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CHARACTERISATIONS OF HARDY GROWTH SPACES WITH DOUBLING WEIGHTS

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Abstract

Let $H(\mathbb{D})$ denote the space of holomorphic functions on the unit disc $\mathbb{D}$. Given $p > 0$ and a weight ω, the Hardy growth space $H(p, \omega)$ consists of those $f \in H(\mathbb{D})$ for which the integral means $M_p(f, r)$ are estimated by $C\omega(r)$, $0 < r < 1$. Assuming that $p > 1$ and ω satisfies a doubling condition, we characterise $H(p, \omega)$ in terms of associated Fourier blocks. As an application, extending a result by Bennett *et al.* ['Coefficients of Bloch and Lipschitz functions', *Illinois J. Math.* **25** (1981), 520–531], we compute the solid hull of $H(p, \omega)$ for $p \geq 2$.

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1. Introduction

Let $H(\mathbb{D})$ denote the space of holomorphic functions on the unit disc $\mathbb{D}$. For $p > 0$ and $f \in H(\mathbb{D})$, put

$$M_p(f, r) = \left(\int_{\mathbb{T}} |f(r\zeta)|^p \, dm(\zeta) \right)^{1/p}, \quad 0 \leq r < 1,$$

where m denotes the normalised Lebesgue measure on the unit circle $\mathbb{T} = \partial\mathbb{D}$.

1.1. Hardy growth spaces. A function $\omega : [0, 1) \to (0, +\infty)$ is called a weight if ω is increasing, unbounded and continuous. Given $p > 0$ and a weight ω, the Hardy growth space $H(p, \omega)$ consists of those $f \in H(\mathbb{D})$ for which

$$\|f\|_{H(p,\omega)} = \sup_{0 \leq r < 1} \frac{M_p(f, r)}{\omega(r)} < \infty. \tag{1.1}$$

The spaces $H(p, \omega)$ were introduced in [10]. For $\omega \equiv 1$, (1.1) defines the classical Hardy space $H^p = H^p(\mathbb{D})$. However, every weight ω is assumed to be unbounded; hence, H^p is formally excluded from the scale of Hardy growth spaces.

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1.2. Blocking technique. Let $1 < p < \infty$ and let $\Omega_\alpha(t) = (1-t)^{-\alpha}$, $\alpha > 0$. Then it is known that equivalent definitions of $H(p, \Omega_\alpha)$ are related to the so-called blocking technique (see [7]). Namely, given a function $f(z) = \sum_{k=0}^{\infty} a_k z^k \in H(p, \Omega_\alpha)$, the following Fourier blocks are useful:

$$\Delta_0 f(z) = a_0 + a_1 z,$$
$$\Delta_j f(z) = \sum_{k=2^j}^{2^{j+1}-1} a_k z^k, \quad j \geq 1.$$

THEOREM 1.1 [8, Theorem 2.1]. *Let $\alpha > 0$, $1 < p < \infty$ and $f \in H(\mathbb{D})$. Then $f \in H(p, \Omega_\alpha)$ if and only if*

$$\|\Delta_j f\|_{H^p} \leq C\Omega_\alpha(1 - 2^{-j}) = C2^{\alpha j}, \quad j = 0, 1, \ldots,$$

for a constant $C > 0$.

In fact, the above theorem extends to the weights ω that are normal in the sense of [9]; see [8] for details. However, if $\omega_\beta(t) = (\log(2/(1-t)))^\beta$, $\beta > 0$, then it is natural to consider different Fourier blocks. For $f(z) = \sum_{k=0}^{\infty} a_k z^k$, put

$$\delta_0 f(z) = a_0 + a_1 z + a_2 z^2,$$
$$\delta_j f(z) = \sum_{k=2^{2^j}}^{2^{2^{j+1}}-1} a_k z^k, \quad j \geq 1.$$

The following characterisation of the property $f \in H(p, \omega_\beta)$ is known.

THEOREM 1.2 [6, Theorem 5.1]. *Let $\beta > 0$, $1 < p < \infty$ and $f \in H(\mathbb{D})$. Then $f \in H(p, \omega_\beta)$ if and only if*

$$\|\delta_j f\|_{H^p} \leq C2^{\beta j}, \quad j = 0, 1, \ldots,$$

for a constant $C > 0$.

1.3. Doubling weights. In the present paper, we obtain analogues of Theorems 1.1 and 1.2 for all doubling weights. By definition, a weight $\omega : [0, 1) \to (0, +\infty)$ is called *doubling* if there exists a constant $A > 1$ such that

$$\omega\Big(1 - \frac{s}{2}\Big) \leq A\omega(1 - s), \quad 0 < s \leq 1. \tag{1.2}$$

The doubling property (1.2) is a natural technical assumption (see, for example, [1, 3–5]). In particular, Ω_α with $\alpha > 0$, the normal weights and ω_β with $\beta > 0$ are doubling weights. On the one hand, (1.2) is a restriction on the growth of ω; on the other hand, the class of doubling weights contains functions that grow arbitrarily slowly. Also, it is worth mentioning that the standard doubling property of the measure $\omega(r)\,dr$ is not related to (1.2).

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In what follows, ω denotes a doubling weight. In Section 2, we construct an increasing sequence $\{n_j\}$ that is adapted to ω via the doubling constant $A > 1$ from estimate (1.2). In Section 3, we use the associated Fourier blocks

$$\Delta_j^A f(z) = \sum_{k=n_j}^{n_{j+1}-1} a_k z^k$$

to characterise the property $f \in H(p, \omega)$. As an application, we characterise the Hadamard lacunary series in $H(p, \omega)$ and the solid hull of $H(p, \omega)$, $2 \le p < \infty$; see Section 4.

2. Doubling weights as lacunary series with positive coefficients

Given two functions $u, v : [0, 1) \to (0, +\infty)$, we say that u and v are equivalent and we write $u \asymp v$ if

$$C_1 u(t) \le v(t) \le C_2 u(t), \quad 0 \le t < 1,$$

for some constants $C_1, C_2 > 0$.

Let ω be a doubling weight. In this section, we construct an increasing sequence $\{n_j\}$ of positive integers such that

$$\omega(t) \asymp \sum_{j=0}^{\infty} b_j t^{n_j}, \quad 0 \le t < 1,$$

for appropriate coefficients b_j, $b_j > 0$.

Without loss of generality, assume that $\omega(0) = 1$. We use the auxiliary function

$$\Phi(x) = \omega\Big(1 - \frac{1}{x}\Big), \quad x \ge 1.$$

Thus, $\Phi(1) = 1$ and $\omega(t) = \Phi(1/(1 - t))$, $0 \le t < 1$. The doubling condition (1.2) becomes

$$\Phi(2x) \le A\Phi(x), \quad x \ge 1. \tag{2.1}$$

For $j = 1, 2, \dots$, put

$$n_j = \max\{k \in \mathbb{N} : \Phi(k) \le A^j\}. \tag{2.2}$$

Below we often use the definition of n_j without explicit reference.

The sequence $\{n_j\}_{j=1}^{\infty}$ and its analogues are known to be useful in constructions of holomorphic or harmonic lacunary series in the growth spaces defined by the weight ω (see [1, 5]). In particular, certain arguments in the present section are similar to those in [1, Lemma 1].

By the definition of n_j, we have $\Phi(n_j + 1) > A^j$. Hence, by (2.1),

$$\Phi(n_j) > A^{j-1}. \tag{2.3}$$

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Also, observe that $\Phi(2n_j) \le A\Phi(n_j) \le A^{j+1} < \Phi(n_{j+1}+1)$. Since Φ is an increasing function, $n_{j+1} + 1 > 2n_j$. Therefore,

$$\frac{n_{j+1}}{n_j} \ge 2, \tag{2.4}$$

for $j = 1, 2, \ldots$.

Lemma 2.1. *Let ω be a doubling weight with a doubling constant $A > 1$. Put*

$$\Omega(t) = \sum_{j=0}^{\infty} A^j t^{n_j}, \quad 0 \le t < 1,$$

where $n_0 = 0$ and the sequence $\{n_j\}_{j=1}^{\infty}$ is defined by (2.2). Then $\Omega \asymp \omega$.

Proof. Put

$$t_j = 1 - \frac{1}{n_j}, \quad j = 1, 2, \ldots.$$

Fix an integer m, $m \ge 1$. Let $t_m \le t < t_{m+1}$. First, applying (2.3),

$$\sum_{j=0}^{m} A^j t^{n_j} \le \sum_{j=0}^{m} A^j = \frac{A^{m+1}}{A-1} < \frac{A^2}{A-1}\Phi(n_m) \le C\Phi\Big(\frac{1}{1-t}\Big),$$

because Φ is increasing. Second, applying (2.3) and (2.4),

$$\begin{aligned}
\sum_{j=m+1}^{\infty} A^j t^{n_j} &\le \sum_{j=m+1}^{\infty} A^j \Big(1 - \frac{1}{n_{m+1}}\Big)^{n_j} \\
&\le A^2\Phi(n_m) \sum_{k=0}^{\infty} A^k \Big(1 - \frac{1}{n_{m+1}}\Big)^{n_{m+1}\cdot n_{m+1+k}/n_{m+1}} \\
&\le A^2\Phi\Big(\frac{1}{1-t}\Big) \sum_{k=0}^{\infty} \frac{A^k}{\exp(2^k)} \\
&\le C\Phi\Big(\frac{1}{1-t}\Big).
\end{aligned}$$

In sum, we obtain

$$\Omega(t) = \sum_{j=0}^{\infty} A^j t^{n_j} \le C\Phi\Big(\frac{1}{1-t}\Big) = C\omega(t), \quad t_1 \le t < 1.$$

Since $\omega(t) \ge 1$ for $0 \le t \le t_1$, we conclude that

$$\Omega(t) \le C\omega(t), \quad 0 \le t < 1.$$

To prove the reverse estimate, fix an integer m, $m \ge 1$. If $t_m \le t < t_{m+1}$, then

$$\Omega(t) \ge A^m t^{n_m} \ge \frac{\Phi(n_{m+1})}{4A} \ge \frac{1}{4A}\Phi\Big(\frac{1}{1-t}\Big) = \frac{\omega(t)}{4A}.$$

Also, $\Omega(t) \ge 1$ for $0 \le t \le t_1$. Hence, $\omega(t) \le C\Omega(t)$, $0 \le t < 1$. Therefore, $\omega \asymp \Omega$, as required. □

3. Decomposition theorems

Let ω be a doubling weight with a doubling constant $A > 1$ and let $\{n_j\}_{j=1}^{\infty}$ be the associated sequence of integers defined by (2.2). For $f(z) = \sum_{k=0}^{\infty} a_k z^k \in H(\mathbb{D})$, put

$$\Delta_j^A f(z) = \sum_{k=n_j}^{n_{j+1}-1} a_k z^k, \quad z \in \mathbb{D}, \ j = 0, 1, \ldots, \tag{3.1}$$

where $n_0 = 0$. To work with the blocks $\Delta_j^A f$, we need the following lemma.

Lemma 3.1 [8, Lemma 3.1]. *Let $p > 0$ and let $g(z) = \sum_{k=n}^{m} a_k z^k$, $n < m$, $z \in \mathbb{D}$. Then*

$$r^m \|g\|_{H^p} \le M_p(g, r) \le r^n \|g\|_{H^p}, \quad 0 < r < 1.$$

The following result generalises Theorems 1.1 and 1.2.

Theorem 3.2. *Let ω be a doubling weight with a doubling constant $A > 1$. Assume that $1 < p < \infty$ and $f \in H(\mathbb{D})$. Let the blocks $\Delta_j^A f$ be defined by (3.1). Then $f \in H(p, \omega)$ if and only if*

$$\|\Delta_j^A f\|_{H^p} \le C A^j, \quad j = 0, 1, \ldots, \tag{3.2}$$

for a constant $C > 0$.

Proof. Let $f \in H(p, \omega)$. The Riesz projection theorem and Lemma 3.1 guarantee that

$$M_p(f, r) \ge C M_p(\Delta_j^A f, r) \ge C r^{n_{j+1}} \|\Delta_j^A f\|_{H^p}, \quad 0 < r < 1, \ j = 0, 1, \ldots,$$

where $C > 0$ is a constant that depends only on p, $1 < p < \infty$. Applying the above estimate,

$$\begin{aligned} \sup_{0<r<1} \frac{M_p(f, r)}{\omega(r)} &\ge C \sup_{0<r<1} \frac{r^{n_{j+1}} \|\Delta_j^A f\|_{H^p}}{\omega(r)} \\ &\ge C \frac{\left(1 - \frac{1}{n_{j+1}}\right)^{n_{j+1}} \|\Delta_j^A f\|_{H^p}}{\Phi(n_{j+1})} \\ &\ge \frac{C}{A} \frac{\|\Delta_j^A f\|_{H^p}}{A^j}. \end{aligned}$$

So, the property $f \in H(p, \omega)$ implies (3.2).

To prove the reverse implication, assume that (3.2) holds. Applying the triangle inequality, Lemma 3.1, property (3.2) and Lemma 2.1, we obtain the following chain of inequalities:

$$\begin{aligned} M_p(f, r) &\le \sum_{j=0}^{\infty} M_p(\Delta_j^A f, r) \le \sum_{j=0}^{\infty} r^{n_j} \|\Delta_j^A f\|_{H^p} \\ &\le C \sum_{j=0}^{\infty} A^j r^{n_j} \le C\omega(r), \quad 0 < r < 1. \end{aligned}$$

The proof of the theorem is finished. ◻

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For $n \in \mathbb{N}$ and $f(z) = \sum_{k=0}^{\infty} a_k z^k$, consider the standard partial sums

$$S_n f(z) = \sum_{k=0}^{n-1} a_k z^k, \quad z \in \mathbb{D}.$$

Replacing the blocks $\Delta_j^A f$ by the partial sums $S_n f$, we obtain a more explicit description of the space $H(p, \omega)$.

COROLLARY 3.3. *Let $1 < p < \infty$, $f \in H(\mathbb{D})$ and let ω be a doubling weight. Then $f \in H(p, \omega)$ if and only if*

$$\|S_n f\|_{H^p} \le C\omega\Big(1 - \frac{1}{n}\Big), \quad n = 1, 2, \ldots, \tag{3.3}$$

for a constant $C > 0$.

PROOF. Assume that (3.3) holds. Since $1 < p < \infty$, the Riesz projection theorem and (3.3) guarantee that

$$\|\Delta_j^A f\|_{H^p} \le C\|S_{n_j} f\|_{H^p} \le C\Phi(n_{j+1}) \le CA^{j+1}.$$

Hence, $f \in H(p, \omega)$ by Theorem 3.2.

To prove the reverse implication, assume that $f \in H(p, \omega)$. Applying Theorem 3.2,

$$\|S_{n_{j+1}} f\|_{H^p} \le \sum_{k=0}^{j+1} \|\Delta_k^A f\|_{H^p} \le C \sum_{k=0}^{j+1} A^k \le CA^2 \cdot A^{j-1} \le C\Phi(n_j)$$

by (2.3). Thus, for $n_j < k \le n_{j+1}$,

$$\|S_k f\|_{H^p} \le C\|S_{n_{j+1}} f\|_{H^p} \le C\Phi(n_j) \le C\Phi(k)$$

by the Riesz projection theorem. So, (3.3) holds. The proof of the corollary is finished. □

4. Applications

4.1. Hadamard lacunary series. By definition, the growth space $H(\infty, \omega)$ consists of those $f \in H(\mathbb{D})$ for which $|f(z)| \le C\omega(|z|)$, $z \in \mathbb{D}$.

Assume that $f \in H(\mathbb{D})$ is represented by a Hadamard lacunary series, that is,

$$f(z) = \sum_{j=1}^{\infty} a_{m_j} z^{m_j}, \quad z \in \mathbb{D},$$

where $m_{j+1} \ge \lambda m_j$, $j = 1, 2, \ldots,$ for some $\lambda > 1$. Then, by [4, Theorem 2.2], $f \in H(\infty, \omega)$ if and only if

$$\sum_{m_j \le M} |a_{m_j}| \le C\omega\Big(1 - \frac{1}{M}\Big), \quad M = 1, 2, \ldots.$$

Replacing the norm in ℓ^1 by that in ℓ^2, we obtain an analogous result for $H(p, \omega)$ with $0 < p < \infty$.

Corollary 4.1. *Assume that* $0 < p < \infty$, $f \in H(\mathbb{D})$ *and* f *is represented by a Hadamard lacunary series. Then* $f \in H(p, \omega)$ *if and only if*

$$\Big(\sum_{m_j \le M} |a_{m_j}|^2\Big)^{1/2} \le C\omega\Big(1 - \frac{1}{M}\Big), \quad M = 1, 2, \ldots.$$

Proof. Given $p > 0$, we have $M_p(f, r) \asymp M_2(f, r)$, $0 \le r < 1$, because f is represented by a Hadamard lacunary series. It remains to apply Corollary 3.3 with $p = 2$. □

4.2. The solid hull of $H(p, \omega)$, $2 \le p \le \infty$. To define the solid hull $S(H(p, \omega))$, we identify a function $f(z) = \sum_{j=0}^{\infty} a_j z^j \in H(p, \omega)$ and its sequence $\{a_j\}_{j=0}^{\infty}$ of Taylor coefficients.

Recall that a sequence space X is called *solid* if $\{b_j\} \in X$ whenever $\{a_j\} \in X$ and $|b_j| \le |a_j|$ (see [2]). The solid hull $S(X)$ is the smallest solid space containing X. Formally,

$$S(X) = \{\{\lambda_j\} : \text{there exists } \{a_j\} \in X \text{ such that } |\lambda_j| \le |a_j| \text{ for all } j\}.$$

Let S_ω denote the space of sequences $\{b_j\}_{j=0}^{\infty}$ such that

$$\Big(\sum_{j=0}^{n-1} |b_j|^2\Big)^{1/2} \le C\omega\Big(1 - \frac{1}{n}\Big), \quad n = 1, 2, \ldots.$$

Corollary 4.2. *If* $2 \le p \le \infty$, *then* $S(H(p, \omega)) = S_\omega$.

Proof. Since $\infty \ge p \ge 2$, we have $H(p, \omega) \subset H(2, \omega)$ and $S(H(p, \omega)) \subset S(H(2, \omega)) = S_\omega$ by Corollary 3.3. It remains to observe that $S_\omega \subset S(H(\infty, \omega))$ by [3, Theorem 1.8(b)]. □

We remark that Corollary 4.2 was proved in [3] for $p = \infty$. In particular, a different approach was used in [3] to prove the property $S(H(\infty, \omega)) \subset S_\omega$. Also, it would be interesting to compute the solid hull $S(H(p, \omega))$ for $1 < p < 2$.

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