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ENDPOINT ESTIMATES FOR COMMUTATORS OF RIESZ TRANSFORMS ASSOCIATED WITH SCHRÖDINGER OPERATORS

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Abstract

In this paper, we discuss the H^1_L-boundedness of commutators of Riesz transforms associated with the Schrödinger operator $L = -\triangle + V$, where $H^1_L(R^n)$ is the Hardy space associated with L. We assume that $V(x)$ is a nonzero, nonnegative potential which belongs to B_q for some $q > n/2$. Let $T_1 = V(x)(-\triangle + V)^{-1}$, $T_2 = V^{1/2}(-\triangle + V)^{-1/2}$ and $T_3 = \nabla(-\triangle + V)^{-1/2}$. We prove that, for $b \in \mathrm{BMO}(R^n)$, the commutator $[b, T_3]$ is not bounded from $H^1_L(R^n)$ to $L^1(R^n)$ as T_3 itself. As an alternative, we obtain that $[b, T_i]$, $(i = 1, 2, 3)$ are of $(H^1_L, L^1_{\text{weak}})$-boundedness.

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1. Introduction

Let $L = -\triangle + V$ be the Schrödinger operator on R^n, $n \geq 3$. Throughout this paper, we assume that V is a nonzero, nonnegative potential which belongs to B_q for some $q > n/2$. Let T_i $(i = 1, 2, 3)$ be the Riesz transform associated with Schrödinger operators, specifically, $T_1 = V(-\triangle + V)^{-1}$, $T_2 = V^{1/2}(-\triangle + V)^{-1/2}$ and $T_3 = \nabla(-\triangle + V)^{-1/2}$. The L^p-boundedness of T_i $(i = 1, 2, 3)$ was widely studied in [7, 9]. In [3], using a pointwise estimate of the kernel of T_i $(i = 1, 2, 3)$, the authors proved the L^p-boundedness of commutators $[b, T_i]$ $(i = 1, 2, 3)$ for some $p > 1$. In this paper, we discuss the boundedness of $[b, T_i]$ $(i = 1, 2, 3)$ at the endpoint $p = 1$.

A nonnegative locally L^q integrable function $V(x)$ on R^n is said to belong to B_q $(1 < q < \infty)$, if there exists $C > 0$, such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V^q(x)\,dx\right)^{1/q} \leq C\left(\frac{1}{|B|}\int_B V(x)\,dx\right) \tag{1.1}$$

holds for every ball B in R^n.

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By Hölder's inequality, we have $B_{q_1} \subseteq B_{q_2}$ for $q_1 \geq q_2 > 1$. One remarkable feature of the B_q class is that if $V \in B_q$ for some $q > 1$, then there exists an $\varepsilon > 0$ which depends only on n and the constant C in (1.1) such that $V \in B_{q+\varepsilon}$. It is also well known that if $V \in B_q$, $q > 1$, then $V(x)\,dx$ is a doubling measure, namely for any $r > 0$, $x \in R^n$ and some constant C_0,

$$\int_{B(x,2r)} V(y)\,dy \leq C_0 \int_{B(x,r)} V(y)\,dy. \tag{1.2}$$

For such a Schrödinger operator L, Shen [7] studied the L^p-boundedness of Riesz transforms associated with L. He obtained the following result.

THEOREM 1.1 [7, Theorem 0.5, Theorem 3.1, Theorem 5.10].

(i) *Suppose that* $V \in B_q$ *and* $q \geq n/2$. *Then for* $q' \leq p < \infty$,

$$\|(-\triangle + V)^{-1} V f\|_p \leq C_p \|f\|_p.$$

(ii) *Suppose that* $V \in B_q$ *and* $q \geq n/2$. *Then for* $(2q)' \leq p < \infty$,

$$\|(-\triangle + V)^{-1/2} V^{1/2} f\|_p \leq C_p \|f\|_p.$$

(iii) *Suppose that* $V \in B_q$ *and* $n/2 \leq q < n$. *Then for* $p_1' \leq p < \infty$,

$$\|(-\triangle + V)^{-1/2} \nabla f\|_p \leq C_p \|f\|_p$$

where $1/p_1 = 1/q - 1/n$.

By duality, we can easily obtain the L^p-boundedness of T_i $(i = 1, 2, 3)$. Take $T_3 = \nabla(-\triangle + V)^{-1/2}$ for example; using (iii) of Theorem 1.1, we find that T_3 is bounded on $L^p(R^n)$, $1 < p \leq p_1$. So an interesting problem is the boundedness of T_i $(i = 1, 2, 3)$ at the endpoint $p = 1$. In Section 2, we prove that the T_i $(i = 1, 2, 3)$ are bounded from $L^1(R^n)$ to $L^1_{\text{weak}}(R^n)$. It was pointed out in [7] that if $V \in B_n$, then T_3 is a Calderón–Zygmund operator. So when considering $[b, T_3]$, we restrict ourselves to the case where $V \in B_q$ $(n/2 < q < n)$.

In [3] the authors proved that for $b \in \mathrm{BMO}(R^n)$, the commutators $[b, T_i]$ $(i = 1, 2, 3)$ are bounded on $L^p(R^n)$ for some $p > 1$. Another problem we are interested in is the boundedness of commutators $[b, T_i]$ $(i = 1, 2, 3)$ at endpoint $p = 1$ for $b \in \mathrm{BMO}(R^n)$. In [6] Pérez proved that if $b \in \mathrm{BMO}(R^n)$, the commutator $[b, T]$ may not be of weak-type (1, 1) where T is a Calderón–Zygmund operator. In [4] Harboure *et al.* proved that, even if we restrict $f \in H^1(R^n) \subset L^1(R^n)$, $[b, T]f$ still may not be in $L^1(R^n)$.

In [2] Dziubanski and Zienkiewicz studied the Hardy space H^1_L associated with the Schrödinger operator $L = -\triangle + V$, for $V \in B_q$, $q > n/2$. Actually they showed that if $f \in H^1_L(R^n)$, then $T_3 f \in L^1(R^n)$. So a natural question is whether the commutator $[b, T_3]$ is bounded from $H^1_L(R^n)$ into $L^1(R^n)$ when $b \in \mathrm{BMO}(R^n)$? Unfortunately, in Section 3, we get a negative result. We give a counterexample to imply that the commutators $[b, T_i]$ $(i = 1, 2, 3)$ may not be bounded from $H^1_L(R^n)$ to $L^1(R^n)$.

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These facts imply that, in order to get the H^1_L-boundedness of the commutators $[b, T_i]$ $(i = 1, 2, 3)$, we need to replace of the space $L^1(R^n)$ by a larger class. In Section 4, we prove that, if $b \in \mathrm{BMO}(R^n)$, the commutators $[b, T_i]$ $(i = 1, 2, 3)$ are bounded from $H^1_L(R^n)$ into $L^1_{\mathrm{weak}}(R^n)$.

In the rest of this section, we list some notation and properties for later use.

Definition 1.2. For $x \in R^n$, the function $m(x, V)$ is defined by

$$\frac{1}{m(x, V)} = \sup_{r>0}\left\{r : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y)\, dy \le 1\right\}. \tag{1.3}$$

Clearly, $0 < m(x, V) < \infty$ for every $x \in R^n$ and if $r = 1/m(x, V)$, then $1/r^{n-2} \int_{B(x,r)} V(y)\, dy = 1$. For simplicity, we sometimes denote $1/m(x, V)$ by $\rho(x)$ in proofs.

The function $m(x, V)$ has many useful properties. We list them in the following lemmas.

Lemma 1.3 [7, Lemma 1.4]. *There exist $C > 0$, $c > 0$ and $k_0 > 0$ such that for $x, y \in R^n$:*

(1) $m(x, V) \sim m(y, V)$, *if* $|x - y| \le C/m(x, V)$;
(2) $m(y, V) \le C\{1 + |x - y| m(x, V)\}^{k_0} m(x, V)$;
(3) $m(y, V) \ge c m(x, V)/\{1 + |x - y| m(x, V)\}^{k_0/(k_0+1)}$.

Lemma 1.4 [7, Lemma 1.8]. *There exist $C > 0$ and $k_0 > 0$ such that if $Rm(x, V) \ge 1$, then*

$$\frac{1}{R^{n-2}} \int_{B(x,R)} V(y)\, dy \le C\{Rm(x, V)\}^{k_0}.$$

When we estimate the integral of the kernels of T_i $(i = 1, 2, 3)$, we need the following lemma.

Lemma 1.5 [3, Lemma 1]. *Suppose that $V \in B_q$ for some $q > n/2$. Let $N > \log_2 C_0 + 1$, where C_0 is the constant in (1.2). Then for any $x_0 \in R^n$ and $R > 0$,*

$$\frac{1}{\{1 + m(x_0, V)R\}^N} \int_{B(x_0,R)} V(\xi)\, d\xi \le R^{n-2}.$$

2. The $(L^1, L^1_{\mathrm{weak}})$-boundedness of T_i $(i = 1, 2, 3)$

In this section, we discuss the $(L^1, L^1_{\mathrm{weak}})$-boundedness of T_i $(i = 1, 2, 3)$. For the operator $T_3 = \nabla(-\triangle + V)^{-1/2}$, Li [5] proved the $(L^1, L^1_{\mathrm{weak}})$-boundedness of the Riesz transform $X_j L^{-1/2}$ associated with a Schrödinger operator on a nilpotent group. So we need only give the proof of T_i for $i = 1, 2$. For the proof, we need the well-known Calderón–Zygmund decomposition as follows.

Lemma 2.1 [8]. *Let $f \in L^1$ and $\alpha > 0$; there exist a decomposition of f as $f = g + b$, where $b = \sum_k b_k$, and a sequence of balls $\{B^*_k\}$ such that:*

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(i) $|g(x)| \le c\alpha$ *a.e. for* x;
(ii) $\operatorname{supp} b_k \subset B_k^*$, $\int |b_k(x)|\,dx \le c\alpha |B_k^*|$;
(iii) $\int b_k(x)\,dx = 0$;
(iv) $\sum_k |B_k^*| \le (c/\alpha) \int |f(x)|\,dx$.

THEOREM 2.2. *Suppose that* $V \in B_q$ *for some* $q \ge n/2$. *If* $T_1 = V(x)(-\triangle + V)^{-1}$, *then* T_1 *is bounded from* $L^1(R^n)$ *into* $L^1_{\text{weak}}(R^n)$.

For the proof of Theorem 2.2, we need the following pointwise estimate of the kernel of T_1.

LEMMA 2.3 [3, Lemma 2]. *Suppose that* $V \in B_q$ *for some* $q > n/2$. *Then there exists* $\delta > 0$ *such that for any integer* $K > 0$, $0 < h < |x-y|/16$,

$$|K_1(x, y)| \le \frac{C_K}{\{1 + m(y, V)|x-y|\}^K} \frac{1}{|x-y|^{n-2}} V(x), \tag{2.1}$$

$$|K_1(x, y+h) - K_1(x, y)| \le \frac{C_K}{\{1 + m(y, V)|x-y|\}^K} \frac{|h|^\delta}{|x-y|^{n-2+\delta}} V(x). \tag{2.2}$$

PROOF OF THEOREM 2.2. By Calderón–Zygmund decomposition,

$$|\{x : |T_1 f(x)| > \alpha\}| \le |\{x : |T_1 g(x)| > \alpha/2\}| + |\{x : |T_1 b(x)| > \alpha/2\}|.$$

Using (i) and (iv) of Lemma 2.1,

$$\begin{aligned}\int |g(x)|^p\,dx &\le \int_{(\cup B_k^*)^c} |g(x)|^p\,dx + \int_{\cup B_k^*} |g(x)|^p\,dx \\ &\le c\alpha^{p-1} \int_{(\cup B_k^*)^c} |f(x)|\,dx + c\alpha^p |\cup B_k^*| \\ &\le c\alpha^{p-1} \|f\|_1.\end{aligned}$$

Then by (i) of Theorem 1.1 and $1 < p < q$,

$$\begin{aligned}|\{x : |T_1 g(x)| > \alpha/2\}| &\le \frac{C}{\alpha^p} \int_{\{x : |T_1 g| > \alpha/2\}} |T_1 g(x)|^p\,dx \\ &\le \frac{C}{\alpha^p} \|g\|_p^p \le \frac{C}{\alpha} \|f\|_1.\end{aligned}$$

Now we estimate $|\{x : |T_1 b(x)| > \alpha/2\}|$:

$$\begin{aligned}|\{x : |T_1 b(x)| > \alpha/2\}| &\le |\{x \in (\cup 16 B_k^*) : |T_1 b(x)| > \alpha/2\}| \\ &\quad + |\{x \in (\cup 16 B_k^*)^c : |T_1 b(x)| > \alpha/2\}| \\ &\le \sum_k |16 B_k^*| + |\{x \in (\cup 16 B_k^*)^c : |T_1 b(x)| > \alpha/2\}| \\ &\le \frac{c}{\alpha} \int |f(x)|\,dx + |\{x \in (\cup 16 B_k^*)^c : |T_1 b(x)| > \alpha/2\}|.\end{aligned}$$

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By the cancelling property of b_k, we let $B_k^* = B(x_k, r_k)$. Then

$$\begin{aligned}
&|\{x \in (\cup 16B_k^*)^c : |T_1 b(x)| > \alpha/2\}| \\
&\quad \le \frac{c}{\alpha} \int_{(\cup 16B_k^*)^c} |T_1 b(x)|\, dx \\
&\quad \le \frac{c}{\alpha} \sum_k \int_{(\cup 16B_k^*)^c} \left| \int_{B_k^*} [K_1(x, y) - K_1(x, x_k)] b_k(y)\, dy \right| dx \\
&\quad \le \frac{c}{\alpha} \sum_k \int_{B_k^*} |b_k(y)|\, dy \int_{(\cup 16B_k^*)^c} |K_1(x, y) - K_1(x, x_k)|\, dx.
\end{aligned}$$

Because $y \in B_k^*$, then $|y - x_k| < r_k < |x - x_k|/16$. In Lemma 2.3, set $h = |y - x_k|$. Then

$$|K_1(x, y) - K_1(x, x_k)| \le \frac{C_K}{\{1 + m(x_k, V)|x - x_k|\}^K} \frac{|y - x_k|^\delta}{|x - x_k|^{n-2+\delta}} V(x).$$

By Lemma 1.5,

$$\begin{aligned}
&\int_{(\cup 16B_k^*)^c} |K_1(x, y) - K_1(x, x_k)|\, dx \\
&\quad \le \int_{(\cup B_k^*)^c} \frac{C_K}{\{1 + m(x_k, V)|x - x_k|\}^K} \frac{|y - x_k|^\delta}{|x - x_k|^{n-2+\delta}} V(x)\, dx \\
&\quad \le \sum_{j=4}^{\infty} \int_{2^j r_k \le |x - x_k| < 2^j r_k} \frac{C_K}{\{1 + m(x_k, V)|x - x_k|\}^K} \frac{|y - x_k|^\delta}{|x - x_k|^{n-2+\delta}} V(x)\, dx \\
&\quad \le \sum_{j=4}^{\infty} \frac{C_K}{\{1 + m(x_k, V) 2^j r_k\}^K} \frac{r_k^\delta}{(2^j r_k)^{n-2+\delta}} \int_{|x - x_k| < 2^{j+1} r_k} V(x)\, dx \\
&\quad \le \sum_{j=4}^{\infty} \frac{r_k^\delta}{(2^j r_k)^{n-2+\delta}} (2^j r_k)^{n-2} \le C.
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
|\{x : |T_1 b(x)| > \alpha/2\}| &\le \frac{c}{\alpha} \int |f(x)|\, dx + \frac{c}{\alpha} \sum_k \int_{B_k^*} |b_k(x)|\, dx \\
&\le \frac{c}{\alpha} \int |f(x)|\, dx.
\end{aligned}$$

This completes the proof of Theorem 2.2. □

For the (L^1, L^1_{weak})-boundedness of T_2, we need the following lemma.

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LEMMA 2.4 [3, Lemma 3]. *Suppose that $V \in B_q$ for some $q > n/2$. Then there exists $\delta > 0$ such that for any integer $K > 0$, $0 < h < |x-y|/16$,*

$$|K_2(x, y)| \le \frac{C_K}{\{1+m(y, V)|x-y|\}^K} \frac{1}{|x-y|^{n-1}} V^{1/2}(x), \tag{2.3}$$

$$|K_2(x, y+h) - K_2(x, y)| \le \frac{C_K}{\{1+m(y, V)|x-y|\}^K} \frac{|h|^\delta}{|x-y|^{n-1+\delta}} V^{1/2}(x). \tag{2.4}$$

We now prove the (L^1, L^1_{weak})-boundedness of T_2.

THEOREM 2.5. *Suppose $V \in B_q$ for some $q > n/2$. If $T_2 = V^{1/2}(x)(-\triangle + V)^{-1/2}$, then T_2 is bounded from $L^1(R^n)$ into $L^1_{\text{weak}}(R^n)$.*

PROOF. By the Calderón–Zygmund decomposition,

$$|\{x : |T_2 f(x)| > \alpha\}| \le |\{x : |T_2 g(x)| > \alpha/2\}| + |\{x : |T_2 b(x)| > \alpha/2\}|.$$

Similarly, we only need to estimate $|\{x \in (\cup 16B_k^*)^c : |T_2 b(x)| > \alpha/2\}|$. Set $B_k^* = B(x_k, r_k)$. Then by the cancelling of $b_k(x)$,

$$\begin{aligned}
&|\{x \in (\cup 16B_k^*)^c : |T_2 b(x)| > \alpha/2\}| \\
&\quad \le \frac{C}{\alpha} \sum_k \int_{(\cup 16B_k^*)^c} \left| \int_{B_k^*} [K_2(x, y) - K_2(x, x_k)] b(y)\, dy \right| dx \\
&\quad \le \frac{C}{\alpha} \sum_k \int_{B_k^*} |b(y)|\, dy \int_{(\cup 16B_k^*)^c} |K_2(x, y) - K_2(x, x_k)|\, dx.
\end{aligned}$$

Since $y \in B_k^*$ and $x \in (\cup 16B_k^*)^c$, then $|y - x_k| < r_k < |x - x_k|/16$. Let $h = |y - x_k|$, by Lemma 2.4 and Hölder's inequality,

$$\begin{aligned}
&\int_{(\cup 16B_k^*)^c} |K_2(x, y) - K_2(x, x_k)|\, dx \\
&\quad \le \sum_{j=4}^{\infty} \int_{2^j r_k < |x-x_k| \le 2^{j+1} r_k} \frac{C_K}{\{1+m(x_k, V)|x-x_k|\}^K} \frac{|y-x_k|^\delta}{|x-x_k|^{n-1+\delta}} V^{1/2}(x)\, dx \\
&\quad \le \sum_{j=4}^{\infty} \frac{C_K}{\{1+m(x_k, V)2^j r_k\}^K} \frac{r_k^\delta}{(2^j r_k)^{n-1+\delta}} (2^{j+1} r_k)^{n/(2q)'} \\
&\qquad \times \left(\int_{|x-x_k|<2^{j+1} r_k} V^q(x)\, dx \right)^{1/q} \\
&\quad \le \sum_{j=4}^{\infty} \frac{C_K}{\{1+m(x_k, V)2^j r_k\}^K} \frac{r_k^\delta}{(2^j r_k)^{n-1+\delta}} (2^{j+1} r_k)^{n/(2q)' + n/2q - n/2} \\
&\qquad \times \left(\int_{|x-x_k|<2^{j+1} r_k} V(x)\, dx \right)^{1/2}
\end{aligned}$$

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$$\leq C\sum_{j=4}^{\infty}\frac{r_k^{\delta}}{(2^j r_k)^{n-1+\delta}}(2^{j+1}r_k)^{n/(2q)'+n/(2q)-n/2}(2^{j+1}r_k)^{n/2-1}$$
$$\leq C\sum_{j=4}^{\infty}\frac{1}{2^{j\delta}}\leq C.$$

Finally, we obtain

$$|\{x : |T_2 b(x)| > \alpha/2\}| \leq \frac{C}{\alpha}\|f\|_1 + \frac{C}{\alpha}\sum_k \int_{B_k}|b_k(x)|\,dx \leq \frac{C}{\alpha}\|f\|_1.$$

This completes the proof of Theorem 2.5. □

In a similar manner to the two previous theorems, and using the following lemma, we can prove the $(L^1, L^1_{\mathrm{weak}})$-boundedness of T_3.

LEMMA 2.6 [3, Lemma 4]. *Suppose that $V \in B_q$ for some $n/2 < q < n$. Then there exists $\delta > 0$ and for any integer $K > 0$, $0 < h < |x-y|/16$,*

$$|K_3(x,y)| \leq \frac{C_K}{\{1+m(y,V)|x-y|\}^K}\frac{1}{|x-y|^{n-1}} \times\bigg(\int_{B(x,|x-y|)}\frac{V(\xi)}{|x-\xi|^{n-1}}\,d\xi + \frac{1}{|x-y|}\bigg), \tag{2.5}$$

$$|K_3(x,y+h) - K_3(x,y)| \leq \frac{C_K}{\{1+m(y,V)|x-y|\}^K}\frac{|h|^{\delta}}{|x-y|^{n-1+\delta}} \times\bigg(\int_{B(x,|x-y|)}\frac{V(\xi)}{|x-\xi|^{n-1}}\,d\xi + \frac{1}{|x-y|}\bigg). \tag{2.6}$$

THEOREM 2.7. *Suppose that $V \in B_q$, $n/2 < q < n$. Letting $T_3 = \nabla(-\Delta + V)^{-1/2}$, then T_3 is bounded from $L^1(R^n)$ into $L^1_{\mathrm{weak}}(R^n)$.*

3. Failure for (H_L^1, L^1)-boundedness of $[b, T_3]$

In [2] Dziubanski and Zienkiewicz studied the Hardy space H_L^1 associated with a Schrödinger operator L. In that paper they constructed the atomic Hardy space as follows.

DEFINITION 3.1 (H_L^1-atom). For $n \in \mathbb{Z}$, define the set $\mathfrak{B}_n$ by

$$\mathfrak{B}_n = \{x : 2^{n/2} \leq m(x,V) < 2^{(n+1)/2}\}.$$

Since $0 < m(x,V) < \infty$, then $R^n = \bigcup_n \mathfrak{B}_n$.

A function $a(x)$ is an atom for the Hardy space $H_L^1(R^n)$ associated with a ball $B(x_0, r)$, if the following conditions hold:

(i) $\operatorname{supp} a(x) \subset B(x_0, r)$;
(ii) $\|a\|_{L^\infty} \leq 1/|B(x_0,r)|$;
(iii) if $x_0 \in \mathfrak{B}_n$, then $r \leq 2^{1-n/2}$;
(iv) if $x_0 \in \mathfrak{B}_n$ and $r \leq 2^{-1-n/2}$, then $\int a(x)\,dx = 0$.

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The atomic norm in $H^1_L(R^n)$ is defined by $\|f\|_{L-\text{atom}} = \inf\{\sum_j |\lambda_j|\}$, where the infimum is taken over all decompositions $f = \sum_j \lambda_j a_j$ where a_j are H^1_L-atoms.

In [2] the authors obtained the atomic decomposition of H^1_L as follows.

THEOREM 3.2 [2, Theorem 1.5]. *Assuming that V is a nonnegative potential such that $V \in B_{n/2}$, then the norms $\|f\|_{H^1_L}$ and $\|f\|_{L-\text{atom}}$ are equivalent, that is, there exists a constant $C > 0$ such that*

$$C^{-1}\|f\|_{H^1_L} \le \|f\|_{L-\text{atom}} \le C\|f\|_{H^1_L}.$$

Using atomic decomposition, the authors obtained the following result.

THEOREM 3.3 [2, Theorem 1.7]. *If $V \in B_{n/2}$ is a nonnegative potential, then there is a constant $C > 0$ such that*

$$C^{-1}\|f\|_{H^1_L} \le \|f\|_{L^1} + \sum_{j=1}^{d} \|R^L_j f\|_{L^1} \le C\|f\|_{H^1_L}$$

where R^L_j denotes the jth component of the operator $T_3 = \nabla(-\triangle + V)^{-1/2}$.

Theorem 3.3 implies that the Riesz transform R^L_j is bounded from $H^1_L(R^n)$ into $L^1(R^n)$. A natural question is whether the commutator $[b, R^L_j]$ is bounded from $H^1_L(R^n)$ into $L^1(R^n)$ for $b \in \text{BMO}(R^n)$. For Calderón–Zygmund operators, the answer is negative. In [4], Harboure *et al.* proved that for a singular integral operator T, if $[b, T]$ is bounded from $H^1(R^n)$ into $L^1(R^n)$, then b must be a constant. In this section we prove in a similar manner that for $T_3 = \nabla(-\triangle + V)^{-1/2}$, the commutator $[b, T_3]$ may not be bounded from $H^1_L(R^n)$ into $L^1(R^n)$.

First we state the definition of the dual space of $H^1_L(R^n)$ which was introduced in [1].

DEFINITION 3.4. We shall say that a locally integrable function f belongs to $\text{BMO}_L(R^n)$ whenever there is a constant $C > 0$ such that

$$\frac{1}{|B_s|}\int_{B_s} |f(y) - f_{B_s}|\,dy \le C \quad \text{and} \quad \frac{1}{|B_r|}\int_{B_r} |f(y)|\,dy \le C,$$

for all balls $B_s = B_s(x)$, $B_r = B_r(x)$ such that $s \le \rho(x) \le r$. We let $\|f\|_{\text{BMO}_L}$ denote the smallest C in the above inequalities. Here and subsequently, we set $f_B = (1/|B|)\int_B f(x)\,dx$.

THEOREM 3.5. *Let $T_3 = \nabla(-\triangle + V)^{-1/2}$ be the Riesz transform associated with the Schrödinger operator and let $b \in \text{BMO}_L(R^n)$. Then the following two statements are equivalent.*

(i) *The commutator $[b, T_3]$ is bounded from $H^1_L(R^n)$ into $L^1(R^n)$.*

(ii) *For any atom a supported in a ball with center x_0 and radius $r < \rho(x_0)$, for $u \in B$,*

$$\int_{(33B)^c} |K_3(x, u)| \left|\int_B b(y)a(y)\,dy\right| dx \le C.$$

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PROOF. Because $a(x)$ is an H_L^1-atom, we assume that the support of $a(x)$ is $B(x_0, r)$. In order to estimate the L^1 norm of $T_3a(x)$, we divide the discussion into two cases as follows.

Case I. For $\rho(x_0) \le r \le 4\rho(x_0)$,

$$\begin{aligned}[b, T_3]a(x) &= \chi_{2B}(x)[b, T_3]a(x) + \chi_{(2B)^c}(x)[b, T_3]a(x) \\ &= \chi_{2B}(x)[b, T_3]a(x) + \chi_{(2B)^c}(x)b(x)T_3a(x) - \chi_{(2B)^c}(x)T_3(ba)(x) \\ &=: M_1 + M_2 + M_3.\end{aligned}$$

For M_1, by the L^p-boundedness $[b, T_3]$, we get

$$\begin{aligned}\|M_1\|_{L^1} &= \int_{2B} |[b, T_3]a(x)|\, dx \\ &\le C\bigg(\int_{2B} |[b, T_3]a(x)|^p\, dx\bigg)^{1/p} |B|^{1-1/p} \\ &\le C\|a\|_p |B|^{1-1/p} \|b\|_{\mathrm{BMO}_L} \\ &\le C\|b\|_{\mathrm{BMO}_L}.\end{aligned}$$

For M_2, we have

$$\|M_2\|_{L^1} = \int_{(2B)^c} |b(x)||T_3a(x)|\, dx \le \int_B |a(y)|\, dy \int_{(2B)^c} |b(x)||K_3(x, y)|\, dx.$$

Using Lemma 2.6,

$$\begin{aligned}&\int_{(2B)^c} |b(x)||K_3(x, y)|\, dx \\ &\quad\le \int_{(2B)^c} |b(x)| \frac{C_K}{\{1 + m(y, V)|x - y|\}^K} \frac{1}{|x - y|^{n-1}} \\ &\qquad\times \bigg(\int_{B(x,|x-y|)} \frac{V(z)}{|x - z|^{n-1}}\, dz\bigg)\, dx \\ &\qquad+ \int_{(2B)^c} |b(x)| \frac{1}{\{1 + m(y, V)|x - y|\}^K} \frac{1}{|x - y|^n}\, dx \\ &\quad=: M_{21} + M_{22}.\end{aligned}$$

For M_{22}, because $y \in B$ and $|x - x_0| > 2^k r$ imply $|x - y| > |x - x_0| - |y - x_0| > 2^k r - r > 2^{k-1} r$,

$$\begin{aligned}M_{22} &\le \sum_{k=1}^{\infty} \int_{2^k < |x-x_0| \le 2^{k+1} r} |b(x)| \frac{1}{|x - y|^n} \frac{C_K}{\{1 + m(y, V)|x - y|\}^K}\, dx \\ &\le \sum_{k=1}^{\infty} \frac{C_K}{\{1 + m(y, V)2^{k-1} r\}^K} \frac{1}{(2^{k-1} r)^n} \int_{|x-x_0| \le 2^{k+1} r} |b(x)|\, dx.\end{aligned}$$

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Because $y \in B$ implies that $|y - x_0| < r < 4\rho(x_0)$, then $\rho(x_0) \sim \rho(y)$. We have $m(y, V)r \geq m(y, V)\rho(x_0) = 1$ for $r > \rho(x_0)$. Therefore,

$$M_{22} \leq \sum_{k=1}^{\infty} \frac{C_K}{\{1 + 2^{k-1}\}^K} \|b\|_{\mathrm{BMO}_L}.$$

Because $|x - z| < |x - y|$ implies that $|z - x_0| \leq |z - x| + |x - x_0| \leq |x - y| + |x - x_0| \leq 2|x - x_0| + |y - x_0| < 2^{k+2}r + r < 2^{k+3}r$, then

$$\begin{aligned}
M_{21} &\leq \sum_{k=1}^{\infty} \int_{2^k r < |x - x_0| \leq 2^{k+1} r} \frac{C_K |x - y|^{1-n}}{\{1 + m(y, V)|x - y|\}^K} \\
&\quad \times \bigg(\int_{B(x, |x-y|)} \frac{V(z)}{|x - z|^{n-1}}\, dz \bigg) |b(x)|\, dx \\
&\leq \sum_{k=1}^{\infty} \frac{C_K}{\{1 + 2^{k-1}\}^K} \frac{1}{(2^{k-1} r)^{n-1}} (2^{k+1} r)^{n/p_1'} \|b\|_{\mathrm{BMO}_L} \\
&\quad \times \bigg\| \int \frac{V(z) \chi_{B(x_0, 2^{k+3} r)}(z)}{|z - x|^{n-1}}\, dz \bigg\|_{L^{p_1}(dx)} \\
&\leq \sum_{k=1}^{\infty} \frac{C_K}{\{1 + 2^{k-1}\}^K} \frac{1}{(2^{k-1} r)^{n-1}} (2^{k+1} r)^{n/p_1'} \|b\|_{\mathrm{BMO}_L} (2^{k+3} r)^{n/q - n} \\
&\quad \times \int_{B(x_0, 2^{k+3} r)} V(z)\, dz \\
&\leq \sum_{k=1}^{\infty} \frac{C_K}{\{1 + 2^{k-1}\}^K} \|b\|_{\mathrm{BMO}_L} \frac{1}{(2^{k-1} r)^{n-2}} \int_{B(x_0, 2^{k+3} r)} V(z)\, dz.
\end{aligned}$$

Because $2^{k+3} r > r \geq \rho(x_0)$ for $k \geq 1$, $2^{k+3} r m(x_0, V) > 1$. Then by Lemma 1.4, the double property of $V(x)\, dx$ and $r m(x_0, V) \leq 4$ for $r \leq 4\rho(x_0)$,

$$\frac{1}{(2^k r)^{n-2}} \int_{B(x_0, 2^{k+3} r)} V(z)\, dz \leq C (2^k r m(x_0, V))^{k_0} \leq C 2^{k k_0}.$$

Therefore, choosing K large enough, we obtain

$$M_{21} \leq C \|b\|_{\mathrm{BMO}_L} \sum_{k=1}^{\infty} \frac{C_K}{\{1 + 2^{k-1}\}^K} \cdot 2^{k k_0} \leq C \|b\|_{\mathrm{BMO}_L}.$$

This implies that $\|M_2\|_{L^1} \leq C \|b\|_{\mathrm{BMO}_L}$.

Finally, we estimate M_3:

$$\begin{aligned}
\|M_3\|_{L^1} &= \int_{(2B)^c} \bigg| \int_B K_3(x, y) b(y) a(y)\, dy \bigg|\, dx \\
&\leq \int_B |b(y)||a(y)|\, dy \int_{|x - x_0| > 2r} \frac{C_K |x - y|^{1-n}}{\{1 + m(y, V)|x - y|\}^K}
\end{aligned}$$

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$$\times \left[\int_{B(x,|x-y|)} \frac{V(z)}{|x-z|^{n-1}} + \frac{1}{|x-y|} \right] dx$$
$$=: \int_B |b(y)||a(y)|(M_{31} + M_{32})\, dy.$$

For $y \in B$, $|x - x_0| > 2^k r$, we have $|x - y| > |x - x_0| - |y - x_0| > 2^k r - r > 2^{k-1} r$, where $k \geq 1$. Then

$$\begin{aligned}
M_{32} &= \int_{(2B)^c} \frac{C_K}{\{1 + m(y, V)|x - y|\}^K} \frac{1}{|x - y|^n}\, dx \\
&\leq \sum_{k=1}^{\infty} \int_{2^k r < |x - x_0| \leq 2^{k+1} r} \frac{C_K}{\{1 + m(y, V)|x - y|\}^K} \frac{1}{|x - y|^n}\, dx \\
&\leq \sum_{k=1}^{\infty} \frac{C_K}{\{1 + 2^{k-1} m(y, V) r\}^K} \frac{1}{(2^{k-1} r)^n} \int_{|x - x_0| \leq 2^{k+1} r} dx \\
&\leq C \sum_{k=1}^{\infty} \frac{C_K}{(1 + 2^{k-1})^K} \leq C.
\end{aligned}$$

Here we have used the fact that, for $4\rho(x_0) \geq r > \rho(x_0)$ and any $|y - x_0| < r < 4\rho(x_0)$, we have $m(y, V) r \geq r\rho(x_0) \sim 1$. For M_{31}, since $|y - x_0| < r$, $|x - x_0| > 2^k r$, then $|x - y| > |x - x_0| - |y - x_0| \geq 2^{k-1} r$. Then

$$\begin{aligned}
M_{31} &= \int_{(2B)^c} \frac{C_K}{\{1 + m(y, V)|x - y|\}^K} \frac{1}{|x - y|^n} \left(\int_{B(x,|x-y|)} \frac{V(z)}{|x - z|^{n-1}}\, dz \right) dx \\
&\leq \sum_{k=1}^{\infty} \frac{C_K}{\{1 + m(y, V) 2^{k-1} r\}^K} \frac{1}{(2^{k-1} r)^{n-1}} \\
&\qquad \times \int_{2^k r < |x - x_0| \leq 2^{k+1} r} \left(\int_{B(x,|x-y|)} \frac{V(z)}{|x - z|^{n-1}}\, dz \right) dx.
\end{aligned}$$

For $z \in B(x, |x - y|)$, $|z - x| \leq |x - y|$. So for every $y \in B(x_0, r)$ and $|x - x_0| \leq 2^{k+1} r$,

$$\begin{aligned}
|z - x_0| &\leq |z - x| + |x - x_0| \\
&\leq |x - y| + |x - x_0| \leq 2|x - x_0| + |y - x_0| \\
&\leq 2^{k+2} r + r \leq 2^{k+3} r.
\end{aligned}$$

Then by Lemma 1.4, choosing K large enough,

$$\begin{aligned}
M_{31} &\leq \sum_{k=1}^{\infty} \frac{C_K}{\{1 + 2^{k-1}\}^K} \frac{1}{(2^{k-1} r)^{n-1}} (2^{k+1} r)^{n/p_1'} \\
&\qquad \times \left\| \int \frac{V(z) \chi_{B(x_0, 2^{k+3} r)}(z)}{|z - x_0|^{n-1}}\, dz \right\|_{L^p(dx)}
\end{aligned}$$

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$$
\begin{aligned}
&\leq \sum_{k=1}^{\infty} \frac{C_K}{\{1+2^{k-1}\}^K} \frac{1}{(2^k r)^{n-1}} (2^k r)^{n/p_1'} \bigg(\int_{B(x_0, 2^{k+3} r)} V^q(z)\, dz \bigg)^{1/q} \\
&\leq \sum_{k=1}^{\infty} \frac{C_K}{\{1+2^{k-1}\}^K} \frac{1}{(2^k r)^{n-1}} (2^k r)^{n/p_1'+n/q-n} \int_{B(x_0, 2^{k+3} r)} V(z)\, dz \\
&\leq C \sum_{k=1}^{\infty} \frac{C_K}{\{1+2^{k-1}\}^K} (2^k r m(x_0, V))^{k_0} \\
&\leq \sum_{k=1}^{\infty} \frac{C_K}{(1+2^{k-1})^K} 2^{k k_0} \leq C.
\end{aligned}
$$

Here we have used the fact that, because $\rho(x_0) \leq r \leq 4\rho(x_0)$, then $1 \leq r m(x_0, V) \leq 4$. Then for $y \in B$, $|y - x_0| \leq r \leq 4\rho(x_0)$. Therefore we have $m(x_0, V) \sim m(y, V)$ and $1 \leq r m(y, V) \leq 4$. Finally, using (ii) of Definition 3.1, we obtain

$$
\|M_3\|_{L^1} \leq \int_B |b(y)||a(y)|(M_{31} + M_{32})\, dy \leq C \frac{1}{|B|} \int_B |b(y)|\, dy \leq C \|b\|_{\mathrm{BMO}_L}.
$$

In fact, we have proved that for an H^1_L-atom $a(x)$ with support $B(x_0, r)$ with $\rho(x_0) \leq r \leq 4\rho(x_0)$, if $b \in \mathrm{BMO}_L(R^n)$, then $\|[b, T_3]a\|_{L^1} \leq C\|b\|_{\mathrm{BMO}_L}$.

Case II. For $r < \rho(x_0)$, the atom $a(x)$ has the cancelling condition $\int_B a(x)\, dx = 0$. For any $u \in B$,

$$
\begin{aligned}
[b, T_3]a(x) &= \chi_{33B}(x)[b, T_3]a(x) + \chi_{(33B)^c}(x)[b, T_3]a(x) \\
&= \chi_{33B}(x)[b, T_3]a(x) + \chi_{(33B)^c}(x)(b(x) - b_B)T_3 a(x) \\
&\quad - \chi_{(33B)^c}(x) T_3((b - b_B))a(x) \\
&= \chi_{33B}(x)[b, T_3]a(x) + \chi_{(33B)^c}(x)(b(x) - b_B)T_3 a(x) \\
&\quad - \chi_{(33B)^c}(x) \int [K_3(x, y) - K_3(x, u)](b(y) - b_B)a(y)\, dy \\
&\quad - \chi_{(33B)^c}(x) \int K_3(x, u)[b(y) - b_B]a(y)\, dy \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

Clearly we can see that I_4 is the term in the integral of (ii) of Theorem 3.5. So we need only estimate I_i $(i = 1, 2, 3)$ separately.

Because $[b, T_3]$ is bounded on L^p for $1 < p < p_1$, then we have

$$
\begin{aligned}
\|I_1\|_{L^1} &\leq \int_{33B} |[b, T_3]a(x)|\, dx \\
&\leq C|B|^{1-1/p} \bigg(\int_{33B} |[b, T_3]a(x)|^p\, dx \bigg)^{1/p} \\
&\leq C|B|^{1-1/p} \|a\|_p \|b\|_{\mathrm{BMO}_L} \\
&\leq C\|b\|_{\mathrm{BMO}_L}.
\end{aligned}
$$

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By the cancelling property of $a(x)$,

$$\begin{aligned}\|I_2\|_{L^1} &\le \int_{(33B)^c} |b(x)-b_B||T_3a(x)|\,dx\\ &\le \int_{(33B)^c} |b(x)-b_B| \int_B |K_3(x,y)-K_3(x,x_0)||a(y)|\,dy\\ &\le \int_B |a(y)|\,dy \int_{(33B)^c} |b(x)-b_B||K_3(x,y)-K_3(x,x_0)|\,dx.\end{aligned}$$

Because $y \in B(x_0, r)$ and $x \in (33B)^c$, we have $|y-x_0| < r < |x-x_0|/16$. By Lemma 2.6, setting $h = |y-x_0|$,

$$\begin{aligned}|K_3(x,y)-K_3(x,x_0)| \le{}& \frac{C_K}{\{1+m(x_0,V)|x-x_0|\}^K}\frac{|y-x_0|^\delta}{|x-x_0|^{n-1+\delta}}\\ &\times\Big(\int_{B(x,|x-x_0|)} \frac{V(\xi)}{|x-\xi|^{n-1}}\,d\xi + \frac{1}{|x-x_0|}\Big).\end{aligned}$$

Naturally we divide the integral into two parts,

$$\begin{aligned}&\int_{(33B)^c} |b(x)-b_B||K_3(x,y)-K_3(x,x_0)|\,dx\\ &\quad\le \int_{(33B)^c} \frac{C_K|b(x)-b_B|}{\{1+m(x_0,V)|x-x_0|\}^K}\frac{|y-x_0|^\delta}{|x-x_0|^{n-1+\delta}}\\ &\qquad\times\Big(\int_{B(x,|x-x_0|)} \frac{V(\xi)}{|x-\xi|^{n-1}}\,d\xi\Big)\,dx\\ &\qquad+ \int_{(33B)^c} |b(x)-b_B|\frac{C_K}{\{1+m(x_0,V)|x-x_0|\}^K}\frac{|y-x_0|^\delta}{|x-x_0|^{n+\delta}}\,dx\\ &\quad=: I_{21}+I_{22}.\end{aligned}$$

For I_{22}, because $\mathrm{BMO}_L(R^n)$ is a subspace of $\mathrm{BMO}(R^n)$, then $\|b\|_{\mathrm{BMO}} \le \|b\|_{\mathrm{BMO}_L}$. We have

$$\begin{aligned}I_{22} &\le \sum_{k=5}^\infty \int_{2^kr<|x-x_0|\le 2^{k+1}r} |b(x)-b_B|\frac{C_K}{\{1+m(x_0,V)|x-x_0|\}^K}\frac{|y-x_0|^\delta}{|x-x_0|^{n+\delta}}\,dx\\ &\le \sum_{k=5}^\infty \frac{C_K}{\{1+m(x_0,V)2^kr\}^K}\frac{r^\delta}{(2^kr)^{n+\delta}}(2^{k+1}r)^n(k+2)\|b\|_{\mathrm{BMO}}\\ &\le C\|b\|_{\mathrm{BMO}_L}\sum_{k=5}^\infty \frac{(k+2)}{2^{k\delta}}\\ &\le C\|b\|_{\mathrm{BMO}_L}.\end{aligned}$$

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For I_{21}, by Hölder's inequality and Lemma 1.5,

$$\begin{aligned}
I_{21} &\le \sum_{k=5}^{\infty} \frac{C_K}{\{1+2^k r m(x_0, V)\}^K} \int_{|x-x_0|\le 2^{k+1}r} \frac{r^\delta |b(x)-b_B|}{(2^k r)^{n-1+\delta}} \\
&\quad \times \bigg(\int_{B(x,|x-x_0|)} \frac{V(\xi)}{|x-\xi|^{n-1}}\, d\xi \bigg)\, dx \\
&\le \sum_{k=5}^{\infty} \frac{C_K \|b\|_{\mathrm{BMO}}}{\{1+2^k r m(x_0, V)\}^K} \frac{(k+2) r^\delta}{(2^k r)^{n-1+\delta}} (2^{k+1} r)^{n/p_1'} \\
&\quad \times \bigg\| \int \frac{V(\xi)\chi_{B(x_0,2^{k+2}r)}(\xi)}{|x-\xi|^{n-1}}\, d\xi \bigg\|_{L^{p_1}(dx)} \\
&\le \sum_{k=5}^{\infty} \frac{C_K \|b\|_{\mathrm{BMO}_L}}{\{1+2^k r m(x_0, V)\}^K} \frac{(k+2) r^\delta}{(2^k r)^{n-1+\delta}} (2^{k+1} r)^{n/p_1'} \\
&\quad \times \bigg(\int_{B(x_0,2^{j+2}r)} V^q(\xi)\, d\xi \bigg)^{1/q} \\
&\le \sum_{k=5}^{\infty} \frac{C_K \|b\|_{\mathrm{BMO}_L}}{\{1+2^k r m(x_0, V)\}^K} \frac{(k+2) r^\delta}{(2^k r)^{n-1+\delta}} (2^{k+1} r)^{n/p_1'+n/q-n} \\
&\quad \times \int_{B(x_0,r)} V(\xi)\, d\xi \\
&\le C\|b\|_{\mathrm{BMO}_L} \sum_{k=5}^{\infty} \frac{(k+2) r^\delta}{(2^k r)^{n-1+\delta}} (2^{k+1} r)^{n/p_1'+n/q-n+n-2} \\
&\le C\|b\|_{\mathrm{BMO}_L}.
\end{aligned}$$

Finally, for $\|I_3\|_{L^1}$, we get

$$\begin{aligned}
\|I_3\|_{L^1} &\le \int_{(33B)^c} \int_B |K_3(x,y)-K_3(x,u)||b(y)-b_B||a(y)|\, dy\, dx \\
&= \int_B |b(y)-b_B||a(y)|\, dy \int_{(16B)^c} |K_3(x,y)-K_3(x,u)|\, dx.
\end{aligned}$$

On the one hand, because $u \in B$, we have $|y-u| \le |y-x_0| + |x_0-u| \le 2r$. On the other hand, for $x \in (33B)^c$, we have $|x-u| > |x-x_0| - |u-x_0| > 32r$. Therefore $|y-u| \le 2r \le |x-u|/16$. By Lemma 2.6, setting $h = |y-u|$,

$$\begin{aligned}
|K_3(x,y)-K_3(x,u)| &\le \frac{C_K}{\{1+m(u,V)|x-u|\}^K} \frac{|y-u|^\delta}{|x-u|^{n-1+\delta}} \\
&\qquad \times \bigg(\int_{B(x,|x-u|)} \frac{V(\xi)}{|x-\xi|^{n-1}}\, d\xi + \frac{1}{|x-u|} \bigg).
\end{aligned}$$

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Similarly, we divide the integral of the above inequality into

$$\int_{(33B)^c} |K_3(x, y) - K_3(x, u)|\, dx = I_{31} + I_{32}.$$

For I_{32}, we have

$$\begin{aligned} I_{32} &\le \sum_{k=5}^{\infty} \int_{2^k r<|x-u|\le 2^{k+1}r} \frac{C_K}{\{1+m(u, V)|x-u|\}^K} \frac{|y-u|^{\delta}}{|x-u|^{n+\delta}}\, dx \\ &\le C \sum_{k=5}^{\infty} \frac{r^{\delta}}{(2^j r)^{n+\delta}} (2^{j+1} r)^n \\ &\le C. \end{aligned}$$

For I_{31}, notice that every $\xi \in B(x, |x-u|)$, and $|\xi - u| \le 2|x-u|$. If $|x-u| \le 2^k r$, then $|\xi - u| \le 2^{k+2} r$. So we have

$$\begin{aligned} I_{31} &\le \sum_{k=5}^{\infty} \frac{C_K}{\{1+m(u, V)2^k r\}^K} \frac{r^{\delta}}{(2^k r)^{n-1+\delta}} (2^{k+1} r)^{n/p_1'} \\ &\quad \times \bigg\| \int \frac{V(\xi)\chi_{B(u,2^{k+2}r)}(\xi)}{|x-\xi|^{n-1}}\, d\xi \bigg\|_{L^{p_1}(dx)} \\ &\le \sum_{k=5}^{\infty} \frac{C_K}{\{1+m(u, V)2^k r\}^K} \frac{r^{\delta}}{(2^k r)^{n-1+\delta}} (2^{k+1} r)^{n/p_1'} \bigg(\int_{B(u,2^{k+2}r)} V^q(\xi)\, d\xi \bigg)^{1/q} \\ &\le \sum_{k=5}^{\infty} \frac{C_K}{\{1+m(u, V)2^k r\}^K} \frac{r^{\delta}}{(2^k r)^{n-1+\delta}} (2^{k+1} r)^{n/p_1'} (2^{k+2} r)^{n/q-n} \\ &\quad \times \int_{B(u,2^{k+2}r)} V(\xi)\, d\xi \\ &\le \sum_{k=5}^{\infty} \frac{r^{\delta}}{(2^k r)^{\delta}} (2^{k+1} r)^{n/p_1'+n/q-n+n-2} \le C. \end{aligned}$$

Then we have $\|I_3\|_{L^1} \le \int_B |b(y) - b_B||a(y)|\, dy \le (1/|B|) \int_B |b(y) - b_B|\, dy \le \|b\|_{\mathrm{BMO}_L}$. Finally, the estimate of $\|I_i\|_{L^1}$ $(i = 1, 2, 3)$ implies that, for an H^1_L-atom $a(x)$, $\|T_3 a(x)\|_{L^1} \le C$ if and only if $\|I_4\|_{L^1} \le C$. This completes the proof of Theorem 3.5. □

Counterexample 3.6. From Theorem 3.5, we find that the commutator $[b, T_3]$ may not be bounded from $H^1_L(R^n)$ into $L^1(R^n)$. We use a simple example to imply this conclusion. If we choose r small enough such that $33r < \rho(x_0)$,

$$\begin{aligned} &\int_{|x-x_0|>33r} |K_3(x, x_0)|\, dx \\ &\quad \ge \int_{|x-x_0|>33r} |R(x, x_0)|\, dx - \int_{|x-x_0|>33r} |K_3(x, x_0) - R(x, x_0)|\, dx \end{aligned}$$

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$$\begin{aligned}
&\geq \int_{|x-x_0|>33r} |R(x, x_0)|\,dx - \int_{|x-x_0|>\rho(x_0)} |K_3(x, x_0)|\,dx \\
&\quad - \int_{|x-x_0|>\rho(x_0)} |R(x, x_0)|\,dx \\
&\quad - \int_{33r<|x-x_0|\leq\rho(x_0)} |K_3(x, x_0) - R(x, x_0)|\,dx \\
&\geq \int_{33r<|x-x_0|<\rho(x_0)} |R(x, x_0)|\,dx - \int_{|x-x_0|>\rho(x_0)} |K_3(x, x_0)|\,dx \\
&\quad - \int_{33r<|x-x_0|\leq\rho(x_0)} |K_3(x, x_0) - R(x, x_0)|\,dx \\
&=: M_1 - M_2 - M_3.
\end{aligned}$$

Shen [7] proved that there exist constants C_1, C_2 such that $M_2 \leq C_1$ and $M_3 \leq C_2$. Then by Theorem 3.5, if $[b, T_3]$ is bounded from H_L^1 to L^1, then

$$\bigg(\int_{33r<|x-x_0|<\rho(x_0)} |R(x, x_0)|\,dx - C_1 - C_2\bigg)\bigg|\int b(y)a(y)\,dy\bigg| \leq C$$

where $|R(x, x_0)| = 1/|x - x_0|^n$. If we set $V(x) = 1$ for convenience, then by Definition 3.1, it is easy to see that $\rho(x_0) = 1$. By Definition 3.1, because r is the radius of the atom $a(x)$, then $r \leq 2^{1-n/2}$. This means that if n is large enough,

$$\bigg(C\frac{n}{2} - C_1 - C_2\bigg)\bigg|\int b(y)a(y)\,dy\bigg| \leq \bigg(\ln\frac{1}{33r} - C_1 - C_2\bigg)\bigg|\int b(y)a(y)\,dy\bigg| \leq C,$$

that is,

$$\bigg|\int b(y)a(y)\,dy\bigg| \to 0 \quad \text{when } r \to 0\ (n \to \infty). \tag{$*$}$$

Unfortunately the conclusion $(*)$ is not true for a general atom $a(x)$. For example, we set

$$b(x) = \log|x|, \text{ when } |x| \leq 1, \qquad b(x) = 0, \text{ otherwise};$$
$$a_k(x) = -2^k, \text{ when } x \in \bigg(0, \frac{1}{2^{k+1}}\bigg), \qquad a_k(x) = 2^k, \text{ when } x \in \bigg(\frac{1}{2^{k+1}}, \frac{1}{2^k}\bigg).$$

It can be proved that $b(x) \in \mathrm{BMO}_L(R^n)$ and $a_k(x)$, $k \in Z^+$ are H_L^1-atoms. We have, for every $k \in Z^+$, $|\int b(y)a_k(y)\,dy| = \ln 2$, which is contrary to the conclusion $(*)$.

4. $(H_L^1, L_{\mathrm{weak}}^1)$-boundedness of $[b, T_i]$, $i = 1, 2, 3$

The counterexample in Section 3 implies that, if $b \in \mathrm{BMO}_L(R^n)$ and b is nonzero in the BMO_L norm, we cannot guarantee that the commutators $[b, T_i]$ $(i = 1, 2, 3)$ are bounded from $H_L^1(R^n)$ into $L^1(R^n)$. In this section we prove that if L^1 is replaced by a larger space, namely $L_{\mathrm{weak}}^1(R^n)$, then the $[b, T_i]$ $(i = 1, 2, 3)$ are bounded on $H_L^1(R^n)$.

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THEOREM 4.1. *Suppose that* $V \in B_q$*,* $q > n/2$*. Let* $T_1 = V(x)(-\triangle + V)^{-1}$, $T_2 = V^{1/2}(-\triangle + V)^{-1/2}$ *and* $T_3 = \nabla(-\triangle + V)^{-1/2}$*. For* $b \in \mathrm{BMO}$*, the commutators* $[b, T_i]$ *(*$i = 1, 2, 3$*) are bounded from* $H_L^1(R^n)$ *into* $L^1_{\mathrm{weak}}(R^n)$*.*

PROOF. For convenience, we prove the $(H_L^1, L^1_{\mathrm{weak}})$-boundedness of $[b, T_3]$. The proofs for $[b, T_1]$ and $[b, T_2]$ are similar. From Theorem 3.2, we know that for every $f \in H_L^1$, there exist a sequence of H_L^1-atoms $\{a_j(x)\}$ and a sequence of $\{\lambda_j\}$ for $j \in Z$ such that $f = \sum_j \lambda_j a_j(x)$ and $\sum_j |\lambda_j| \le \|f\|_{H_L^1}$. If we set the support of $a_j(x)$ as $B_j = B(x_j, r_j)$, then $r_j \le 4\rho(x_0)$ by Definition 3.1. Therefore,

$$
\begin{aligned}
[b, T_3]f(x) &= \sum_j \lambda_j [b, T_3]a_j(x) \\
&= \sum_{r_j < \rho(x_j)} \lambda_j [b, T_3]a_j(x) + \sum_{\rho(x_j) \le r_j < 4\rho(x_j)} \lambda_j [b, T_3]a_j(x) \\
&=: \sum\nolimits_1 \lambda_j [b, T_3]a_j(x) + \sum\nolimits_2 \lambda_j [b, T_3]a_j(x),
\end{aligned}
$$

where we denote

$$
\sum_{r_j < \rho(x_j)} \lambda_j [b, T_3]a_j(x) \quad \text{by} \ \sum\nolimits_1 \lambda_j [b, T_3]a_j(x)
$$

and

$$
\sum_{\rho(x_j) \le r_j < 4\rho(x_j)} \lambda_j [b, T_3]a_j(x) \quad \text{by} \ \sum\nolimits_2 \lambda_j [b, T_3]a_j(x).
$$

Then

$$
\begin{aligned}
|\{x : |[b, T_3]f(x)| > \lambda\}| \le{}& \left|\left\{x : \left|\sum\nolimits_1 \lambda_j [b, T_3]a_j(x)\right| > \lambda/2\right\}\right| \\
&+ \left|\left\{x : \left|\sum\nolimits_2 \lambda_j [b, T_3]a_j(x)\right| > \lambda/2\right\}\right|.
\end{aligned}
$$

Hence we need to estimate $|\{x : |\sum_i \lambda_j [b, T_3]a_j(x)| > \lambda/2\}|$, $i = 1, 2$, separately.

Step I. First, we estimate $|\{x : |\sum_1 \lambda_j [b, T_3]a_j(x)| > \lambda/2\}|$. We have

$$
\begin{aligned}
&\left|\left\{x : \left|\sum\nolimits_1 \lambda_j [b, T_3]a_j(x)\right| > \lambda/2\right\}\right| \\
&\quad \le \left|\left\{x : \left|\sum\nolimits_1 \lambda_j (b(x) - b_{B_j}) T_3 a_j(x) \chi_{(16B_j)}(x)\right| > \lambda/6\right\}\right| \\
&\qquad + \left|\left\{x : \left|\sum\nolimits_1 \lambda_j (b(x) - b_{B_j}) T_3 a_j(x) \chi_{(16B_j)^c}(x)\right| > \lambda/6\right\}\right| \\
&\qquad + \left|\left\{x : \left|\sum\nolimits_1 \lambda_j T_3((b - b_{B_j})a_j)(x)\right| > \lambda/6\right\}\right| \\
&\quad =: I_1 + I_2 + I_3.
\end{aligned}
$$

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For I_1, because T_3 is bounded on $L^p(R^n)$, $1 < p < p_1$, $1/p_1 = 1/q - 1/n$,

$$\begin{aligned} I_1 &= \Big|\Big\{x : \Big|\sum\nolimits_1 \lambda_j (b(x) - b_{B_j}) T_3 a_j(x) \chi_{(16B_j)}(x)\Big| > \lambda/6\Big\}\Big| \\ &\le \frac{C}{\lambda} \sum\nolimits_1 |\lambda_j| \int_{16B_j} |b(x) - b_{B_j}||T_3 a_j(x)|\, dx \\ &\le \frac{C}{\lambda} \sum\nolimits_1 |\lambda_j| \Big(\int_{16B_j} |b(x) - b_{B_j}|^2\, dx\Big)^{1/2} \Big(\int_{16B_j} |T_3 a_j(x)|^2\, dx\Big)^{1/2} \\ &\le \frac{C}{\lambda} \sum\nolimits_1 |\lambda_j||B_j|^{1/2} \|b\|_{\mathrm{BMO}} \|a_j\|_2 \\ &\le \frac{C}{\lambda} \sum\nolimits_1 |\lambda_j| \|b\|_{\mathrm{BMO}}. \end{aligned}$$

For I_3, by Theorem 2.7, T_3 is of weak-type $(1, 1)$. Using Hölder's inequality,

$$\begin{aligned} \Big|\Big\{x : \Big|\sum\nolimits_1 \lambda_j T_3((b - b_{B_j}) a_j)(x)\Big| > \lambda/6\Big\}\Big| &\le \frac{C}{\lambda} \sum\nolimits_1 \int_{B_j} |b(x) - b_{B_j}||a_j(x)|\, dx \\ &\le \frac{C}{\lambda} \sum\nolimits_1 |\lambda_j| \|b\|_{\mathrm{BMO}}. \end{aligned}$$

For I_2, the atom a_j has the cancelling property when $r_j \le \rho(x_j)$. We have

$$\begin{aligned} &\Big|\Big\{x : \Big|\sum\nolimits_1 \lambda_j (b(x) - b_{B_j}) T_3 a_j(x) \chi_{(16B_j)^c}(x)\Big| > \lambda/6\Big\}\Big| \\ &\quad\le \frac{C}{\lambda} \sum\nolimits_1 |\lambda_j| \int_{(16B_j)^c} |b(x) - b_{B_j}| \times |T_3 a_j(x)|\, dx \\ &\quad\le \frac{C}{\lambda} \sum\nolimits_1 |\lambda_j| \int_{(16B_j)^c} |b(x) - b_{B_j}| \Big| \int_{B_j} [K_3(x, y) - K_3(x, x_j)] a_j(y)\, dy \Big|\, dx \\ &\quad\le \frac{C}{\lambda} \sum\nolimits_1 |\lambda_j| \int_{B_j} |a_j(y)|\, dy \int_{(16B_j)^c} |b(x) - b_{B_j}||K_3(x, y) - K_3(x, x_j)|\, dx. \end{aligned}$$

We set $I_{2,y} = \int_{(16B_j)^c} |b(x) - b_{B_j}||K_3(x, y) - K_3(x, x_j)|\, dx$. Because $y \in B_j$, $|y - x_j| < r_j$ and $x \in (16B_j)^c$, $|x - x_j| > 16r_j$, then $|y - x_j| \le |x - x_j|/16$. By (2.6) of Lemma 2.6,

$$\begin{aligned} |K_3(x, y) - K_3(x, x_j)| \le{}& \frac{C_K}{\{1 + m(x_j, V)|x - x_j|\}^K} \frac{|y - x_j|^\delta}{|x - x_j|^{n-1+\delta}} \\ &\times \Big(\int_{B(x,|x-x_j|)} \frac{V(u)}{|x - u|^{n-1}}\, du + \frac{1}{|x - x_j|}\Big). \end{aligned}$$

Then

$$\begin{aligned}
I_{2,y} &= \int_{(16B_j)^c} |b(x)-b_{B_j}||K_3(x,y)-K_3(x,x_j)|\,dx \\
&\le \int_{(16B_j)^c} \frac{C_K|b(x)-b_{B_j}|}{\{1+m(x_j,V)|x-x_j|\}^K}\frac{|y-x_j|^\delta}{|x-x_j|^{n-1+\delta}} \\
&\qquad\times\bigg(\int_{B(x,|x-x_j|)}\frac{V(u)}{|x-u|^{n-1}}\,du\bigg)\,dx \\
&\qquad+\int_{(16B_j)^c}|b(x)-b_{B_j}|\frac{|y-x_j|^\delta}{|x-x_j|^{n+\delta}}\,dx \\
&=: I^1_{14,y}+I^2_{14,y}.
\end{aligned}$$

For $I^2_{2,y}$, we have

$$\begin{aligned}
I^2_{2,y} &= \int_{(16B_j)^c} |b(x)-b_{B_j}|\frac{|y-x_j|^\delta}{|x-x_j|^{n+\delta}}\,dx \\
&\le \sum_{k=4}^{\infty}\int_{2^k r_j\le|x-x_j|<2^{k+1}r_j}|b(x)-b_{B_j}|\frac{|y-x_j|^\delta}{|x-x_j|^{n+\delta}}\,dx \\
&\le \sum_{k=4}^{\infty}\frac{r_j^\delta}{(2^k r_j)^{n+\delta}}\int_{|x-x_j|<2^{k+1}r_j}|b(x)-b_{B_j}|\,dx \\
&\le C\|b\|_{\mathrm{BMO}}\sum_{k=4}^{\infty}\frac{(k+2)r^\delta}{(2^k r_j)^{n+\delta}}(2^{k+1}r_j)^n \\
&\le C\|b\|_{\mathrm{BMO}}.
\end{aligned}$$

For $I^1_{2,y}$, we have

$$\begin{aligned}
I^1_{2,y} &\le \sum_{k=4}^{\infty}\int_{2^k r_j\le|x-x_j|<2^{k+1}r_j}\frac{C_K|b(x)-b_{B_j}|}{\{1+m(x_j,V)|x-x_j|\}^K}\frac{|y-x_j|^\delta}{|x-x_j|^{n-1+\delta}} \\
&\qquad\times\bigg(\int_{B(x,|x-x_j|)}\frac{V(u)}{|x-u|^{n-1}}\,du\bigg)\,dx \\
&\le \sum_{k=4}^{\infty}\frac{C_K}{\{1+m(x_j,V)2^k r_j\}^K}\frac{r_j^\delta}{(2^k r_j)^{n-1+\delta}}\int_{2^k r_j\le|x-x_j|<2^{k+1}r_j}|b(x)-b_{B_j}| \\
&\qquad\times\bigg(\int_{B(x,|x-x_j|)}\frac{V(u)}{|x-u|^{n-1}}\,du\bigg)\,dx.
\end{aligned}$$

Because every $u\in B(x,|x-x_j|)$ implies that $|u-x_j|\le 2|x-x_j|\le 2^{k+2}r_j$ for $|x-x_j|<2^{k+1}r_j$, then by Hölder's inequality and Lemma 1.5,

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$$\begin{aligned}
I^1_{2,y} &\le \sum_{k=4}^{\infty} \frac{C_K}{\{1+m(x_j, V)2^k r_j\}^K} \frac{r_j^\delta}{(2^k r_j)^{n-1+\delta}} \\
&\qquad \times \bigg(\int_{|x-x_j|<2^{k+1}r_j} |b(x)-b_{B_j}|^{p_1'}\,dx\bigg)^{1/p_1'} \\
&\qquad \times \bigg\| \int \frac{V(u)\chi_{B(x_j,2^{k+2}r_j)}(u)}{|x-u|^{n-1}}\,du \bigg\|_{L^{p_1}(dx)} \\
&\le C\sum_{k=4}^{\infty} \frac{\|b\|_{\mathrm{BMO}}}{\{1+m(x_j, V)2^k r_j\}^K} \frac{(k+2)r_j^\delta}{(2^k r_j)^{n-1+\delta}} \\
&\qquad \times (2^{k+1}r_j)^{n/p_1'} \bigg(\int_{B(x_j,2^{k+2}r_j)} V^q(u)\,du\bigg)^{1/q} \\
&\le C\sum_{k=4}^{\infty} \frac{(k+2)r_j^\delta}{(2^k r_j)^{n-1+\delta}} (2^{k+1}r_j)^{n/p_1'+n/q-n} \frac{\|b\|_{\mathrm{BMO}}}{\{1+m(x_j, V)2^k r_j\}^K} \\
&\qquad \times \int_{B(x_j,2^{k+2}r_j)} V(u)\,du \\
&\le C\|b\|_{\mathrm{BMO}} \sum_{k=4}^{\infty} \frac{(k+2)r_j^\delta}{(2^k r_j)^{n-1+\delta}} (2^{k+1}r_j)^{n/p_1'+n/q-n+n-2} \\
&\le C\|b\|_{\mathrm{BMO}}
\end{aligned}$$

where we have used the fact that, for $1/q = 1/p - 1/n$, $n/p_1' + n/q - n + n - 2 = n - 1$. Then

$$I_2 \le \frac{C}{\lambda} \sum\nolimits_1 |\lambda_j| \int_{B_j} |a_j(y)|(I_{14,y})\,dy \le \frac{C}{\lambda}\|b\|_{\mathrm{BMO}} \sum\nolimits_1 |\lambda_j|.$$

Step II. We estimate $|\{x : |\sum_2 \lambda_j [b, T_3]a_j(x)| > \lambda/2\}|$. Notice that in this case, $\rho(x_j) \le r_j \le \rho(x_0)$, the atom $a_j(x)$ has no cancelling property. Similarly,

$$\begin{aligned}
&\Big|\Big\{x : \Big|\sum\nolimits_2 \lambda_j [b, T_3]a_j(x)\Big| > \lambda/2\Big\}\Big| \\
&\quad \le \Big|\Big\{x : \Big|\sum\nolimits_2 \lambda_j (b(x)-b_{B_j})T_3 a_j(x)\chi_{(2B_j)}(x)\Big| > \lambda/6\Big\}\Big| \\
&\qquad + \Big|\Big\{x : \Big|\sum\nolimits_2 \lambda_j (b(x)-b_{B_j})T_3 a_j(x)\chi_{(2B_j)^c}(x)\Big| > \lambda/6\Big\}\Big| \\
&\qquad + \Big|\Big\{x : \Big|\sum\nolimits_2 \lambda_j T_3((b-b_{B_j})a_j)(x)\Big| > \lambda/6\Big\}\Big| \\
&\quad =: I_4 + I_5 + I_6.
\end{aligned}$$

Similar to the proof of step I, using the L^p- and $(L^1, L^1_{\mathrm{weak}})$-boundedness of T_3,

$$I_4 \le \frac{C}{\lambda}\|b\|_{\mathrm{BMO}} \sum\nolimits_2 |\lambda_j| \quad \text{and} \quad I_6 \le \frac{C}{\lambda}\|b\|_{\mathrm{BMO}} \sum\nolimits_2 |\lambda_j|.$$

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For I_5, we have

$$
\begin{aligned}
I_5 &= \Big|\Big\{x : \Big|\sum\nolimits_2 \lambda_j (b(x) - b_{B_j}) T_3 a_j(x) \chi_{(2B_j)^c}(x)\Big| > \lambda/6\Big\}\Big| \\
&\le \frac{C}{\lambda} \sum\nolimits_2 |\lambda_j| \int_{(2B_j)^c} |b(x) - b_{B_j}||T a_j(x)|\, dx \\
&\le \frac{C}{\lambda} \int_{B_j} |a_j(y)|\, dy \int_{(2B_j)^c} |b(x) - b_{B_j}||K_3(x, y)|\, dx.
\end{aligned}
$$

We set $I_{5,y} = \int_{(2B_j)^c} |b(x) - b_{B_j}||K_3(x, y)|\, dx$. By (2.5) of Lemma 2.6,

$$
\begin{aligned}
|K_3(x, y)| \le {} & \frac{C_K}{\{1 + m(y, V)|x - y|\}^K} \frac{1}{|x - y|^{n-1}} \\
& \times \bigg(\int_{B(x,|x-y|)} \frac{V(u)}{|x - u|^{n-1}}\, du + \frac{1}{|x - y|}\bigg).
\end{aligned}
$$

Then

$$
\begin{aligned}
I_{5,y} &= \int_{(2B_j)^c} |b(x) - b_{B_j}||K_3(x, y)|\, dx \\
&\le \int_{(2B_j)^c} \frac{C_K |b(x) - b_{B_j}|}{\{1 + m(y, V)|x - y|\}^K} \frac{1}{|x - y|^{n-1}} \bigg(\int_{B(x,|x-y|)} \frac{V(u)}{|x - u|^{n-1}}\, du\bigg)\, dx \\
&\quad + \int_{(2B_j)^c} \frac{C_K |b(x) - b_{B_j}|}{\{1 + m(y, V)|x - y|\}^K} \frac{1}{|x - y|^n}\, dx \\
&=: I^1_{5,y} + I^2_{5,y}.
\end{aligned}
$$

For $I^2_{5,y}$, we have

$$
\begin{aligned}
I^2_{5,y} &= \int_{(2B_j)^c} \frac{C_K |b(x) - b_{B_j}|}{\{1 + m(y, V)|x - y|\}^K} \frac{1}{|x - y|^n}\, dx \\
&\le \sum_{k=1}^{\infty} \frac{C_K}{\{1 + m(y, V) 2^{k-1} r_j\}^K} \frac{1}{(2^{k-1} r_j)^n} \int_{2^k r_j \le |x - x_j| < 2^{k+1} r_j} |b(x) - b_{B_j}|\, dx \\
&\le \sum_{k=1}^{\infty} (k + 2) \frac{C_K \|b\|_{\mathrm{BMO}}}{\{1 + m(y, V) 2^{k-1} r_j\}^K} \\
&\le C \|b\|_{\mathrm{BMO}} \sum_{k=1}^{\infty} \frac{C_K (k + 2)}{\{1 + 2^{k-1}\}^K} \\
&\le C \|b\|_{\mathrm{BMO}}.
\end{aligned}
$$

Here, in the second inequality, we used the fact that because $y \in B_j$, $|y - x_j| < r_j$, then $|x - y| > |x - x_j| - |y - x_j| > 2^{k-1} r_j$ for $2^k r_j \le |x - x_j| < 2^{k+1} r_j$. In the fourth inequality, we used the fact that because $\rho(x_j) \le r_j \le \rho(x_0)$, then $|y - x_j| < r_j < 4\rho(x_j)$, $m(y, V) \sim m(x_j, V)$ and $1 \le r_j m(x_j, V) \le 4$.

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Finally, we estimate $I^1_{5,y}$. For every $u \in B(x, |x-y|)$, $|u-x| < |y-x_j| + |x-x_j|$, then for $2^k r_j \le |x-x_j| < 2^k r_j$, we have $|x-y| > 2^{k-1} r_j$ and $|u-x_j| < |x-u| + |x-x_j| < |y-x_j| + 2|x-x_j| < 2^{k+3} r_j$. Using Hölder's inequality,

$$\begin{aligned}
I^1_{5,y} &= \int_{(2B_j)^c} \frac{C_K |b(x)-b_{B_j}|}{\{1+m(y,V)|x-y|\}^K} \frac{1}{|x-y|^{n-1}} \bigg(\int_{B(x,|x-y|)} \frac{V(u)}{|x-u|^{n-1}}\, du \bigg)\, dx \\
&\le \sum_{k=1}^{\infty} \frac{C_K}{\{1+m(y,V)2^{k-1}r_j\}^K} \frac{1}{(2^{k-1}r_j)^{n-1}} \\
&\quad \times \int_{2^k r_j \le |x-x_j| < 2^k r_j} |b(x)-b_{B_j}| \cdot \bigg(\int_{B(x,|x-y|)} \frac{V(u)}{|x-u|^{n-1}}\, du \bigg)\, dx \\
&\le \sum_{k=1}^{\infty} \frac{C_K}{\{1+m(y,V)2^{k-1}r_j\}^K} \frac{1}{(2^{k-1}r_j)^{n-1}} \\
&\quad \times \bigg(\int_{|x-x_j|<2^{k+1}r_j} |b(x)-b_{B_j}|^{p_1'}\, dx \bigg)^{1/p_1'} \\
&\quad \times \bigg\| \int \frac{V(u)\chi_{B(x_j,2^{k+3}r_j)}(u)}{|x-u|^{n-1}}\, du \bigg\|_{L^{p_1}(dx)}.
\end{aligned}$$

Because $y \in B(x_j, r)$, we have $|y-x_j| < 4\rho(x_j)$ and $m(x_j, V) \sim m(y, V)$. For $\rho(x_j) \le r_j \le 4\rho(x_j)$, we have $1 \le m(x_j, V) r_j \le 4$. By Lemma 1.4 and the fractional integral,

$$\begin{aligned}
I^1_{5,y} &\le \sum_{k=1}^{\infty} \frac{C_K \|b\|_{\mathrm{BMO}}}{\{1+m(y,V)2^{k-1}r_j\}^K} \frac{(k+2)}{(2^{k-1}r_j)^{n-1}} (2^{k+1}r_j)^{n/p_1'} \\
&\quad \times \bigg(\int_{|x-x_j|<2^{k+3}r_j} V^q(x)\, dx \bigg)^{1/q} \\
&\le \sum_{k=1}^{\infty} \frac{C_K \|b\|_{\mathrm{BMO}}}{\{1+m(y,V)2^{k-1}r_j\}^K} \frac{(k+2)}{(2^{k-1}r_j)^{n-1}} (2^{k+1}r_j)^{n/p_1'+n/q-n} \\
&\quad \times \int_{|x-x_j|<2^{k+3}r_j} V(x)\, dx \\
&\le C \sum_{k=1}^{\infty} \frac{C_K \|b\|_{\mathrm{BMO}}}{\{1+m(y,V)2^{k-1}r_j\}^K} (k+2)(2^{k-1}r_j m(x_j,V))^{k_0} \\
&\le C\|b\|_{\mathrm{BMO}} \sum_{k=1}^{\infty} \frac{C_K}{\{1+2^{k-1}\}^K} (k+2)(2^{k-1})^{k_0} \le C\|b\|_{\mathrm{BMO}}.
\end{aligned}$$

Finally, we obtain

$$I_5 \le \frac{C}{\lambda} \sum\nolimits_2 |\lambda_j| \int_{B_j} |a_j(y)|(I_{5,y})\, dy \le \frac{C}{\lambda} \|b\|_{\mathrm{BMO}} \sum\nolimits_2 |\lambda_j|.$$

This completes the proof of Theorem 4.1. □

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