \tag{3.7}$$

(c) *The operator* D_P *is invertible if and only if the* boundary integral

$$P \circ P_+ \colon \text{ range } P_+ \longrightarrow \text{range } P$$

is invertible. Denote by $\widetilde{Q}_P$ *its inverse and put* $Q_P := \widetilde{Q}_P P$. *Then*

$$\begin{aligned} D_P^{-1} &= D_{P_+}^{-1} - K Q_P \varrho^+ D_{P_+}^{-1} \\ &= (I - K Q_P \varrho^+)(r^+ \widetilde{D}^{-1} e^+ - K P_+ \widetilde{\varrho} \widetilde{D}^{-1} e^+). \end{aligned} \tag{3.8}$$

Lemma 3.6 *Let* H *be a Hilbert space. For an invertible pair* (P, R) *of orthogonal projections let* $\widetilde{Q}(P, R)$ *denote the inverse of* $PR \colon$ range $R \longrightarrow$ range P *and put*

$$Q(P, R) := \widetilde{Q}(P, R) P.$$

Then the map

$$(P, R) \mapsto Q(P, R) \in \mathcal{B}(H)$$

is continuous in the operator norm.

Proof (P, R) is an invertible pair if and only if

$$T(P, R) := PR + (I - P)(I - R)$$

is an invertible operator. Obviously, $(P, R) \mapsto T(P, R)$ is continuous on the set of invertible pairs. From

$$T(P, R)R = PR = PT(P, R), \quad T(P, R)(I - R) = (I - P)T(P, R)$$

we infer

$$RT(P, R)^{-1} = T(P, R)^{-1}P, \quad (I - R)T(P, R)^{-1} = T(P, R)^{-1}(I - P)$$

and so $Q(P, R) = T(P, R)^{-1}P$, and we reach the conclusion. ■

Corollary 3.7 *For fixed D the mapping*

$$\mathrm{Gr}^{\mathrm{sa}}(D) \ni P \mapsto D_P \in \mathcal{CF}^{\mathrm{sa}}(L^2(M; E))$$

is continuous from the operator norm to the gap metric.

Proof It follows immediately from (3.8), Lemma 3.6, and Theorem 1.1(a) (see also Remark 1.4(a) that

$$\{P \in \mathrm{Gr}^{\mathrm{sa}}(D) \mid (P, P_+) \text{ invertible}\} \ni P \mapsto D_P \in \mathcal{CF}^{\mathrm{sa}}(L^2(M; E))$$

is continuous. Now consider $P_0 \in \mathrm{Gr}^{\mathrm{sa}}(D)$ such that D_{P_0} is not invertible. Since $D_{P_0} \in \mathcal{CF}^{\mathrm{sa}}(L^2(M; E))$, the operator $D_{P_0} + \varepsilon = (D + \varepsilon)_{P_0}$ is invertible for any real $\varepsilon > 0$ small enough. Obviously $D + \varepsilon$ also satisfies Assumptions 3.1(1)–(3) and its invertible extension, $\widetilde{D + \varepsilon}$, may be chosen as $\widetilde{D} + \varepsilon$ which depends continuously on ε. In view of (3.6) the Calderón projector $P_+(D + \varepsilon)$ depends continuously on ε (see also Theorem 3.9 below). Thus for ε small enough we have $P_0 \in \mathrm{Gr}^{\mathrm{sa}}(D+\varepsilon)$ with $\big(P_0, P_+(D+\varepsilon)\big)$ invertible and the above argument shows that $P \mapsto (D+\varepsilon)_P = D_P+\varepsilon$ is continuous at P_0. Since $\varepsilon = \varepsilon \cdot I$ is bounded, also $P \mapsto D_P$ is continuous at P_0. ■

3.3 The Variation of the Operator D

We now assume that D depends on an additional parameter s. More precisely, let $(D_s)_{s \in X}$, X a metric space, be a family of differential operators satisfying the Assumption 3.1(1). We assume moreover that

(3.9) *in each local chart, the coefficients of D_s depend continuously on s.*

In a collar $U \approx [0, \varepsilon) \times \Sigma$ the operator $D_s|_U$ takes the form

$$D_s|_U = \sigma_s(y, \tau)\Big(\frac{\partial}{\partial \tau} + A_{s,\tau} + B_{s,\tau}\Big) \tag{3.10}$$

with σ, A, B depending continuously on s and smoothly on τ. By the very definition of smoothness on a manifold with boundary we find extensions of σ, A, B to

$$(s, \tau) \in X \times [-\delta, \varepsilon)$$

for some $\delta > 0$, such that (3.2) and (3.3) are preserved and such that the operator

$$(3.11)\qquad D_s{}' := \begin{cases} D_s & \text{on } M, \\ \sigma_s\big(\frac{\partial}{\partial\tau} + A_{s,\tau} + B_{s,\tau}\big) & \text{on } [-\delta, \varepsilon) \times \Sigma, \end{cases}$$

is a first order elliptic differential operator on the manifold

$$M_\delta := \big([-\delta, 0] \times \Sigma\big) \cup_\Sigma M.$$

We fix $s_0 \in X$. We choose δ so small that the operator

$$(3.12)\qquad \Big[t\sigma_{s_0}(y, -\delta)\Big(\frac{\partial}{\partial\tau} + A_{s_0,-\delta} + B_{s_0,-\delta}\Big) + (1 - t)D_s{}'\Big]\Big|_{[-\delta,\delta]\times\Sigma}$$

is elliptic for all $t \in [0, 1]$.

Next we choose a cut-off function $\varphi \in C^\infty(\mathbb{R})$ with

$$(3.13)\qquad \varphi(x) = \begin{cases} 1 & x \le -\frac{2}{3}\delta, \\ 0 & x \ge -\frac{1}{3}\delta. \end{cases}$$

Then we put

$$(3.14)\qquad D_s{}'' := \varphi\sigma_{s_0}(y, -\delta)\Big(\frac{\partial}{\partial\tau} + A_{s_0,-\delta} + B_{s_0,-\delta}\Big) + (1 - \varphi)D_s{}'.$$

Clearly, $D_{s_0}{}''$ is an elliptic differential operator on M_δ which satisfies assumption (3.1). Moreover, in the collar $U'' := [-\delta, -\frac{2}{3}\delta] \times \Sigma$ of ∂M_δ we have

$$(3.15)\qquad D_s{}'' = \sigma_{s_0}(y, -\delta)\Big(\frac{\partial}{\partial\tau} + A_{s_0,-\delta} + B_{s_0,-\delta}\Big) =: \sigma''\Big(\frac{\partial}{\partial\tau} + A'' + B''\Big),$$

where σ'', A'', B'' are *independent* of s and τ.

By construction, $D_s{}''$ preserves (3.9). Hence there is an open neighbourhood X_0 of s_0 such that for $s \in X_0$ the operator $D_s{}''$ is elliptic.

For $\{D_s{}''\}_{s\in X_0}$ we now apply the construction of the invertible double of [5, Chapter 9]. In view of (3.15), the invertible double will be a first order elliptic differential operator on a closed manifold which depends continuously on the parameter s.

Summing up we have proved

Theorem 3.8 *Let M be a compact Riemannian manifold with boundary. Let $\{D_s\}_{s\in X}$, X a metric space, be a family of differential operators satisfying Assumption* 3.1(1) *and which depends continuously on s in the sense of* (3.9)*. Then for each $s_0 \in X$ there exists an open neighbourhood X_0 of s_0 and a continuous family $\{\widetilde{D}_s\}_{s\in X_0}$ of invertible elliptic differential operators $\widetilde{D}_s : C^\infty(\widetilde{M}; \widetilde{E}) \to C^\infty(\widetilde{M}; \widetilde{E})$ with $\widetilde{D}_s|_M = D_s$. Here $\widetilde{M}$ is a closed Riemannian manifold with $\widetilde{M} \supset M$; and $\widetilde{E} \to \widetilde{M}$ a smooth Hermitian vector bundle with $\widetilde{E}|_M = E$.*

The continuity of $s \mapsto \widetilde{D}_s$ is understood in the sense of (3.9). However, since $\widetilde{M}$ is closed this implies that $\{\widetilde{D}_s\}_{s\in X_0}$ is a graph continuous family of invertible self-adjoint operators.

Theorem 3.9 *Under the assumptions of Theorem* 3.8 *we have*

(a) *The Poisson operator K_s of D_s depends continuously on s.*
(b) *The Calderón projector $P_+(s)$ of D_s depends continuously on s.*
(c) *The family*

$$X \ni s \mapsto (D_s)_{P_+(s)} \in \mathcal{CF}^{\mathrm{sa}}(L^2(M; E))$$

is continuous.

(d) *Let $\{P_t\}_{t\in Y}$ be a norm-continuous path of orthogonal projections in $L^2(\Sigma; E|_\Sigma)$. If*

$$P_t \in \bigcap_{s\in X} \mathrm{Gr}^{\mathrm{sa}}(D_s), \quad t \in Y,$$

then

$$X \times Y \ni (s,t) \mapsto (D_s)_{P_t} \in \mathcal{CF}^{\mathrm{sa}}(L^2(M; E))$$

is continuous.

Proof (a) follows from Theorem 3.8 and (3.5); (b) follows from Theorem 3.8 and (3.6); (c) follows from Theorem 3.8 and (3.5). (d) Similarly as in the proof of Lemma 3.6 it suffices to prove the claim for $(D_s)_{P_t}$ invertible. Now the assertion follows from Lemma 3.6 and (3.8).[1] ■

Remark 3.10

(a) By different methods, somewhat related results have been obtained in [3] under the additional assumption of a fixed principal symbol of the family $\{D_s\}$ and a fixed boundary condition.
(b) Corollary 3.7 for fixed D and the preceding Theorem 3.9 for variation of D yield a well-defined and homotopy invariant spectral flow by Proposition 2.3, resp. Propositions 2.16, 2.17. The surprising facts are that
    (1) gap continuity suffices to establish spectral flow and
    (2) gap continuity is obtainable from continuous variation of the operator and the boundary condition without any restrictions and without any need to fix the domains of the unbounded L^2-extensions by unitary transformations.

    Roughly speaking, these constitute the differences between the present approach and Nicolaescu's approach in [15] which requires the continuity of the Riesz map and to achieve that, additional properties of the families of boundary problems.

[1]Added in Proof: The proof of Theorem 3.9 is incomplete. The continuous dependence of formula (3.8) on all input data is obvious only if K is viewed as a map from $H^s(E|_\Sigma)$ to $Z^{s+1/2}$ for $s < 0$. In the critical case $s = 0$ additional considerations are necessary (for the Trace theorem in the critical Sobolev case $s = 0$ see e.g. [5, Theorem 12.4]). Nevertheless, Theorem 3.9 is correct, though the proof is more involved. For a perturbative approach see [11, Section 3].

(c) In some important applications in topology, families of Dirac operators are considered on non-compact manifolds. The L^2-extensions of these operators are self-adjoint Fredholm operators but do not have a compact resolvent and therefore require a light modification of our preceding arguments to establish the continuity in the gap metric.

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