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ON THE PEANO DERIVATIVES

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1. Introduction. Let f be a real valued function defined in some neighbourhood of a point x. If there are numbers $\alpha_1, \alpha_2, \ldots \alpha_{r-1}$, independent of h such that

$$f(x+h) = f(x) + h\alpha_1 + \frac{h^2}{2}\alpha_2 + \ldots \frac{h^{r-1}}{(r-1)!}\alpha_{r-1} + o(h^{r-1})$$

then the number α_k is called the kth Peano derivative (also called kth de la Vallée Poussin derivative [**6**]) of f at x and we write $\alpha_k = f_k(x)$. It is convenient to write $\alpha_0 = f_0(x) = f(x)$. The definition is such that if the mth Peano derivative exists so does the nth for $0 \leqq n \leqq m$. Also if $f^{(n)}(x)$, the ordinary nth derivative of f at x, exists then necessarily $f_n(x)$ exists and equals $f^{(n)}(x)$ and hence also $f_k(x)$ exists and equals $f^{(k)}(x)$ for $0 \leqq k \leqq n$. The converse is true only for $n = 1$.

Let us suppose that $f_{r-1}(x)$ exists. Then the upper and the lower rth Peano derivatives of f at x are defined as the upper and the lower limits of

$$\frac{r!}{h^r}\left\{f(x+h) - \sum_{k=0}^{r-1}\frac{h^k}{k!}\alpha_k\right\}$$

as h tends to 0. They will be denoted by $\overline{f}_r(x)$ and $\underline{f}_r(x)$ respectively. When they are equal we shall say that the rth Peano derivative of f at x exists. (We are allowing $f_r(x)$ to be infinite, although for the existence of $f_r(x)$ all the previous derivatives $f_0(x), f_1(x), \ldots . f_{r-1}(x)$ should be finite).

In a recent paper [**12**] Verblunsky proved that for $n \geqq 2$ (i) if f_{n-1} is defined in $[a, b]$ and $\overline{f}_n > 0$ except on a denumerable subset in $[a, b]$ then f_{n-1} is continuous and nondecreasing in $[a, b]$, and, (ii) if f_n is defined and bounded on one side in $[a, b]$ then $f^{(n)}$ exists and $f^{(n)} = f_n$ in $[a, b]$. The last result of Verblunsky is due to Oliver [**8**]. It may be noted that a similar result in this direction has also been obtained by Bullen [**2**] which asserts that for $n \geqq 2$, if f_{n-1} exists in $[a, b]$, the right hand upper Peano derivative f_n^+ (i.e. restricting h to be positive while finding $\overline{f}_n$) is nonnegative almost everywhere in $[a, b]$ and $f_n^+ > -\infty$ except on a denumerable subset of $[a, b]$ then f is n-convex (or, equivalently f_{n-1} is nondecreasing) in $[a, b]$.

The purpose of the present note is to obtain sufficient conditions implying the monotonicity of the function f_{n-1} and to study some consequences.

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2. Terminology and notations. For convenience we are stating here certain known definitions which will be useful in the proof of our results. For details of these definitions and notations we refer the reader to the book of Saks [**9**].

(i) A function f is said to satisfy *Banach condition* (T_2), if almost every value taken by f is taken at most a denumerable number of times;

(ii) A function f is said to satisfy *Luzin condition* (N) on a set S if for every measurable set $E \subset S$ of measure zero, the set $f(E)$ is also of measure zero;

(iii) A function f is said to be of *generalized bounded variation* (VBG) on a set E if E can be expressed as a denumerable union of sets E_i on each of which f is of bounded variation (VB).

(Here, and elsewhere, denumerable allows finite as a possibility.)

Throughout, f will denote a real function, $[a, b]$ and (a, b) will denote the closed and the open intervals $a \leqq x \leqq b$ and $a < x < b$ respectively, and $m(E)$ will denote the Lebesgue measure of the measurable set E and $m^*(E)$ will denote Lebesgue outer measure for any set E.

(iv) A function f is said to be *n-convex* on $[a, b]$ if for all choices of $(n + 1)$ distinct points $x_0, x_1, \ldots x_n$ in $[a, b]$ the nth divided difference of f at these points is nonnegative. (For details of the definitions and references see [**2**]). So, for $n = 0$, the class of n-convex functions is the class of nonnegative functions, for $n = 1$, it is the class of nondecreasing functions and for $n = 2$, it is the class of usual convex functions. It can be shown that f_{n-1} is nondecreasing, if and only if f is n-convex [**2**].

3. We begin with the following known results.

THEOREM A. *Let $\mathscr{P}$ be any function - theoretic property. A necessary and sufficient condition that every Darboux function of Baire class* 1 *possessing property $\mathscr{P}$ on an interval $[a, b]$ be nondecreasing in $[a, b]$ is that the property $\mathscr{P}$ be sufficiently strong to satisfy the following conditions:*

(i) *Every continuous function of bounded variation possessing property $\mathscr{P}$ on some interval is nondecreasing in that interval, and*

(ii) *Every Darboux function of Baire class* 1 *possessing property $\mathscr{P}$ is* VBG.

This theorem is due to Bruckner (for a proof see [**1**]).

LEMMA B. *If f_k is defined in $[a, b]$, then given any nonempty closed set $H \subset [a, b]$ there is a portion of H on which f_k is bounded.*

This lemma is due to Verblunsky [**12**].

THEOREM C. *If f_n is defined in $[a, b]$ and is bounded on one side at least, then $f_n = f^{(n)}$.*

This is proved in [**8**; **12**].

LEMMA 1. *If at every point x of a set E, except perhaps at the points of a denu-*

merable subset, a function f satisfies any one of the following conditions:

$$\overline{f}_2(x) < \infty, \underline{f}_2(x) > -\infty$$

then f_1 is VBG *on E.*

Proof. Let us suppose that $\overline{f}_2(x) < \infty$ holds for all points x in E, except possibly on a denumerable subset of E and let

$$A = \{x : x \in E; \overline{f}_2(x) < \infty\}.$$

For each positive integer n let A_n denote the set of all points x of A such that

(1) $$|h| \leqq 1/n \text{ implies } f(x + h) - f(x) - hf_1(x) \leqq \tfrac{1}{2}nh^2.$$

For each integer i let $A_{ni} = [i/n, (i + 1)/n] \cap A_n$. Then

(2) $$A = \bigcup_{n=1}^{\infty} \bigcup_{i=-\infty}^{\infty} A_{ni}.$$

Let $g(x) = f_1(x) - nx$. Then for any two points $x_1, x_2, x_1 < x_2$, of A_{ni}, we have $|x_2 - x_1| \leqq 1/n$ and hence by (1)

$$f(x_2) - f(x_1) - (x_2 - x_1)f_1(x_1) \leqq \tfrac{1}{2}n(x_2 - x_1)^2$$

and

$$f(x_1) - f(x_2) - (x_1 - x_2)f_1(x_2) \leqq \tfrac{1}{2}n(x_2 - x_1)^2$$

and hence

$$f_1(x_2) - f_1(x_1) \leqq n(x_2 - x_1),$$

i.e.,

$$g(x_2) \leqq g(x_1).$$

So, the function g is nonincreasing on each A_{ni} and hence A_{ni} can be expressed as the union of a sequence of sets A_{nij} on each of which g is monotone and bounded. So, g is VB on each A_{nij} and hence f_1 is VB on each A_{nij}, from which we conclude that f_1 is VBG on A_{ni} for each n and i. From (2) it follows that f_1 is VBG on A. Since $E - A$ is at most denumerable f_1 is VBG on E.

It can similarly be shown that if $\underline{f}_2(x) > -\infty$ holds for points x in E, except possibly on a denumerable set then f_1 is VBG on E.

COROLLARY. *If $f_2(x)$ exists, finitely or infinitely, on a set E, except perhaps on a denumerable subset, then f_1 is* VBG *on E.*

LEMMA 2. *Let f be a function in $[a, b]$ satisfying the conditions:*

(i) *f_1 is continuous in $[a, b]$,*

(ii) *f_2 exists, finitely or infinitely, except on a denumerable subset in $[a, b]$, and*

(iii) *$f_2 \geqq 0$ almost everywhere in $[a, b]$.*

Then f_1 is nondecreasing in $[a, b]$.

Proof. Since f_1 is continuous in $[a, b]$, we conclude that f is continuous in $[a, b]$. Let E be the set of all points x in $[a, b]$ such that f_1 is not monotone in any neighbourhood of x. Then E is closed. So, $[a, b] - E$ is open in $[a, b]$.

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Since f_1 is continuous, f_1 is nondecreasing in each component interval of $[a, b] - E$. Hence E has no isolated point. For, if E has an isolated point, say x_0, then f_1 is nondecreasing in an interval having x_0 as right hand end point and f_1 is also nondecreasing in an interval having x_0 as left hand end point. But since f_1 is continuous, $x_0 \notin E$ which is a contradiction. Thus E is a perfect set. We show that $E = 0$.

If possible, suppose $E \neq 0$. For each n let P_n denote the set of all points x in $[a, b]$ such that

$$(1) \qquad |t - x| < 1/n \text{ implies } f(t) - f(x) - (t - x)f_1(x) \leqq -\tfrac{1}{2}(t - x)^2$$

and let Q_n denote the set of all points x in $[a, b]$ such that

$$(2) \qquad |t - x| < 1/n \text{ implies } f(t) - f(x) - (t - x)f_1(x) \geqq -(t - x)^2.$$

Since f and f_1 are continuous, the sets P_n and Q_n are closed for each n. Also if f_2 exists, finitely or infinitely, at a point ξ then $\xi \in (\bigcup P_n) \cup (\bigcup Q_n)$. So, the set $[a, b] - (\bigcup P_n) \cup (\bigcup Q_n)$ is at most denumerable and hence the set $(\bigcup P_n) \cup (\bigcup Q_n)$ is residual in $[a, b]$ and, *a fortiori*, is residual in E. Since the set $E \cap [(\bigcup P_n) \cup (\bigcup Q_n)]$ is a residual subset of the complete metric space E there is a portion of E in which one of the sets $E \cap P_n$ or $E \cap Q_n$ is dense. Since P_n and Q_n are closed, we conclude further that there is a portion of E which is contained in one of the sets P_n or Q_n. Let I be an open interval such that $I \cap E \neq 0$ and $I \cap E$ is contained in one of the sets P_n or one of the sets Q_n. Let $I \cap E \subset P_{n_0}$ for some n_0. We may suppose $\delta(I) < 1/n_0$, where $\delta(I)$ denotes the diameter of I. Since $f_2 \geqq 0$ almost everywhere in $[a, b]$ and since the set P_{n_0} is closed, we conclude that P_{n_0} is nondense and hence $I \cap E$ is nondense. Let (α, β) be any interval contiguous to $I \cap E$. Then f_1 is nondecreasing in (α, β) and by continuity of f_1 it is nondecreasing in $[\alpha, \beta]$. But $\alpha, \beta \in P_{n_0}$ and $\beta - \alpha < 1/n_0$, and hence from (1),

$$f(\beta) - f(\alpha) - (\beta - \alpha)f_1(\alpha) \leqq -\tfrac{1}{2}(\beta - \alpha)^2$$

and

$$f(\alpha) - f(\beta) - (\alpha - \beta)f_1(\beta) \leqq -\tfrac{1}{2}(\beta - \alpha)^2$$

which gives

$$f_1(\beta) - f_1(\alpha) \leqq -(\beta - \alpha) < 0,$$

which is a contradiction.

Let us now suppose that $I \cap E \subset Q_{n_0}$ for some n_0. Then $f_2(x) \geqq -2$ for all $x \in I \cap E$, but if $x \in I - E$ then f_1 is monotone in some neighbourhood of x and since $f_2 \geqq 0$ a.e., f_1 is nondecreasing and hence $f_2 \geqq 0$; so $f_2(x) \geqq -2$ for all $x \in I$; also $f_2(x) \geqq 0$ almost everywhere in I. So by applying the result of [**2**] mentioned earlier we conclude that f_1 is nondecreasing I. But this also contradicts the fact that $I \cap E \neq 0$.

So, we conclude that $E = 0$ and hence f_1 is nondecreasing in $[a, b]$.

THEOREM 1. *Let f be a function satisfying the following conditions in the interval $[a, b]$:*

(i) *f is continuous in* $[a, b]$,

(ii) f_{n-1} *exists finitely everywhere in* $[a, b]$,

(iii) f_n *exists, finitely or infinitely, except on a denumerable subset in* $[a, b]$, *and*

(iv) $f_n \geqq 0$ *almost everywhere in* $[a, b]$.

Then f_{n-1} *is continuous and nondecreasing* (*or equivalently, f is n-convex*) *in* $[a, b]$.

If $n = 1$, then the theorem reduces to the theorem of Goldowski and Tonelli [**9**, p. 206] for the ordinary derivative f'. So, we prove the theorem for $n \geqq 2$. We mention that for $n \geqq 2$ the condition (i) is a consequence of the existence of f_1 and hence is superfluous.

Proof of the theorem for $n = 2$. Let a finite function g be said to satisfy property $\mathscr{P}$ in the interval $[a, b]$ if g is the first Peano derivative of a function G such that the second Peano derivative G_2 of G exists, finitely or infinitely, everywhere in $[a, b]$ except on a denumerable subset, and is nonnegative almost everywhere in $[a, b]$. Let f satisfy the hypothesis of Theorem 1. Then f_1 satisfies property $\mathscr{P}$ on $[a, b]$. Also the property $\mathscr{P}$ is such that if it is possessed by a continuous function in an interval then that function becomes nondecreasing in that interval (by Lemma 2) and if a function satisfies the property $\mathscr{P}$ then that function must be VBG (by the Corollary of Lemma 1). Since f_1 is the ordinary derivative of the continuous function f, we conclude f_1 is a Darboux function of Baire class 1, and hence from Theorem A it follows that f_1 is nondecreasing in $[a, b]$. The Darboux property of f_1 implies the continuity of f_1 also. This completes the proof of Theorem 1 for $n = 2$.

Proof of the theorem for $n > 2$. Since f_{n-1} is defined and finite in $[a, b]$, it follows from Lemma B that any nonempty closed subset of $[a, b]$ contains a portion on which f_{n-1} is bounded. Let E be the set of all points x in $[a, b]$ such that in every neighbourhood of x the function f_{n-1} is unbounded. Then E is closed. So, $[a, b] - E$ is open in $[a, b]$. Let (c, d) be any component interval of $[a, b] - E$ and let $c < \alpha < \beta < d$. Then f_{n-1} is bounded in $[\alpha, \beta]$. Hence by Theorem C, $f_{n-1} = f^{(n-1)}$ in $[\alpha, \beta]$. So, we conclude

$$f_2 = f^{(2)}, f_3 = f^{(3)}, \ldots\ldots f_{n-2} = f^{(n-2)}$$

in $[\alpha, \beta]$. Also since $f^{(n-1)}$ exists finitely in $[\alpha, \beta]$, all the derivatives $f^{(1)}, f^{(2)}, \ldots\ldots f^{(n-2)}$ are continuous in $[\alpha, \beta]$ and hence

$$f_n(x) = \lim_{h \to 0} \frac{f_{n-2}(x + h) - f_{n-2}(x) - hf_{n-1}(x)}{h^2/2}.$$

Set $g(x) = f_{n-2}(x)$. Then f_{n-1} is the derivative of g and g is continuous in $[\alpha, \beta]$. Also $g_2 = f_n$ in $[\alpha, \beta]$. Hence the function g satisfies all the conditions of Theorem 1 for $n = 2$. So, we conclude that g_1 is continuous and nondecreasing in $[\alpha, \beta]$. Since $[\alpha, \beta]$ is any closed subinterval of (c, d), the function f_{n-1} is continuous and nondecreasing in (c, d). Since f_{n-1} possesses Darboux property [**8**], f_{n-1} is continuous and nondecreasing in $[c, d]$.

Now if E is empty then $[a, b] - E$ is the same as $[a, b]$ and in this case c and d coincide with a and b respectively, and the theorem is proved. So, we suppose that E is not empty. Then E cannot have an isolated point. For, if x_0 is an isolated point of E then f_{n-1} is continuous and nondecreasing in a closed interval having x_0 as a right hand end point and f_{n-1} is continuous and nondecreasing in a closed interval having x_0 as a left hand end point which contradicts the fact that $x_0 \in E$. So, we conclude that E is perfect. Applying Lemma B there is a portion of E say $[\alpha, \beta] \cap E$ where $\alpha, \beta \in E$ such that f_{n-1} is bounded on $[\alpha, \beta] \cap E$. Now by our above argument f_{n-1} is continuous and nondecreasing on the closure of each contiguous interval of $[\alpha, \beta] - E$. So, f_{n-1} is bounded in $[\alpha, \beta]$. But this is a contradiction since $[\alpha, \beta]$ contains points of E. Thus we conclude that E is empty and the theorem is proved.

4. We shall require the following theorem.

THEOREM D. *If f is a Darboux function of Baire class 1 satisfying the condition* (T_2) *in the interval* $[a, b]$ *and if*

$$P = \{x : x \in [a, b]; 0 \leqq f'(x) \leqq \infty\}$$
$$Q = \{x : x \in [a, b]; -\infty \leqq f'(x) \leqq 0\},$$

then the set $P \cup Q$ is nondenumerable and the sets $f(P)$ and $f(Q)$ are measurable. If $f(a) < f(b)$, then $m(f(P)) \geqq f(b) - f(a)$. If $f(a) > f(b)$ then

$$m(f(Q)) \geqq f(a) - f(b).$$

(This is proved in [**1**].)

LEMMA 3. *Let f_1 exist finitely at each point in $[a, b]$. Then the set*

$$E = \{x : f_2(x) = \pm\infty\}$$

is of measure zero.

Proof. Since f_1 exists finitely in $[a, b]$ it is integrable in the Perron sense in $[a, b]$ and

$$f(x) = \int_a^x f_1(x)dx + f(a)$$

where the integral is in the Perron sense. So,

$$\begin{aligned} f_2(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x) - hf_1(x)}{h^2/2} \\ &= \lim_{h\to 0} \frac{2}{h^2} \int_x^{x+h} [f_1(t) - f_1(x)]dt \\ &= \mathrm{CD}\, f_1(x) \end{aligned}$$

where $\mathrm{CD}f_1$ denotes the Cesaro derivative of f_1, [**10**]. Hence for each $x \in E$, $\mathrm{CD}f_1(x)$ exists and equals $+\infty$ or $-\infty$. So, by a known result [**10**] the set E is of measure zero.

LEMMA 4. *If f_1 is finite and if the inequalities*

$$-M \leqq {}_-f_2 < {}^-f_2 \leqq M$$

where M is a finite nonnegative number, hold at each point of a set E, then $m^*(f_1(E)) \leqq Mm^*(E)$.

Proof. Let $\epsilon > 0$ be arbitrary. For each positive integer n let E_n denote the set of all points x of E such that

$$(1) \qquad |t - x| \leqq 1/n \Rightarrow |f(t) - f(x) - (t - x)f_1(x)| \leqq \tfrac{1}{2}(M + \epsilon)(t - x)^2.$$

The sets E_n are such that $E_n \subset E_m$ whenever $n \leqq m$ and

$$(2) \qquad E = \bigcup_{n=1}^{\infty} E_n.$$

To each E_n we associate a sequence of intervals $\{I_{n,k}; k = 1, 2, \ldots\}$ which covers E_n and satisfies

$$(3) \qquad \sum_{k=1}^{\infty} m(I_{nk}) \leqq m^*(E_n) + \epsilon.$$

We may suppose that $m(I_{nk}) \leqq 1/n$ for all k. Let x_1 and x_2 be any two points of $E_n \cap I_{nk}$. Then from (1)

$$|f(x_2) - f(x_1) - (x_2 - x_1)f_1(x_1)| \leqq \tfrac{1}{2}(M + \epsilon) \cdot (x_2 - x_1)^2$$

and

$$|f(x_1) - f(x_2) - (x_1 - x_2)f_1(x_2)| \leqq \tfrac{1}{2}(M + \epsilon) \cdot (x_2 - x_1)^2.$$

Hence $|x_1 - x_2|\,|f_1(x_1) - f_1(x_2)| \leqq (M + \epsilon) \cdot (x_1 - x_2)^2$, i.e.,

$$\begin{aligned}|f_1(x_1) - f_1(x_2)| &\leqq (M + \epsilon) \cdot |x_1 - x_2| \\ &\leqq (M + \epsilon) \cdot m(I_{nk}).\end{aligned}$$

From this we conclude that

$$(4) \qquad m^*(f_1(E_n \cap I_{nk})) \leqq (M + \epsilon) \cdot m(I_{nk})$$

for all k. Since the sequence of intervals $\{I_{nk}\}$ covers E_n, we have

$$\begin{aligned} m^*(f_1(E_n)) &\leqq \sum_{k=1}^{\infty} m^*(f_1(E_n \cap I_{nk})) \\ &\leqq (M + \epsilon) \cdot \sum_{k=1}^{\infty} m(I_{nk}) \quad \text{by (4)} \\ &\leqq (M + \epsilon)(m^*(E_n) + \epsilon) \quad \text{by (3)}. \end{aligned}$$

Since $\{E_n\}$ is an ascending sequence, letting $n \to \infty$ we get from (2)

$$m^*(f_1(E)) \leqq (M + \epsilon)(m^*(E) + \epsilon).$$

Since ϵ is arbitrary,

$$m^*(f_1(E)) \leqq M \cdot m^*(E).$$

COROLLARY. *If* $-\infty < {}_-f_2 \leqq {}^-f_2 < \infty$ *holds on a set* E *except perhaps on a denumerable subset then* f_1 *satisfies the property* (N) *on* E.

Proof. Let E_0 be any subset of E such that $m(E_0) = 0$. We may suppose that at each point of E_0, the following relations hold:

$$-\infty < {}_-f_2 \leqq {}^-f_2 < \infty.$$

For each positive integer n, let

$$E_n = \{x : x \in E_0;\ -n \leqq {}_-f_2 \leqq {}^-f_2 \leqq n\}.$$

Then $E_0 = \bigcup_{n=1}^{\infty} E_n$. So, by the above lemma $m^*(f_1(E_n)) \leqq n \cdot m(E_n) = 0$. Since the sequence $\{E_n\}$ is ascending, taking the limit as $n \to \infty$,

$$m^*(f_1(E_0)) = 0.$$

This completes the proof.

LEMMA 5. *If* f_2 *exists finitely at each point of a measurable set* E *then*

$$m(f_1(E)) \leqq \int_E |f_2| dx.$$

Proof. Let $\epsilon > 0$ be arbitrary. For each positive integer n, let

$$E_n = \{x : x \in E;\ (n-1)\epsilon \leqq |f_2(x)| < n\epsilon\}.$$

Then

$$E = \bigcup E_n.$$

Now the function f_1 is measurable and also by the corollary of Lemma 4, f_1 satisfies the condition (N). So, by [**4**], f_1 transforms every measurable set into a measurable set. Thus, since f_2 is also measurable, by Lemma 4 we have

$$\begin{aligned} m(f_1(E)) &\leqq \sum_{n=1}^{\infty} m(f_1(E_n)) \leqq \sum_{n=1}^{\infty} n\epsilon m(E_n) \\ &= \sum_{n=1}^{\infty} (n-1)\epsilon m(E_n) + \sum_{n=1}^{\infty} \epsilon m(E_n) \\ &\leqq \sum_{n=1}^{\infty} \int_{E_n} |f_2| dx + \epsilon m(E). \end{aligned}$$

Since ϵ is arbitrary,

$$m(f_1(E)) \leqq \int_E |f_2| dx.$$

COROLLARY 1. *If* $E = \{x : f_2(x) = 0\}$, *then* $m(f_1(E)) = 0$.

COROLLARY 2. *If* $f_2 = 0$ *almost everywhere in an interval* $[a, b]$ *and* $-\infty < {}_-f_2 \leqq {}^-f_2 < \infty$ *except perhaps on a denumerable subset of* $[a, b]$, *then* f_1 *is constant in* $[a, b]$.

Proof. Let $E = \{x : x \in [a, b]; f_2(x) = 0\}$, $E_1 = [a, b] - E$. Then $m(E_1) = 0$. By the corollary of Lemma 4, f_1 satisfies the property (N). Hence $m(f_1(E_1)) = 0$. Also by Corollary 1, $m(f_1(E)) = 0$. Thus $m(f_1([a, b])) = 0$.

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This implies that f_1 is constant; for, in the contrary case, f_1 being a Darboux function, $f_1([a, b])$ would contain an interval.

LEMMA 6. *Let f_1 exist and satisfy the condition* (N) *on* $[a, b]$ *and let* g *be a finite summable function in* $[a, b]$ *such that* $|f_2(x)| \leqq |g(x)|$ *for each* $x \in [a, b]$ *where* $f_2(x)$ *exists finitely, except perhaps at those points of a set* E *for which* $m(f_1(E)) = 0$. *Then* f_1 *is* VB *in* $[a, b]$.

Proof. Let $[a_1, b_1]$ by any subinterval of $[a, b]$ and let

$$\begin{aligned} P &= \{x : x \in [a_1, b_1]; 0 \leqq f_2(x) \leqq \infty\}, \\ Q &= \{x : x \in [a_1, b_1]; -\infty \leqq f_2(x) \leqq 0\}, \\ G &= \{x : x \in [a_1, b_1]; f_2(x) = \pm\infty\}. \end{aligned}$$

Then for $x \in P - (E \cup G)$ we have by Lemma 5

$$m(f_1(P - E \cup G)) \leqq \int_{P-(E \cup G)} |f_2| dx \leqq \int_{a_1}^{b_1} |g| dx.$$

Since by Lemma 3 $m(G) = 0$, and since f_1 satisfies the condition (N), we have $m(f_1(G)) = 0$. Also by hypothesis $m(f_1(E)) = 0$ and hence

$$m(f_1(G \cup E)) = 0.$$

So,

$$\begin{aligned} m(f_1(P)) &\leqq m(f_1(P - E \cup G)) + m(f_1(E \cup G)) \\ &= m(f_1(P - E \cup G)) \\ &\leqq \int_{a_1}^{b_1} |g| dx. \end{aligned}$$

Similarly we have

$$m(f_1(Q)) \leqq \int_{a_1}^{b_1} |g| dx.$$

Since f_1 satisfies the condition (N), it also satisfies the condition (T_2) [**4**] and hence by Theorem D if $f_1(b_1) \geqq f_1(a_1)$ then

$$f_1(b_1) - f_1(a_1) \leqq m(f_1(P)) \leqq \int_{a_1}^{b_1} |g| dx,$$

and if $f_1(b_1) \leqq f_1(a_1)$ then

$$f_1(a_1) - f_1(b_1) \leqq m(f_1(Q)) \leqq \int_{a_1}^{b_1} |g| dx.$$

Thus in any case

$$|f_1(b_1) - f_1(a_1)| \leqq \int_{a_1}^{b_1} |g| dx.$$

Since g is summable on $[a, b]$, we conclude that f_1 is VB on $[a, b]$.

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LEMMA 7. *Let f_1 exist and satisfy the condition* (N) *in an interval* $[a, b]$ *and let* $E = \{x : x \in [a, b]; -\infty < f_2(x) < \infty\}$. *If*

$$\int_E |f_2| dx < \infty$$

then f_1 is absolutely continuous in $[a, b]$.

Proof. Let

$$\begin{aligned} g(x) &= f_2(x), \text{ for } x \in E \\ &= 0, \qquad \text{elsewhere.} \end{aligned}$$

Then g is a finite summable function in $[a, b]$. Also $|f_2(x)| = |g(x)|$ for each $x \in E$ where $f_2(x)$ exists finitely. Hence by Lemma 6, f_1 is VB in $[a, b]$. Since f_1 is a Darboux function, f_1 is also continuous in $[a, b]$. Finally, the condition (N) implies the absolute continuity of f_1.

THEOREM 2. *Let $n \geqq 2$ and let f_{n-1} be defined and satisfy the condition* (N) *in the interval* $[a, b]$. *Let $f_n \geqq 0$ at almost every point where f_n exists finitely and let*

$$\int_P f_n dx < \infty$$

where P denotes the set of points where f_n exists finitely and is nonnegative.

Then f_{n-1} is nondecreasing and continuous in $[a, b]$.

Proof. Suppose $n = 2$. Then by Lemma 7, f_{n-1} is absolutely continuous in $[a, b]$ and hence the ordinary derivative $(f_{n-1})'$ exists almost everywhere in $[a, b]$ and since $f_n \geqq 0$ at almost every point where f_n exists finitely we conclude $(f_{n-1})' \geqq 0$ almost everywhere in $[a, b]$ and hence f_{n-1} is nondecreasing in $[a, b]$. The continuity of f_{n-1} follows from the Darboux property of f_{n-1}.

The proof for $n > 2$ can be made in the same manner as in Theorem 1 and so we omit it.

5. It is well known that the derivative, finite or infinite, of a continuous function belongs to the class $\mathscr{M}_2$ and a finite derivative belongs to the class $\mathscr{M}_3$ of Zahorski [**14**]. It is interesting to study the nature of the Peano derivative in the light of the above classification. We mention that Weil [**13**] proved that a finite f_n belongs to the class $\mathscr{M}_3$. Here we shall show that if f_n exists, finitely or infinitely, for a continuous function f then f_n belongs to the class $\mathscr{M}_2$. For completeness we state the definition of the class $\mathscr{M}_2$. A set $E \in M_2$ if and only if E is an F_σ and every one sided neighbourhood of each point of E intersects E in a set of positive measure; $f \in \mathscr{M}_2$ if and only if for every α and β, the sets $\{x : f(x) > \alpha\}$ and $\{x : f(x) < \beta\}$ belong to the class $\mathscr{M}_2$.

THEOREM 3. *If f is continuous and f_n exists finitely or infinitely, then* $f_n \in \mathscr{M}_2$.

Proof. Since

$$f_n(x) = \lim_{\nu \to \infty} \nu^n \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f(n + j/\nu),$$

f_n is a function of Baire class 1 (see also [**3**]). Hence for each α the sets $E_\alpha = \{x : f_n(x) < \alpha\}$ and $E^\alpha = \{x : f_n(x) > \alpha\}$ are F_σ. Also if $\xi \in E_\alpha$ and if $\delta > 0$ is arbitrary, then $m([\xi, \xi + \delta] \cap E_\alpha) > 0$; for, if $m([\xi, \xi + \delta] \cap E_\alpha) = 0$ then the function $f_{n-1}(x) - \alpha x$ is nondecreasing and continuous in $[\xi, \xi + \delta]$ by Theorem 1 and hence all the derivatives $f^{(1)}, f^{(2)}, \ldots f^{(n-1)}$ exist and are continuous in $[\xi, \xi + \delta]$ and hence

$$f_n(\xi) = \lim_{h \to 0} \frac{f_{n-1}(\xi + h) - f_{n-1}(\xi)}{h} \geqq \alpha.$$

But $\xi \in E_\alpha$ and so we arrive at a contradiction. Similarly

$$m([\xi - \delta, \xi] \cap E_\alpha) > 0.$$

Thus $E_\alpha \in M_2$. In a similar manner it can be proved that $E^\alpha \in M_2$. This completes the proof.

COROLLARY. *The* nth *Peano derivative* f_n, *finite or infinite of a continuous* f *possesses the Denjoy property, viz., for any two reals* α *and* β, $\alpha < \beta$, *the set* $\{x : \alpha < f_n(x) < \beta\}$ *is either void or is of positive measure.*

Proof. Let $[a, b]$ be any interval and let α and β be arbitrary. Then the sets $[a, b] \cap \{x : f_n(x) < \alpha\}$ and $[a, b] \cap \{x : f_n(x) > \beta\}$ are such that they are either void or are of positive measure and hence by a known result [**7**] the set $\{x : \alpha < f_n(x) < \beta\}$ is either void or is of positive measure.

The above result is proved in [**8**].

6. In this section we consider certain generalizations of the results of Verblunsky [**12**]. His results mostly depend on an interesting lemma, i.e., Lemma 1 of [**12**]. We consider the following generalization of the above lemma. In the following $G_r^+(x)$, $G_{r,-}(x)$ will denote respectively the upper rth Peano derivative on the right at x and the lower rth Peano derivative on the left at x, of the function G, which are obtained from the definitions of $\overline{G}_r(x)$ and $\underline{G}_r(x)$ by suitably restricting the sign of h while taking limits.

LEMMA 8. *Let* ϕ *be an upper semi-continuous Darboux function in* (α, β) *and let* $D^+\phi \geqq D_-\phi$ *hold everywhere in* (α, β). *Let* $E = \{x : x \in (\alpha, \beta)$; $D^+\phi(x)$ *finite and* $D^+\phi(x) = D_-\phi(x) = m(x)$, *say*$\}$. *Suppose that for all* $\xi \in E$, *except perhaps at the points of a subset* $G \subset E$ *such that* $m(G)$ *does not contain an interval, there are, in every neighbourhood of* $(\xi, \phi(\xi))$, *points of the graph of* ϕ *above the line* $y - \phi(\xi) = m(\xi)(x - \xi)$. *Then* ϕ *is convex in* (α, β).

Proof. If possible, suppose that there are points c, d, $\alpha < c < d < \beta$, such that the arc $y = \phi(x)$ $(c \leqq x \leqq d)$ has points above the chord joining $(c, \phi(c))$ and $(d, \phi(d))$. Let $k = (\phi(d) - \phi(c))/(d - c)$. Now the function $\phi(x) - \phi(c) - k(x - c)$ is upper semi-continuous and so it will attain its supremum at some point γ in $[c, d]$. By our assumption $c < \gamma < d$. Let $\mu = (\phi(\gamma) - \phi(c))/(\gamma - c)$. Then $\mu > k$. Since $(\phi(x) - \phi(c))/(x - c)$ is an upper semi-

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continuous and Darboux function (being the product of a continuous function and a Darboux Baire -1 function (see A. M. Bruckner, J. G. Ceder and M. Weiss, Colloq. Math. (1966), 65–77) it will assume all the values between k and μ as x assumes the values between γ and d. Let μ' be such that $k < \mu' < \mu$ and $\mu' \notin m(G)$. This is possible, because $m(G)$ does not contain interval. By the above argument there exists $\xi' \in (\gamma, d)$ such that

$$\mu' = (\phi(\xi') - \phi(c))/(\xi' - c).$$

Now the function $\phi(x) - \phi(c) - \mu'(x - c)$ is upper semi-continuous and so it will attain a supremum at some point η in $[c, \xi']$. Since $\mu > \mu'$, we conclude $c < \eta < \xi'$. Hence $D^+\phi(\eta) \leqq \mu' \leqq D_-\phi(\eta)$. So, by the given condition we conclude

$$D^+\phi(\eta) = D_-\phi(\eta) = \mu'$$

which gives $\eta \in E$ and $m(\eta) = \mu'$. Now the line $y - \phi(\eta) = m(\eta)(x - \eta)$ has the property that in some neighbourhood of the point $(\eta, \phi(\eta))$, no point of the graph of ϕ is above the line. Hence $\eta \in G$. So, $m(\eta) = \mu' \in m(G)$. But this is a contradiction to the choice of μ'. This completes the proof of the lemma.

THEOREM 4. *Let G be continuous in $[a, b]$ and suppose that, for a positive integer r and a finite function g, we have:*

(i) $G_r^+(x) \geqq g(x)$ *for* $a \leqq x < b$; $G_{r,-}(x) \leqq g(x)$ *for* $a < x \leqq b$;

(ii) $$\overline{\lim_{h\to 0}} \left\{G(x+h) - \sum_0^{r-1} \frac{h^k}{k!} G_k(x) - \frac{h^r}{r!} g(x)\right\} \Big/ h^{r+1} > 0$$

on $[a, b]$, except perhaps on a subset E such that $g(E)$ does not contain an interval;

(iii) *if $r > 1$, then every nonempty closed set contains a portion on which g is bounded on one side.*

Then g is nondecreasing in $[a, b]$.

We omit the proof of the theorem. The proof is similar to that of Theorem 2 of Verblunsky [**12**] except that we are to use our Lemma 8 instead of his Lemma 1. We note that if G has a finite rth Peano derivative G_r in $[a, b]$, then (i) holds with $g = G_r$. Also (iii) holds by Lemma B. Hence in this case Theorem 4 becomes:

THEOREM 5. *If $n \geqq 2$ and f_{n-1} is defined and finite in $[a, b]$ and $^-f_n > 0$, except perhaps on a set $E \subset [a, b]$ such that $f_{n-1}(E)$ does not contain an interval, then f_{n-1} is nondecreasing and continuous in $[a, b]$. (The continuity of f_{n-1} follows from the Darboux property* [**8**] *and monotonicity of f_{n-1}.)*

Theorems 4 and 5 are generalizations of Theorems 2 and 1 respectively of Verblunsky [**12**]. Verblunsky also proved that the condition (iii) of Theorem 4 can be replaced by other similar conditions involving the $C_\lambda P$-integral of g introduced by Burkill [**3**], where λ is a positive integer. For completeness we give the definition of $C_\lambda P$-integral, where λ is a positive integer in the form given by Verblunsky [**12**]. (See also [**5**, Theorems 9.1 and 11.1].)

Let g be a function in $[a, b]$. If there are two functions M and m continuous in $[a, b]$ such that M_λ and m_λ exist and are finite in $[a, b]$ and

(i) $\_M_{\lambda+1}(x) \geqq g(x) \geqq {}^{-}m_{\lambda+1}(x)$ in $[a, b]$

(ii) $\_M_{\lambda+1}(x) \neq -\infty, {}^{-}m_{\lambda+1}(x) \neq \infty$ in $[a, b]$

(iii) $M_\lambda(a) = m_\lambda(a) = 0$,

then the functions M_λ and m_λ are called $C_\lambda P$-major and $C_\lambda P$-minor functions respectively for the function g in $[a, b]$. By the condition (i) it follows that the function $M_\lambda(x) - m_\lambda(x)$ is nondecreasing and continuous and so by (iii) $M_\lambda(b) - m_\lambda(b) \geqq 0$. If $\inf\{M_\lambda(b); M_\lambda \in \mathscr{M}\} = \sup\{m_\lambda(b); m_\lambda \in \mathfrak{m}\}$ where $\mathscr{M}$ and $\mathfrak{m}$ are respectively the class of $C_\lambda P$-major functions and the class of $C_\lambda P$-minor functions of g then g is called $C_\lambda P$-integrable in $[a, b]$ and the common value, denoted by

$$(C_\lambda P) \int_a^b g\, dx,$$

is called the $C_\lambda P$-integral of g in $[a, b]$. If $\lambda = 0$, the above definition reduces to that of Perron integral. Clearly if a finite function f is a Peano derivative in $[a, b]$, i.e. if there is a continuous function F in $[a, b]$ and a positive integer r such that $F_r = f$ in $[a, b]$, then

$$F_{r-1} = (C_{r-1}P) \int_a^b f\, dx.$$

Now returning to Theorem 4 we remark that the condition (iii) can be replaced by any one of the following two conditions:

(iii)′ If $r > 1$ then g is $C_{r-1}P$ integrable in $[a, b]$.

(iii)″ If $r > 1$ then g has a (possibly discontinuous) Perron major or minor function.

We again omit the proof. The proofs are the same as those given by Verblunsky [**12**] to prove his Theorems 3 and 4 except that one is to apply Lemma 8 instead of his Lemma 1.

7. If a function g is $C_{r-1}P$ integrable in $[a, b]$, where $r > 1$, then the rth Cesaro mean of g in $(x, x + h) \subset (a, b]$ is given by

$$C_r(g, x, x + h) = \frac{r}{h^r} \int_x^{x+h} (x + h - t)^{r-1} g(t) dt$$

where the integral is taken in the $C_{r-1}P$-sense. The C_r- right hand upper limit and the C_r- right hand upper derivate of g at x are defined to be

$$C_r\text{-}\limsup_{h\to 0+} g(x + h) = \limsup_{h\to 0+} C_r(g, x, x + h)$$

and

$$C_r D^+ g(x) = \limsup_{h\to 0+} \frac{C_r(g, x, x + h) - g(x)}{h/(r + 1)},$$

respectively, with similar definitions for other C_r- limits and C_r- derivates. The C_r- upper derivate ${}^{-}C_r D$ is the maximum of $C_r D^+$ and $C_r D^-$. We prove the following result which is more general than those of Verblunsky [**12**] and of Sargent [**11**] and is analogous to that of Zygmund for ordinary derivatives [**9**, p. 203].

THEOREM 6. *Let the finite function g be $C_{r-1}P$-integrable in* $[a, b]$ *and let*

$$C_r\text{-}\liminf_{h\to 0+} g(x-h) \leqq g(x) \leqq C_r\text{-}\limsup_{h\to 0+} g(x+h).$$

If the set of values assumed by g at the points where $^{-}C_rDg \leqq 0$ *does not contain an interval, then g is nondecreasing in* $[a, b]$.

Proof. Since g is $C_{r-1}P$-integrable in $[a, b]$, there is a function G continuous in $[a, b]$ such that

$$G_{r-1}(x) = (C_{r-1}P)\int_a^x g\,dx.$$

Also

$$G_r^{+}(x) = C_r\text{-}\limsup_{h\to 0+} g(x+h), \quad a \leqq x < b,$$

$$G_{r,-}(x) = C_r\text{-}\liminf_{h\to 0+} g(x-h), \quad a < x \leqq b,$$

and

$$^{-}C_rDg(x) = \limsup_{h\to 0} \frac{r+1}{h^{r+1}}\left\{G(x+h) - \sum_0^{r-1}\frac{h^k}{k!}G_k(x) - \frac{h^r}{r!}g(x)\right\},$$

so by applying the result of Theorem 4 we get that g is nondecreasing in $[a, b]$. This completes the proof.

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