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SCALAR CURVATURE OF HYPERSURFACES WITH CONSTANT MEAN CURVATURE IN SPHERES

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Abstract. Let M^n be an n-dimensional closed hypersurface with constant mean curvature H satisfying $|H| \leq \varepsilon(n)$ in a unit sphere $S^{n+1}(1)$, $n \leq 8$ and S the square of the length of the second fundamental form of M. There exists a constant $\delta(n, H) > 0$, which depends only on n and H such that if $S_0 \leq S \leq S_0 + \delta(n, H)$, then $S \equiv S_0$ and M is isometric to a Clifford hypersurface, where $\varepsilon(n)$ is a sufficiently small constant depending on n and $S_0 = n + \frac{n^3}{2(n-1)}H^2 + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}$.

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1. Introduction. Let M^n be an n-dimensional closed hypersurface with constant mean curvature H in a unit sphere $S^{n+1}(1)$ of dimension $n+1$, denoted by S the squared norm of the second fundamental form of M^n.

When $H \equiv 0$, Lawson [**16**], Simons [**10**] and Chern et al. [**8**] obtained independently the famous rigidity theorem, which says, if $S \leq n$, then $S \equiv 0$, or $S \equiv n$, i.e. M^n is the great sphere $S^n(1)$, or the Clifford torus. Further discussions in this direction have been carried out by many other authors [**2**, **5**, **7**, **12**, **18**, **19**–**21**]. In [**14**], Peng and Terng proved that if the scalar curvature of M is constant, then there exists a positive constant $\alpha(n)$ depending only on n such that if $n \leq S \leq n + \alpha(n)$, then $S \equiv n$. Later, Cheng and Yang [**6**] improved the pinching constant $\alpha(n)$ to $\frac{n}{3}$. Without the assumption of constant scalar curvature, Peng and Terng [**15**] proved that if $M^n (n \leq 5)$ is a closed minimal hypersurface in S^{n+1}, then there exists a positive constant $\alpha(n)$ depending only on n such that if $n \leq S \leq n + \alpha(n)$, then $S \equiv n$. So they proposed the following attractive problem:

Let $M^n (n \geq 6)$ be a closed minimal hypersurface in S^{n+1}. Does there exist a positive constant $\alpha(n)$ depending only on n such that if $n \leq S \leq n + \alpha(n)$, then $S \equiv n$ and M is isometric to a Clifford torus $S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$?

In [**3**], Cheng gave a positive answer under the additional condition that M has only two distinct principal curvatures. Later, Hasanis and Vlachos [**9**] proved that if M^n is a compact minimal hypersurface in S^{n+1} with two distinct principal curvatures and the squared norm S of the second fundamental form of M^n satisfies $S \geq n$, then M^n is a minimal Clifford torus. In [**5**], Cheng and Ishikawa improved the result of Peng and Terng [**15**] when $n \leq 5$. Later, Wei and Xu [**17**] solved the problem proposed by Peng and Terng [**15**] for $n = 6$ and 7. Recently, we [**22**] obtained a sharper pinching constant of S for $n \leq 7$ and solved this problem for $n = 8$.

When M is a hypersurface with constant mean curvature, Alencar and do Carmo [**1**] proved the first rigidity result under the assumption that the traceless second fundamental form is sufficiently bounded. Later, Li [**11**] extended the result of Peng and Terng [**15**] for minimal hypersurfaces to the case of hypersurfaces with constant mean curvature. That is, Li [**11**] proved the following theorem:

Let M be an n-dimensional closed hypersurface with constant mean curvature H satisfying $|H| \leq \varepsilon(n)$ in a unit sphere S^{n+1}, $n \leq 5$, and S the square of the length of the second fundamental form of M. Then there exists a constant $\delta(n, H) > 0$, which depends only on n and H, such that if $S_0 \leq S \leq S_0 + \delta(n, H)$, then $S \equiv S_0$ and M is isometric to a Clifford torus $S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$ if $H = 0$; M is isometric to a Clifford hypersurface $C_{1,n-1} = S^1(\frac{1}{\sqrt{1+\lambda^2}}) \times S^{n-1}(\frac{\lambda}{\sqrt{1+\lambda^2}})$ if $H \neq 0$, where $\lambda = \frac{nH+\sqrt{n^2H^2+4(n-1)}}{2}$ and $\varepsilon(n)$ is a sufficiently small constant depending on n, $S_0 = n + \frac{n^3}{2(n-1)}H^2 + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}$.

In [**4**], Cheng, He and Li proved the above theorem is valid for the case of $n = 6, 7$. In this paper, we study the case of $n = 8$. We prove the following theorem.

THEOREM 1.1. *Let M be an n-dimensional closed hypersurface with constant mean curvature H satisfying $|H| \leq \varepsilon(n)$ in a unit sphere S^{n+1}, $n \leq 8$, and S the square of the length of the second fundamental form of M. Then there exists a constant $\delta(n, H) > 0$, which depends only on n and H, such that if $S_0 \leq S \leq S_0 + \delta(n, H)$, then $S \equiv S_0$ and M is isometric to a Clifford torus $S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$ if $H = 0$; M is isometric to a Clifford hypersurface*

$$C_{1,n-1} = S^1\left(\frac{1}{\sqrt{1+\lambda^2}}\right) \times S^{n-1}\left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right)$$

if $H \neq 0$, where $\lambda = \frac{nH+\sqrt{n^2H^2+4(n-1)}}{2}$ and $\varepsilon(n)$ is a sufficiently small constant depending on n,

$$S_0 = n + \frac{n^3}{2(n-1)}H^2 + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}. \tag{1.1}$$

2. Fundamental formulas. Let M^n be an n-dimensional hypersurface with constant mean curvature H in an $(n+1)$-dimensional unit sphere $S^{n+1}(1)$. We choose a local orthonormal frame field $e_1, \ldots, e_{n+1}$ in $S^{n+1}(1)$, restricted to M^n, so that $e_1, \ldots, e_n$ are tangent to M^n. Let $\omega_1, \ldots, \omega_{n+1}$ denote the dual coframe field in $S^{n+1}(1)$. Then in M^n, $\omega_{n+1} = 0$. It follows from Cartan's Lemma that

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j. \tag{2.1}$$

The second fundamental form α and the mean curvature H of M^n are defined by

$$\alpha = \sum_{ij} h_{ij}\omega_i\omega_j e_{n+1}, \quad nH = \sum_i h_{ii}, \tag{2.2}$$

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respectively. The connection form ω_{ij} is characterized by the structure equations

$$d\omega_i + \sum_j \omega_{ij} \wedge \omega_j = 0, \quad \omega_{ij} + \omega_{ji} = 0, \tag{2.3}$$

$$d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, \tag{2.4}$$

$$\Omega_{ij} = \frac{1}{2} \sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \tag{2.5}$$

where Ω_{ij} (resp. R_{ijkl}) denotes the curvature form (resp. the components of the curvature tensor) of M^n. The Gauss equation is given by

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}). \tag{2.6}$$

Denote by $h_{ijk}, h_{ijkl}, h_{ijklm}$ components of the first, second and third covariant derivatives of the second fundamental form, respectively. Then

$$h_{ijk} = h_{ikj} = h_{jik}, \tag{2.7}$$

$$h_{ijkl} - h_{ijlk} = \sum_m h_{im}R_{mjkl} + \sum_m h_{mj}R_{mikl}, \tag{2.8}$$

$$h_{ijklm} - h_{ijkml} = \sum_r h_{rjk}R_{rilm} + \sum_r h_{irk}R_{rjlm} + \sum_r h_{ijr}R_{rklm}. \tag{2.9}$$

For any fixed point p in M^n, we take a local orthonormal frame field $e_1, \ldots, e_n$ such that

$$h_{ij} = \begin{cases} \lambda_i, & i = j, \\ 0, & i \neq j. \end{cases} \tag{2.10}$$

We define the squared norm of the second fundamental form S of M, f_3, f_4 to be

$$S = \sum_{i,j} h_{ij}^2, \quad f_3 = \sum_{i,j,k} h_{ij}h_{jk}h_{ki}, \quad f_4 = \sum_{i,j,k,l} h_{ij}h_{jk}h_{kl}h_{li}. \tag{2.11}$$

Then at the point p, we have

$$S = \sum_i \lambda_i^2, \quad f_3 = \sum_i \lambda_i^3, \quad f_4 = \sum_i \lambda_i^4. \tag{2.12}$$

Since the mean curvature H of M is a constant, using the above equations, we easily get

$$\frac{1}{2}\triangle S = \sum_{i,j,k} h_{ijk}^2 - S(S-n) - n^2H^2 + nHf_3, \tag{2.13}$$

$$\begin{aligned} \frac{1}{2}\triangle \sum_{i,j,k} h_{ijk}^2 = \sum_{i,j,k,l} h_{ijkl}^2 + (2n+3-S)\sum_{i,j,k} h_{ijk}^2 + 3(2B - A) \\ + 3nH \sum_{i,j,k} \lambda_i h_{ijk}^2 - \frac{3}{2}|\nabla S|^2, \end{aligned} \tag{2.14}$$

where $A = \sum_{i,j,k} \lambda_i^2 h_{ijk}^2$, $B = \sum_{i,j,k} \lambda_i\lambda_j h_{ijk}^2$.

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3. Proof of Theorem. At first, we give two lemmas which will play a crucial role in the proof of our theorem. For convenience, we define

$$\mu_{ij} = h_{ij} - H\delta_{ij}, \quad \mu_i = \mu_{ii}, \quad \widetilde{A} = \sum_{i,j,k} \mu_i^2 h_{ijk}^2, \quad \widetilde{B} = \sum_{i,j,k} \mu_i \mu_j h_{ijk}^2. \tag{3.1}$$

Then

$$A - 2B = \widetilde{A} - 2\widetilde{B} + 2H \sum_{i,j,k} \lambda_i h_{ijk}^2 + H^2 \sum_{i,j,k} h_{ijk}^2, \tag{3.2}$$

$$\sum_i \mu_i = 0, \quad \sum_i \mu_i^2 = S - nH^2.$$

LEMMA 3.1. *Let M be a closed hypersurface with constant mean curvature H in $S^{n+1}(1)$. Then*

$$\sum_{i,j,k,l} h_{ijkl}^2 \geq \frac{3}{2}\{(Sf_4 - f_3^2 - S^2 + nHf_3) - [S(S-n) + n^2H^2 - nHf_3]\} + \frac{3[S(S-n) + n^2H^2 - nHf_3]^2}{2(n+4)(S-nH^2)}.$$

Proof. From formulae (2.6) and (2.8), we have

$$\begin{aligned} h_{iijj} - h_{jjii} = h_{ijij} - h_{ijji} &= \sum_m h_{im}R_{mjij} + \sum_m h_{jm}R_{miij} \\ &= \lambda_i R_{ijij} + \lambda_j R_{jiij} = (\lambda_i - \lambda_j)R_{ijij} \\ &= (\lambda_i - \lambda_j)(1 + \lambda_i\lambda_j). \end{aligned} \tag{3.3}$$

We define

$$u_{ijkl} = \frac{1}{4}(h_{ijkl} + h_{jkli} + h_{klij} + h_{lijk}). \tag{3.4}$$

Since h_{ijkl} is symmetric in the indices i, j, k, from equation (3.3) we obtain

$$\begin{aligned} \sum_{i,j,k,l} h_{ijkl}^2 &= \sum_{i,j,k,l} u_{ijkl}^2 + \frac{3}{8}\sum_{i,j,k,l}(h_{ijkl} - h_{ijlk})^2 \\ &\geq \sum_{i,j,k,l} u_{ijkl}^2 + \frac{3}{4}\sum_{i,j}(h_{iijj} - h_{jjii})^2 \\ &= \sum_{i,j,k,l} u_{ijkl}^2 + \frac{3}{2}\{(Sf_4 - f_3^2 - 2S^2 + nS - n^2H^2 + 2nHf_3\}. \end{aligned} \tag{3.5}$$

Since $\sum_i h_{iikl} = 0$, we have

$$\sum_{i,j} \mu_i u_{iijj} = \frac{1}{2}(nS - S^2 - n^2H^2 + nHf_3). \tag{3.6}$$

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Since for any $\alpha \in R$,

$$\sum_{i,j,k,l}[u_{ijkl} + \alpha(\mu_{ij}\delta_{kl} + \mu_{ik}\delta_{jl} + \mu_{il}\delta_{jk} + \mu_{jk}\delta_{il} + \mu_{jl}\delta_{ik} + \mu_{kl}\delta_{ij})]^2 \geq 0, \tag{3.7}$$

it follows from equations (3.2) and (3.6) that

$$\sum_{i,j,k,l} u_{ijkl}^2 \geq 6\alpha(S^2 - nS - nHf_3 + n^2H^2) - 6\alpha^2(n+4)(S - n^2H^2). \tag{3.8}$$

Letting

$$\alpha = \frac{S(S-n) + n^2H^2 - nHf_3}{2(n+4)(S - n^2H^2)}, \tag{3.9}$$

we have

$$\sum_{i,j,k,l} u_{ijkl}^2 \geq \frac{3[S(S-n) + n^2H^2 - nHf_3]^2}{2(n+4)(S - nH^2)}. \tag{3.10}$$

Thus we have finished the proof of Lemma 3.1. □

LEMMA 3.2. *Let M be an n-dimensional closed hypersurface with constant mean curvature H in $S^{n+1}(1)$, for $n \leq 8$. Then*

$$3(\widetilde{A} - 2\widetilde{B}) \leq 2.34(S - nH^2)\sum_{i,j,k} h_{ijk}^2.$$

Proof. Since $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = S - nH^2 = \widetilde{S}$, the following equation can be proved in the same method as in our early paper (Lemma 3.4 in [**22**]):

$$\sum_{i(\neq j)} (\mu_j^2 - 4\mu_j\mu_i)h_{ij}^2 - \mu_j^2 h_{jjj}^2 \leq 2.34\widetilde{S}\left(\sum_{i(\neq j)} h_{iij}^2 + \frac{1}{3}h_{jjj}^2\right), \quad \forall j. \tag{3.11}$$

Hence we get

$$\begin{aligned} 3(\widetilde{A} - 2\widetilde{B}) &= \sum_{i\neq j\neq k\neq i} \left[2\left(\mu_i^2 + \mu_j^2 + \mu_k^2\right) - (\mu_i + \mu_j + \mu_k)^2\right]h_{ijk}^2 \\ &\quad - 3\sum_i \mu_i^2 h_{iii}^2 + 3\sum_{i\neq j}\left(\mu_j^2 - 4\mu_i\mu_j\right)h_{iij}^2 \\ &\leq 2\widetilde{S}\sum_{i\neq j\neq k\neq i} h_{ijk}^2 + 3\sum_j\left\{\sum_{i\neq j}\left[\left(\mu_j^2 - 4\mu_i\mu_j\right)h_{iij}^2 - \mu_j^2 h_{jjj}^2\right]\right\} \\ &\leq 2.34\widetilde{S}\left\{\sum_{i\neq j\neq k\neq i} h_{ijk}^2 + 3\sum_{i\neq j} h_{iij}^2 + \sum_j h_{jjj}^2\right\} \\ &= 2.34\widetilde{S}\sum_{i,j,k} h_{ijk}^2. \end{aligned}$$

This completes the proof of Lemma 3.2. □

Proof of Theorem 1.1. Now, we assume

$$S_0 \leq S \leq S_0 + \delta(n, H), \tag{3.12}$$

where S_0 is defined by equation (1.1).

It is not difficult to prove the following elementary inequality (cf. [**13**]):

$$\left|\sum_i (\lambda_i - H)^3\right| \leq \frac{n-2}{\sqrt{n(n-1)}}(S - nH^2)^{\frac{3}{2}}.$$

Since $S \geq S_0$ is equivalent to

$$\sqrt{n + \frac{n^3 H^2}{4(n-1)}} - \sqrt{S - nH^2} + \frac{n(n-2)|H|}{2\sqrt{n(n-1)}} \leq 0,$$

we have

$$\begin{aligned}
&S(S-n) + n^2H^2 - nHf_3 \\
&\quad = -(S - nH^2)\{n + nH^2 - (S - nH^2)\} - nH\sum_i (\lambda_i - H)^3 \\
&\quad \geq -(S - nH^2)\left\{n + nH^2 - (S - nH^2) + \frac{n(n-2)|H|}{\sqrt{n(n-1)}}\sqrt{S - nH^2}\right\} \\
&\quad \geq -(S - nH^2)\left\{\sqrt{n + \frac{n^3H^2}{4(n-1)}} + \sqrt{S - nH^2} - \frac{n(n-2)|H|}{2\sqrt{n(n-1)}}\right\} \\
&\qquad \times \left\{\sqrt{n + \frac{n^3H^2}{4(n-1)}} - \sqrt{S - nH^2} + \frac{n(n-2)|H|}{2\sqrt{n(n-1)}}\right\} \\
&\quad \geq 0.
\end{aligned} \tag{3.13}$$

The following equation can be found in [**22**] or [**11**]:

$$\int_M (A - 2B)dM = \int_M \left[Sf_4 - f_3^2 - S^2 + nHf_3 - \frac{1}{4}|\nabla S|^2\right] dM. \tag{3.14}$$

Integrating equation (2.13) and $S \times$ (2.13) gives

$$\int_M \sum_{i,j,k} h_{ijk}^2 dM = \int_M \left[S(S-n) + n^2H^2 - nHf_3\right] dM. \tag{3.15}$$

$$\int_M \frac{1}{2}|\nabla S|^2 dM = \int_M \left[S^2(S-n) + n^2H^2S - nHSf_3 - S\sum_{i,j,k} h_{ijk}^2\right] dM. \tag{3.16}$$

Noticing that

$$\begin{aligned}
&S(S-n) + n^2H^2 - nHf_3 \\
&\quad = (S - nH^2)(S - S_0) + n^2H^2 - nHf_3 - nH^2S_0 + (S_0 + nH^2 - n)S,
\end{aligned} \tag{3.17}$$

from equations (3.12) and (3.13), there exists some constant α_1 such that

$$\frac{3[S(S-n)+n^2H^2-nHf_3]^2}{2(n+4)(S-nH^2)} \geq \left\{\frac{3(S-S_0)}{2(n+4)} - \alpha_1 H\right\}[S(S-n)+n^2H^2-nHf_3]. \tag{3.18}$$

It follows from equations (3.14), (3.15), (3.18) and Lemma 3.1 that

$$\int_M \sum_{i,j,k,l} h_{ijkl}^2 dM \geq \int_M \left[\frac{3}{2}(A-2B) - \frac{3}{2}\sum_{i,j,k} h_{ijk}^2 + \frac{3}{8}|\nabla S|^2\right] dM \tag{3.19}$$
$$+ \int_M \left[\frac{3(S-S_0)}{2(n+4)} - \alpha_1 H\right][S(S-n)+n^2H^2-nHf_3] dM.$$

From equations (2.14) and (3.2), we have

$$\int_M \sum_{i,j,k,l} h_{ijkl}^2 dM = \int_M \left[(S-2n-3)\sum_{i,j,k} h_{ijk}^2 + \frac{3}{2}|\nabla S|^2 + \frac{3}{2}(A-2B)\right] dM$$
$$+ \int_M \left[\frac{3}{2}(\widetilde{A}-2\widetilde{B}) + 3(1-n)H\sum_{i,j,k}\lambda_i h_{ijk}^2 + \frac{3}{2}H^2\sum_{i,j,k} h_{ijk}^2\right] dM.$$

Since $S_0 \leq S \leq S_0 + \delta(n, H)$, there exists some constant α_2 such that

$$\int_M \sum_{i,j,k,l} h_{ijkl}^2 dM \leq \int_M \left[(S-2n-3+\alpha_2 H)\sum_{i,j,k} h_{ijk}^2 + \frac{3}{2}(A-2B)\right] dM$$
$$+ \int_M \left[\frac{3}{2}(\widetilde{A}-2\widetilde{B}) + \frac{3}{2}|\nabla S|^2\right] dM. \tag{3.20}$$

From equations (3.13), (3.15), (3.16), (3.19), (3.20) and Lemma 3.2, we obtain

$$\begin{aligned}
0 &\leq \int_M \left\{\left[S-2n-\frac{3}{2}+(\alpha_1+\alpha_2)H\right]\sum_{i,j,k} h_{ijk}^2 + \frac{3}{2}(\widetilde{A}-2\widetilde{B})\right\} dM \\
&\quad + \int_M \left\{\frac{9}{8}|\nabla S|^2 - \frac{3(S-S_0)}{2(n+4)}\left[S(S-n)+n^2H^2-nHf_3\right]\right\} dM \\
&\leq \int_M \left[-0.08S-2n-\frac{3}{2}+(\alpha_1+\alpha_2-1.17nH)H\right]\sum_{i,j,k} h_{ijk}^2 dM \\
&\quad + \int_M \left[\frac{9}{4}S - \frac{3(S-S_0)}{2(n+4)}\right]\left[S(S-n)+n^2H^2-nHf_3\right] dM \\
&\leq \int_M \left[-0.08S-2n-\frac{3}{2}+(\alpha_1+\alpha_2-1.17nH)H\right]\sum_{i,j,k} h_{ijk}^2 dM \\
&\quad + \int_M \left\{\frac{9}{4}S_0 + \frac{9n+30}{4(n+4)}\delta(n, H)\right\}\left[S(S-n)+n^2H^2-nHf_3\right] dM \\
&\leq \int_M G\sum_{i,j,k} h_{ijk}^2 dM,
\end{aligned} \tag{3.21}$$

where $G = 2.17S_0 + \frac{9n+30}{4(n+4)}\delta(n, H) - 2n - \frac{3}{2} + (\alpha_1+\alpha_2-1.17nH)H.$

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Since $2.17n - 2n - \frac{3}{2} < 0$ and $|H| \leq \varepsilon(n)$, if $\varepsilon(n)$ is small enough, we can choose $\delta(n, H)$ such that

$$2.17S_0 + \frac{9n+30}{4(n+4)}\delta(n, H) - 2n - \frac{3}{2} + (\alpha_1 + \alpha_2 - 1.17nH)H < 0. \tag{3.22}$$

According to equations (3.21) and (3.22), we infer $\sum_{i,j,k} h_{ijk}^2 \equiv 0$. Hence, all of the above inequalities are equalities. From equation (3.13) and (3.15), we have $S \equiv S_0$ and M is isometric to a Clifford hypersurface. Thus we have finished the proof of Theorem 1.1. □

REMARK 3.3. In the proof of Theorem 1.1, the constants α_1 and α_2 are chosen so that

$$\alpha_1 \geq \frac{3\delta(n, H)}{3(n+4)H}, \qquad \alpha_2 > \frac{3S_0}{2nH} \qquad \text{if } H > 0$$

and so

$$\delta(n, H) := \min\left\{\frac{(n+4)(2n+3/2-2.17n)}{3n+10}, \frac{2(n+4)H}{3}\alpha_1, \frac{2nH}{3}\alpha_2 - S_0\right\}.$$

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