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# IMAGES IN TOPOI

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0. **Introduction.** The construction of images of morphisms in an elementary topos $E$ has hithero required the use of colimits. For example, in [1], Freyd constructs the image of a morphism by taking the equalizer of the cokernel pair of the morphism. In particular, the construction of the direct image functor, or, as it is sometimes referred to, existential quantification along a morphism, has required the use of images, and hence colimits. However, Mikkelsen has defined existential quantification using only limits. This paper verifies that Mikkelsen's definition can be used to construct images without the use of colimits and agrees with that usually given.

We work throughout in an elementary topos $E$, by which we understand a finitely complete category with power-object functor $P: E^{\mathrm{op}} \to E$. The latter means that for each $X \in E_0$ (= objects of $E$), we have, in addition to $P(X)$, a "membership relation" $\varepsilon_X \xrightarrow{\bar{e}_X} P(X) \times X$ such that if $R \xrightarrow{\langle y, x \rangle} Y \times X$ is any relation there exist unique morphisms $f: Y \to P(X)$, $f^*: R \to \varepsilon_X$ such that the induced square is a pullback.

We write $\Omega = P(\mathbb{1})$ ($\mathbb{1}$ = terminal object of $E$), and verify immediately that there is a unique morphism $t: \ \to \Omega$ which makes $(\Omega, t)$ into a subobject classifier, and hence that there is a unique natural morphism $e_X: P(X) \times X \to \Omega$ classifying $\bar{e}_X$.

Our method of proof will utilize the set-theoretical principles as outlined, for example, in [2]. Most of the material contained herein is derived from the author's notes [3]. I am indebted to G. Wraith and C. Mikkelsen for many stimulating conversations.

1. **Set theory in** $E$. Our basic building blocks are: (1) the functor $P$ as described above; (2) the singleton maps $\{\ \}_A = \{\ \}: A \to P(A)$; (3) the "total subset" maps $\Sigma_A: \mathbb{1} \to P(A)$; (4) the "intersection operators" $\bigwedge_A = \bigwedge: P(A) \times P(A) \to P(A)$; and (5) the "inclusion relations" $⊆_A \xrightarrow{\subseteq_A} P(A) \times P(A)$ and $⊇_A \xrightarrow{\supseteq_A} P(A) \times P(A)$. These are all definable using only the power-object condition and/or the subobject classifier, and hence involve only finite limits. We assume their constructions known (see, for example, [3]—this approach does not use any cartesian-closed structure, which we have not assumed).

Further, we assume known:

1.1. PROPOSITION. (i) $\{\}_A$ *is a nonomorphism.*

(ii) *P is a faithful functor.*

(iii) *E is balanced.* $\square$

Our method of proof will utilize the set-theoretic principles of [2]. This eliminates a large number of pullback arguments. Specifically, we use the following terms, notations and results:

1.2. DEFINITION. A morphism $X \to Y$ is an *X-element* of $Y$.

1.3. NOTATION. If $A_0$ is an $X$-element of $P(A)$ and $a$ is an $X$-element of $A$, we write $a \in_X A_0$, or just $a \in A_0$ to mean that $e_A\langle A_0, a\rangle = t!_X$, where $!_X : X \to \mathbb{1}$ is the terminal map.

1.4. NOTATION. If $A_1$ and $A_2$ are $X$-elements of $P(A)$ we write $A_1 \subseteq_A A_2$, or just $A_1 \subseteq A_2$ to mean that $\langle A_1, A_2\rangle$ factors through $\subseteq_A$. The notation $A_1 \supseteq A_2$ is defined similarly.

1.5. PROPOSITION. *Let $A_1$, $A_2$ and $A_3$ be $X$-elements of $P(A)$. Then:*

(i) $A_1 \subseteq A_2$ *iff* $A_2 \supseteq A_1$.

(ii) $A_1 \subseteq A_2$ *and* $A_2 \subseteq A_3$ *implies* $A_1 \subseteq A_3$.

(iii) $A_1 \subseteq A_2$ *and* $A_2 \subseteq A_1$ *iff* $A_1 = A_2$. $\square$

1.6. SINGLETON PRINCIPLE. If $A_0$ is an $X$-element of $P(A)$ and $a$ is an $X$-element of $A$, then:

$$a \in A_0 \quad \text{iff} \quad \{\ \}a \subseteq A_0. \quad \square$$

1.7. INVERSE-IMAGE PRINCIPLE. If $f : A \to B$, $B_0$ an $X$-element of $P(B)$, and $a$ an $X$-element of $A$, then:

$$a \in P(f)B_0 \quad \text{iff} \quad fa \in B_0. \quad \square$$

1.8. EXTENSIONALITY PRINCIPLE. If $A_1$ and $A_2$ are $X$-elements of $P(A)$ then the following are equivalent:

(i) $A_1 \subseteq A_2$.

(ii) Given $Y$-element $\alpha$ of $X$ and a $Y$-element $a$ of $A$ such that $a \in A_1\alpha$, then $a \in A_2\alpha$. $\square$

The proofs of all of these may be found in [2] or [3].

We also need the idea of a *monotone map.* For present purposes, we shall only consider maps $\phi : P(A) \to P(B)$, and call such a map monotone if, given any $X$ and $X$-elements $A_1$ and $A_2$ of $P(A)$ such that $A_1 \subseteq A_2$ it follows that $\phi A_1 \subseteq \phi A_2$. We have:

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1.9. PROPOSITION. *If* $f:A\to B$ *then* $P(f):P(B)\to P(A)$ *is monotone.* $\square$

Following Mikkelson, we introduce the *upper and lower segments* $S_A^{\uparrow}=S^{\uparrow}:P(A)\to P(P(A))$ and $S_A^{\downarrow}=S^{\downarrow}:P(A)\to P(P(A))$ as the power-object maps of the inclusion relations $\subseteq_A$ and $\supseteq_A$ respectively. Using these maps Mikkelsen defines the *intersection operators* by the composition

$$\bigcap:P(P(A))\xrightarrow{S^{\uparrow}}P(P(P(A)))\xrightarrow{P(S^{\uparrow})}P(P(A))\xrightarrow{P(\{\})}P(A).$$

We will need the following properties:

1.10. PROPOSITION. *If* $A_1$ *and* $A_2$ *are* $X$*-elements of* $P(P(A))$ *such that* $A_1\subseteq A_2$ *then* $\bigcap A_2\subseteq\bigcap A_1$.

1.11. PROPOSITION. $\bigcap S^{\uparrow}=1=P(\{\ \})S^{\downarrow}$.

1.12. PROPOSITION. $S^{\uparrow}\Sigma_A=\{\}\Sigma_A$.

These all follow easily. Let us prove 10. to illustrate the method in this context. Let $\xi$ be a $Y$-element of $X$ and $A_0$ a $Y$-element of $P(A)$ such that $A_0\in\bigcap A_2\xi$. By the Inverse-image Principle, this is equivalent to $S^{\uparrow}\{\ \}A_0\in S^{\uparrow}A_2\xi$, or, $e(S^{\uparrow}\times 1)\langle A_2\xi, S^{\uparrow}\{\ \}A_0\rangle=t!$. By definition of $S^{\uparrow}$, we conclude $A_2\xi\subseteq S^{\uparrow}\{\ \}A_0$. Hence, $A_1\xi\subseteq S^{\uparrow}\{\ \}A_0$. Reversing the above argument, we see that $S^{\uparrow}\{\ \}A_0\in S^{\uparrow}A_1\xi$, or $A_0\in\bigcap A_1\xi$. The result follows from the Extensionality Principle.

(Note that one must pass to a pullback argument briefly. As more results become available, the number of such arguments—hopefully—decreases.)

1.13. PROPOSITION. *Let* $A_1$ *and* $A_2$ *be* $X$*-elements of* $P(A)$. *Then the following are equivalent*:

(i) $A_1\subseteq A_2$.
(ii) $A_2\in S^{\uparrow}A_1$.
(iii) $A_1\in S^{\uparrow}A_2$.
(iv) $S^{\downarrow}A_2\subseteq S^{\uparrow}A_1$.
(v) $S^{\downarrow}A_1\subseteq S^{\downarrow}A_2$.

(The proof is quite straightforward.)

Finally, we define, following Mikkelsen, *existential* and *universal quantification,* as follows: Let $f:A\to B$. Then existential quantification along $f$ is defined by the composition:

$$\exists_f:P(A)\xrightarrow{S^{\uparrow}}P(P(A))\xrightarrow{P(P(f))}P(P(B))\xrightarrow{\cap}P(B)$$

and universal quantification by the composition:

$$\forall_f:P(A)\xrightarrow{S^{\downarrow}}P(P(A))\xrightarrow{P(P(f))}P(P(B))\xrightarrow{P(\{\ \})}P(B).$$

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We have only incidental use for universal quantification here. It is intrinsically simpler than existential and for that reason certain results about existential quantification turn out to be more easily proved by using universal quantification to prove auxiliary results first (see Section 3). We verify in Section 4 that Mikkelsen's definition of existential quantification agrees with that usually given (see, for example, [2]).

2. **Images in** $E$. Let $f:A \to B$ be any map in $E$. Consider the pullback

$$\begin{tikzcd}
I(f) \arrow[rrr] \arrow[d, "f_m"'] & & & 1 \arrow[d, "t"] \\
B \arrow[r, "\langle \Sigma_A !, 1\rangle"] & P(A)\times B \arrow[r, "\exists_f \times 1"] & P(B)\times B \arrow[r, "e"] & \Omega
\end{tikzcd}$$

We claim:

2.1. THEOREM. *There is a unique epimorphism* $f_e : A \to I(f)$ *such that* $A \xrightarrow{f_e} I(f) \xrightarrow{f_m} B$ *is the image factorization of* $f$.

The proof will be preceded by three lemmas.

2.2. LEMMA. *There is a unique map* $f_e : A \to I(f)$ *such that* $f_m f_e = f$.

**Proof.** We must show that $f \in \exists_f \Sigma_A !$, which is equivalent to $P(P(f))S^{\uparrow}\Sigma_A ! \subseteq S^{\uparrow}\{\}f$ (by definition of $\exists$, the Principles, and 13). Let $a$ be an $X$-element of $A$, $B_0$ an $X$-element of $P(B)$, and suppose $B_0 \in P(P(f))S^{\uparrow}\Sigma_A !\, a$, or, equivalently, $\Sigma_A !\, a \subseteq P(f)B_0$. Since $a \in \Sigma_A !\, a$, we have $a \in P(f)B_0$, or $\{\}fa \subseteq B_0$, or $B_0 \in S^{\uparrow}\{\}fa$. $\square$

2.3. LEMMA. *Let* $\psi : \mathbb{1} \to P(B)$ *and suppose* $P(f)\psi = \Sigma_A$. *Then* $\exists_f \Sigma_A \subseteq \psi$.

**Proof.** Let $b \in \exists_f \Sigma_A !$, or, $P(P(f))S^{\uparrow}\Sigma_A ! \subseteq S^{\uparrow}\{\}b$. Now, $P(f)\psi ! \in \{\}\Sigma_A !$ by assumption, so, by 12 and extensionality, $\psi ! \in S^{\uparrow}\{\ \}b$, or, $b \in \psi !$. $\square$

2.4. LEMMA. *Let* $A \xrightarrow{k} K \xrightarrow{n} B$ *be a factorization of* $f$ *with* $n$ *monic. Let* $\chi_n : \mathbb{1} \to P(B)$ *be the power-object map for the relation* $K \xrightarrow{\langle !, n\rangle} \mathbb{1} \times B$. *Then* $P(f)\chi_n = \Sigma_A$

**Proof.** We need only show $\Sigma_A \subseteq P(f)f_n$. Let $a$ be an $X$-element of $A$. Clearly $a \in \Sigma_A !$. Then

$$\begin{aligned}
e\langle P(f)x_n !, a\rangle &= e\langle \chi_n !, fa\rangle \quad \text{(since } e \text{ is natural)}\\
&= e(\chi_n \times 1)\langle !, n\rangle ka\\
&= t!. \quad \square
\end{aligned}$$

**Proof of 2.1.** Let $n$, $k$ be as in 2.4. We thus have $\exists_f \Sigma_A \subseteq x_n$. Then, since the

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characteristic map of $n$ is $e\langle\chi_n!, 1\rangle$, there is a unique map from the subobject of $B$ whose characteristic map is $e\langle\exists_f\Sigma_A!, 1\rangle$, namely $f_m:I(f)\to B$ to the subobject $n:K\to B$, say $\theta:I(f)\to K$. Then $nk=f_mf_e=h\theta f_e$ so $k=\theta f_e$. It remains to see that $f_e$ is epic. Let $u_1f_e=u_2f_e$, and let $K\xrightarrow{n} I(f)$ be the equalizer of $u_1$ and $u_2$. Then there is a unique $k:A\to K$ such that $nk=f_e$ and so $A\xrightarrow{k}K\xrightarrow{f_mn}B$ is a factorization of $f$ with $f_mn$ mono. By the above argument, we see $n$ is iso; hence $u_1=u_2$. $\square$

3. **Adjointness and the Beck condition.** We wish to prove that the existential quantifier defined by Mikkelsen agrees with that usually given. To do this, we will need to know that pullbacks of epis in $E$ are epi. Our proof of this depends on the "adjointness" of $\exists$ and $P$ and the Beck condition. We cannot, of course, assume these results since they are usually proved using the "usual" definition of $\exists$ in terms of images.

We will need the following:

3.1. Proposition. *Let $f:A\to B$ be any map, $A_0$ an $X$-element of $P(A)$ and $a$ an $X$-element of $A$. Then if $a\in A_0$ we have $fa\in\exists_fA_0$.*

**Proof.** We must show $P(P(f))S^{\uparrow}A_0\subseteq s^{\uparrow}\{\ \}fa$. Let $B_0\in P(P(f))S^{\uparrow}A_0\alpha$. Then $A_0\alpha\subseteq P(f)B_0$. Hence $a\alpha\in P(f)B_0$ which implies the result. $\square$

The adjointness referred to above is the content of the following:

3.2. Theorem. *Let $f:A\to B$ be any morphism and let $A_0, B_0$ be $X$-element of $P(A), P(B)$ respectively. Then:*

*(i) $\exists_fA_0\subseteq B_0$ iff $A_0\subseteq P(f)B_0$.*

*(ii) $P(f)B_0\subseteq A_0$ iff $B_0\subseteq\forall_fA_0$.*

**Proof.** (i) Suppose the first inclusion true and let $a\in A_0\alpha$. By 3.1 $fa\in\exists_fA_0\alpha$, so $fa\in B_0\alpha$.

Now, suppose $A_0\subseteq P(f)B_0$. Since we obviously have $P(f)B_0\in S^{\uparrow}P(f)B_0$, we also have $B_0\in P(P(f))S^{\uparrow}P(f)B_0$. Now, if $b\in\exists_fA_0\alpha$, we have $P(P(f))S^{\uparrow}A_0\alpha\subseteq S^{\uparrow}\{\ \}b$. By 1.12 and monotonicity of power-maps, we have $P(P(f))S^{\uparrow}P(f)B_0\alpha\subseteq P(P(f))S^{\uparrow}A_0\alpha$, and hence $B_0\alpha\in S^{\uparrow}\{\ \}b$ or, $b\in B_0$.

The proof of (ii) is similar. $\square$

3.3. Theorem. (Beck Condition). *Let*

(1)
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ A' \arrow[r, "f'"] & B' \end{tikzcd}$$

*be a diagram in $E$.*

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(i) *If* (1) *commutes, then* $P(h)\forall_{f'} \subseteq \forall_f P(g)$ *and* $P(f')\forall_h \subseteq \forall_g P(f)$.

(ii) *If also* (1) *is a pullback then*

$$P(h)\forall_{f'} = \forall_f P(g), \qquad P(f')\forall_h = \forall_g P(f), \qquad P(h)\exists_{f'} = \exists_f P(g),$$

and

$$P(f')\exists_h = \exists_g P(f).$$

(iii) *Conversely, if* $P(h)\exists_{f'} = \exists_f P(g)$ *and* $\exists_g P(f) = P(f')\exists_h$ *then* (1) *commutes.*

**Proof.** (i) By 3.2(ii) we have $P(h)\forall_{f'} \subseteq \forall_f P(g)$ iff $P(f)P(h)\forall_{f'} \subseteq P(g)$, or, since (1) commutes, iff $P(g)P(f')\forall_{f'} \subseteq P(g)$. But, again by 3.2, $P(f')\forall_{f'} \subseteq 1$ iff $\forall_{f'} \subseteq \forall_{f'}$ which is true. Since $P(g)$ is monotone, the result follows. (The other inclusion is simply a relabelling.)

(ii) We show first $\forall_f P(g) \subseteq P(h)\forall_{f'}$, and hence the first equality. Let $b \in \forall_f P(g)A_0'$. Then $P(f)\{\}b \subseteq P(g)A_0'$. We wish to show $P(f')\{\}hb \subseteq A_0'$. Let $a' \in P(f')\{\}hb\beta$. Then $f'a' = hb\beta$, so there is a unique $a$ such that $ga = a'$, $fa = b\beta$. We thus have $P(f)\{\}fa \subseteq P(g)A_0'\beta$. Now clearly $a \in P(f)\{\}fa$, hence $ga = a' \in A_0'\beta$. The second equality is again a relabelling. To prove the third equality we proceed as follows:

$$\begin{array}{llcl}
 & P(h)\exists_{f'} \subseteq \exists_f P(g) & \text{and} & \exists_f P(g) \subseteq P(h)\exists_f g' \\
\text{iff} & \exists_{f'} \subseteq \forall_h \exists_f P(g) & \text{and} & P(g) \subseteq P(f)P(h)\exists_{f'} \\
\text{iff} & 1 \subseteq P(f')\forall_h \exists_f P(g) & \text{and} & 1 \subseteq \forall_g P(f)P(h)\exists_{f'} \\
\text{iff} & 1 \subseteq \forall_g P(f)\exists_f P(g) & \text{and} & 1 \subseteq P(f')\forall_h P(h)\exists_{f'} \\
\text{iff} & P(g) \subseteq P(f)\exists_f P(g) & \text{and} & \exists_{f'} \subseteq \forall_h P(h)\exists_{f'} \\
\text{iff} & \exists_f P(g) \subseteq \exists_f P(g) & \text{and} & P(h)\exists_{f'} \subseteq P(h)\exists_{f'}
\end{array}$$

both of which are true. The final equality is again a relabelling.

(iii) To prove this we note first that if $\theta : X \to Y$ then $\exists_\theta\{\ \}_X = \{\ \}_Y\theta$. For, on the one hand $\exists_\theta\{\ \}_X \subseteq \{\ \}_Y\theta$ iff $\{\ \}_X \subseteq P(\theta)\{\ \}_Y\theta$, which is immediate, while on the other hand, if $y \in \{\ \}_Y\theta x$, we have, by 3.1, $y \in \exists_\theta\{\ \}_X x$ since $x \in \{\ \}_X x$. Now, composing the first equality with $\{\ \}f$, we obtain $P(f')\{\ \}hf = \exists_g P(f)\{\ \}f$. Noting that $\{\ \} \subseteq P(f)\{\ \}f$ and $\exists_g$ is monotone (because it is a composition of *two* uppersegments and power maps) we have $\{\ \}g \subseteq P(f')\{\ \}hf$, so $\{\ \}f'g = \exists_{f'}\{\ \}g \subseteq \{\ \}hf$. Using the other equality proves the opposite inclusion, so $\{\ \}f'g = \{\ \}hf$. Since singleton is monic, the result follows. □

(It is not in general true that that the Beck condition implies that (1) is a pullback. Using the Existence Principle of [2], one can prove the following universal property:

If $hb = f'a'$, then there is an epimorphism $\alpha$ and $a$ such that $g(a) = a'\alpha$, $fa = b\alpha$. We have no idea what use this is.)

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Two more auxiliary results:

3.4. PROPOSITION. *For any map* $f: A \to B$ *we have* $\Sigma_A = P(f)\Sigma_B$. $\square$

3.5. PROPOSITION. *Let* $f: A \to B$ *be a map. Then* $f$ *is an epimorphism iff* $\exists_f \Sigma_A = \Sigma_B$.

**Proof.** Suppose $f$ is an epimorphism. Then, in the image factorization of $f$, $f = f_m f_e$, we have $f_m$ iso, for $f_m f_e$ is epi so $f_m$ is also, so $f_m$ iso since $E$ is balanced. Now let $b \in \Sigma_B !$. Then $e\langle \exists_f \Sigma_A !,\ b\rangle = e\langle \exists_f \Sigma_A !,\ 1\rangle f_m f_m^{-1} b = t!$. Hence $\Sigma_B \subseteq \exists_f \Sigma_A$. The opposite inclusion is obvious. On the other hand, if $\exists_f \Sigma_A = \Sigma_B$, then in the image factorization of $f$, we have $f = f_m f_e = 1 \cdot f$. Hence $f = f_e$, or $f$ is epi. $\square$

Finally;

3.6. PROPOSITION. *Pullbacks of epis are epi.*

**Proof.** Let (1) be a pullback with $h$ epi. Then:

$$\begin{aligned} \exists_g \Sigma_A &= \exists_g P(f)\Sigma_B && \text{(by 3.4)} \\ &= P(f')\exists_h \Sigma_B && \text{(by Beck Condition)} \\ &= P(f')\Sigma_{B'} && \text{(by 3.5)} \\ &= \Sigma_{A'} && \text{(by 3.4)} \end{aligned}$$

Hence, $g$ is epi by 3.5.

**4. Comparison of existential quantifications.** The "usual" definition of existential quantification, as given in, for example, [1], or [2], is as follows: Given $f: A \to B$, one considers the map

$$\hat{f}: \varepsilon_A \longrightarrow P(A) \times A \xrightarrow{1 \times f} P(A) \times B$$

and takes its image factorization

$$\hat{f}: \varepsilon_A \xrightarrow{f_e} I(\hat{f}) \xrightarrow{f_m} P(A) \times B$$

thus obtaining a relation on $P(A) \times B$ whose power-object map is $\bar{\exists}_f: P(A) \to P(B)$. We show here that $\bar{\exists}_f = \exists_f$. The proof rests on the following lemma:

4.1. LEMMA. *Let* $R \xrightarrow{\langle x,y\rangle} A \times B$ *be any relation and let* $f: A \to P(B)$ *be its power-object map. Then* $f$ *is given by the composition*

$$f: A \xrightarrow{\{\,\}} P(A) \xrightarrow{P(x)} P(R) \xrightarrow{\exists_y} P(B).$$

**Proof.** To see that $f \subseteq \exists_y P(x)\{\ \}$, let $a$ be an $X$-element of $A$, $b$ an $X$-element of $B$ such that $b \in fa$. By definition of $f$, there is a unique $r: X \to R$ such that $\langle x, y\rangle r = \langle a, b\rangle$. Clearly $r \in P(x)\{\ \}a$. By 3.1 $b = yr \in \exists_y P(x)\{\ \}a$.

Suppose $b \in \exists_y P(x)\{\ \}a$. Then $P(P(y))S^{\uparrow}P(x)\{\ \}a \subseteq S^{\uparrow}\{\ \}b$. We claim $fa \in P(P(y))\, S^{\uparrow}P(x)\{\ \}a$, or, $P(x)\{\ \}a \subseteq P(y)fa$. For, suppose $r \in P(x)\{\ \}a^{a}$; then $xr = a\alpha$. Since $y \in fx$ by definition, we have $yr \in fxr = fa\alpha$, and so $r \in P(y)fa\alpha$, which proves the inclusion. Hence, $fa \in S^{\uparrow}\{\ \}b$, or, $b \in fa$. □

We may now prove that the two existential quantifiers agree. Write $\hat{f}_m = \{f', f''\rangle: I(\hat{f}) \to P(A) \times B$. By 4.1

$$\bar{\exists}_f = \exists_{f''} P(f')\{\ \}.$$

Thus, we will be done if we show the right-hand side is $\exists_f$.

Let $A_0$ be an $X$-element of $P(A)$. We claim:

(1) $A_0 \subseteq P(f)\exists_{f''} P(f')\{\ \}A_0$
(2) $P(f')\{\ \}A_0 \subseteq P(f'')\exists_f A_0$,

since, by 3.2, this is equivalent to the required result.

Verification of (1): Let $a \in A_0\alpha$, $\alpha: Y \to X$, $a: Y \to A$. By definition, there is a unique $\eta: Y \to \varepsilon_A$ such that $\bar{e}_A\eta = \langle A_0\alpha, a\rangle$. Thus, $A_0\alpha = p_1\bar{e}_A\eta = p_1(1 \times f)\bar{e}_A\eta = p_1\langle f', f''\rangle\hat{f}_e\eta = f'\hat{f}_e$. Thus, $\hat{f}_e\eta \in P(f')\{\}A_0\alpha$, so by 4.1, $f''\hat{f}_2\eta \in \exists_{f''} P(f')\{\}A_0\alpha$. But $f''\hat{f}_e\eta = p_2\langle f', f''\rangle\hat{f}_e\eta = p_2(1 \times f)\bar{\varepsilon}_A\eta = fp_2\langle A_0\alpha, a\rangle = fa$. Hence, (1) holds.

Verification of (2): Let $\eta \in P(f')\{\}A_0\alpha$, where $\eta: Y \to I(\hat{f})$, $\alpha: Y \to X$, and consider the pullback

$$\begin{array}{ccc} Z & \xrightarrow{\eta'} & \varepsilon_A \\ {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle f_e} \\ Y & \xrightarrow{\nu} & I(\hat{f}) \end{array}$$

and note $\beta$ is epi since $\hat{f}_e$ is. A lengthy computation verifies that $p_2\bar{e}_A\eta' \in A_0\alpha\beta$, from which we obtain $fp_2\bar{e}_A\eta' \in \exists_f A_0\alpha\beta$, by 3.1. But $fp_2\bar{e}_A\eta' = p_2\langle f', f''\rangle f_e\eta' = f''\eta\beta$. Hence, (2) holds, which verifies the result.

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