Bull. Aust. Math. Soc. (First published online 2024), page 1 of 12*
doi:10.1017/S0004972724001023
*Provisional—final page numbers to be inserted when paper edition is published

EXTENDING CONGRUENCES FOR OVERPARTITIONS WITH ℓ-REGULAR NONOVERLINED PARTS

JAMES A. SELLERS

(Received 17 June 2024; accepted 5 October 2024)

Abstract

Recently, Alanazi *et al.* ['Refining overpartitions by properties of nonoverlined parts', *Contrib. Discrete Math.* **17**(2) (2022), 96–111] considered overpartitions wherein the nonoverlined parts must be ℓ-regular, that is, the nonoverlined parts cannot be divisible by the integer ℓ. In the process, they proved a general parity result for the corresponding enumerating functions. They also proved some specific congruences for the case $\ell = 3$. In this paper we use elementary generating function manipulations to significantly extend this set of known congruences for these functions.

2020 *Mathematics subject classification*: primary 11P83; secondary 05A17.

Keywords and phrases: partitions, congruences, overpartitions, generating functions.

1. Introduction

A *partition* of a positive integer n is a finite nonincreasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. We refer to the integers $\lambda_1, \lambda_2, \ldots, \lambda_k$ as the *parts* of the partition. For example, the number of partitions of the integer $n = 4$ is 5, and the partitions counted in that instance are

$$(4), \quad (3,1), \quad (2,2), \quad (2,1,1), \quad (1,1,1,1).$$

For more information about integer partitions, see [3, 4].

An *overpartition* of a positive integer n is a partition of n wherein the first occurrence of a part may be overlined. The modern study of overpartitions was initiated in the groundbreaking work of Corteel and Lovejoy [11].

As an example, the number of overpartitions of $n = 4$ is 14, and the overpartitions are

$$(4), \quad (\overline{4}), \quad (3,1), \quad (\overline{3},1), \quad (3,\overline{1}), \quad (\overline{3},\overline{1}), \quad (2,2), \quad (\overline{2},2),$$
$$(2,1,1), \quad (\overline{2},1,1), \quad (2,\overline{1},1), \quad (\overline{2},\overline{1},1), \quad (1,1,1,1), \quad (\overline{1},1,1,1).$$

The number of overpartitions of n is often denoted $\overline{p}(n)$, so from the example above we see that $\overline{p}(4) = 14$.

As part of a larger study of overpartitions with restrictions on the nonoverlined parts, Alanazi *et al.* [1] considered the family of functions $\overline{R_\ell^*}(n)$ which counts the number of overpartitions of weight n wherein nonoverlined parts must be ℓ-regular (that is, the nonoverlined parts are not allowed to be divisible by ℓ) and there are no restrictions on the overlined parts. For example, $\overline{R_3^*}(4) = 12$ thanks to the overpartitions

$$(4), \quad (\overline{4}), \quad (\overline{3}, 1), \quad (\overline{3}, \overline{1}), \quad (2, 2), \quad (\overline{2}, 2),$$
$$(2, 1, 1), \quad (\overline{2}, 1, 1), \quad (2, \overline{1}, 1), \quad (\overline{2}, \overline{1}, 1), \quad (1, 1, 1, 1), \quad (\overline{1}, 1, 1, 1).$$

Note that the two overpartitions of $n = 4$ which are 'removed' from those counted by $\overline{p}(4)$ in order to create the above list are $(3, 1)$ and $(3, \overline{1})$; these are not counted by $\overline{R_3^*}(4)$ because they contain a nonoverlined part which is divisible by $\ell = 3$.

We note that the values of $\overline{R_2^*}(n)$ appear in the *On-Line Encyclopedia of Integer Sequences* [24, A022567], where they are interpreted as the number of partitions of n into two different colours of distinct parts.

One can readily see that an overpartition counted by $\overline{R_\ell^*}(n)$ can be viewed as an ordered pair of partitions (μ, ν) where μ is an ℓ-regular partition (which accounts for the nonoverlined parts in an overpartition) and ν is a partition into distinct parts (which accounts for the overlined parts in an overpartition) such that the weight of μ plus the weight of ν equals n. We will use this representation of an overpartition in our discussions below.

Over the last several years, a number of authors have proven divisibility properties satisfied by $\overline{p}(n)$ and several restricted overpartition functions. See [2, 5–10, 12, 13, 15, 16, 18–23, 25–39] for examples of such work.

In [1], Alanazi *et al.* proved the following congruence properties satisfied by overpartitions with ℓ-regular nonoverlined parts:

THEOREM 1.1 [1, Theorem 3.1]. *For all $n \geq 1$, $\overline{R_\ell^*}(n) \equiv 1 \pmod 2$ if and only if $\ell \mid n$ and the number of partitions of n/ℓ into distinct parts is odd.*

THEOREM 1.2 [1, Theorem 3.2]. *For all $n \geq 0$, $\overline{R_3^*}(9n + 4) \equiv \overline{R_3^*}(9n + 7) \equiv 0 \pmod 3$.*

The proofs of Theorems 1.1 and 1.2 given in [1] rely solely on generating function manipulations.

Our overarching goal in this brief work is to revisit the two theorems above, using them as a springboard for additional results. In Section 2 we collect all of the tools necessary to complete the proofs of our results. Section 3 will focus on proofs of results modulo 2 and 4 satisfied by infinitely many functions in this family. In particular, we provide a combinatorial argument for Theorem 1.1 as well as a more effective way to 'test' whether $\overline{R_\ell^*}(n)$ is even or odd for a particular value of n. We will also provide an infinite family of congruences modulo 4 satisfied by this family of functions. In Section 4 we return specifically to the function $\overline{R_3^*}(n)$, which is the focus of Theorem 1.2. We provide a slightly different proof of Theorem 1.2, and we extend the result significantly by finding an infinite family of mod 3 congruences satisfied by this function. We then prove an infinite number of congruences for $\overline{R}_3^*(n)$ modulo 4. Finally, we share a few closing remarks in Section 5.

2. Necessary tools

Based on the fact that $\overline{R}_\ell^*(n)$ counts overpartitions wherein there are no restrictions on the overlined parts while nonoverlined parts must be ℓ-regular, it is straightforward to see that the corresponding generating function is given by

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{R}_\ell^*(n)q^n &= \bigg(\prod_{i\geq 1}(1+q^i)\bigg)\bigg(\prod_{i\geq 1}\frac{(1-q^{\ell i})}{(1-q^i)}\bigg)\\ &= \prod_{i\geq 1}\frac{(1-q^{\ell i})(1-q^{2i})}{(1-q^i)^2} = \frac{f_\ell f_2}{f_1^2}\end{aligned} \tag{2.1}$$

where we define

$$f_k := (1-q^k)(1-q^{2k})(1-q^{3k})\ldots.$$

We will rely heavily on the generating function (2.1) in our work below. We will also use an alternative representation for the generating function for $\overline{R}_\ell^*(n)$ which involves Ramanujan's theta function,

$$\varphi(q) := \sum_{k=-\infty}^{\infty} q^{k^2} = 1 + 2\sum_{k=1}^{\infty} q^{k^2}. \tag{2.2}$$

The connection between $\varphi(q)$ and the generating function for $\overline{R}_\ell^*(n)$ is clear once we note the following two lemmas.

LEMMA 2.1 (Hirschhorn [14, (1.5.8)]). *We have*

$$\varphi(-q) = \frac{f_1^2}{f_2}.$$

LEMMA 2.2 (Hirschhorn [14, (1.5.16)]). *We have*

$$\frac{1}{\varphi(-q)} = \prod_{i\geq 0}\varphi(q^{2^i})^{2^i}.$$

Thanks to Lemmas 2.1 and 2.2, we see that the generating function for $\overline{R}_\ell^*(n)$ can also be written as

https://doi.org/10.1017/S0004972724001023 Published online by Cambridge University Press

$$\sum_{n=0}^{\infty} \overline{R}_\ell^*(n)q^n = f_\ell \prod_{i\geq 0} \varphi(q^{2^i})^{2^i}. \tag{2.3}$$

This representation is extremely beneficial when proving congruences modulo powers of 2 satisfied by $\overline{R}_\ell^*(n)$ since

$$\varphi(q) = 1 + 2\sum_{k=1}^{\infty} q^{k^2}.$$

We will rely on a few other classical results from the theory of q-series.

LEMMA 2.3 (Euler's Pentagonal number theorem; see Hirschhorn [14, (1.6.1)]). *We have*

$$f_1 = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(3j+1)/2}.$$

LEMMA 2.4 (Jacobi; see Hirschhorn [14, (1.7.1)]). *We have*

$$f_1^3 = \sum_{j=0}^{\infty} (-1)^j (2j+1) q^{j(j+1)/2}.$$

We will also utilise the following 3-dissection.

LEMMA 2.5 (Hirschhorn and Sellers, [17]). *We have*

$$f_1 f_2 = \frac{f_6 f_9^4}{f_3 f_{18}^2} - q f_9 f_{18} - 2q^2 \frac{f_3 f_{18}^4}{f_6 f_9^2}.$$

Lastly, we will need the following well-known fact which follows, in essence, from the binomial theorem.

THEOREM 2.6. *For a prime p and positive integers k and l, we have*

$$(1-q^k)^{p^l} \equiv (1-q^{pk})^{p^{l-1}} \pmod{p^l}.$$

3. General results modulo 2 and 4

We are now in a position to give alternative proofs of Theorems 1.1 and 1.2 as well as additional results. We begin by focusing on Theorem 1.1. In doing so, we provide an equivalent statement which provides a more effective way of checking the parity of $\overline{R}_\ell^*(n)$.

THEOREM 3.1. *For all $n \geq 1$, $\overline{R}_\ell^*(n) \equiv 1 \pmod 2$ if and only if $n = \ell \cdot j(3j+1)/2$ for some integer j.*

PROOF. Beginning with (2.3),

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{R}_\ell^*(n) q^n &= f_\ell \prod_{i\geq 0} \varphi(q^{2^i})^{2^i} \\ &\equiv f_\ell \pmod 2 \quad \text{thanks to (2.2)} \\ &= \sum_{j=-\infty}^{\infty} (-1)^j q^{\ell\cdot j(3j+1)/2} \quad \text{thanks to Lemma 2.3.}\end{aligned}$$

The result follows. □

Next, we wish to use Theorem 3.1 as a springboard in two different directions. First, we provide an alternative proof of Theorem 3.1.

COMBINATORIALLY INSIGHTFUL PROOF OF THEOREM 3.1. Consider an overpartition of the positive integer n which contains at least one part which is *not* a multiple of ℓ. Let λ^* be the largest such part. One can then naturally pair this overpartition with the corresponding overpartition wherein λ^* is switched from overlined to nonoverlined (or vice versa, depending on its original status in the overpartition). Clearly, this pairs all of the overpartitions of n wherein at least one part is not a multiple of ℓ. Thus, modulo 2, this set of overpartitions can be ignored. The remaining overpartitions are those wherein all of the parts must be multiples of ℓ. By definition, all such parts must be overlined, and in order for this to take place, it must be the case that each such part appears exactly once (one can only overline the *first* occurrence of any part in a given partition). Hence, the overpartitions which are not naturally paired as described above must be of the form

$$(\ell \cdot \lambda_1, \ell \cdot \lambda_2, \ldots, \ell \cdot \lambda_k)$$

where every part is overlined and $\lambda_1 > \lambda_2 > \cdots > \lambda_k$, that is, where the parts are all distinct. Clearly, the sum of these parts must be a multiple of ℓ, and it is well known that the generating function for such partitions is given by

$$\prod_{i\geq 1} (1 + q^{\ell\cdot i}).$$

This is congruent, modulo 2, to

$$\prod_{i\geq 1} (1 - q^{\ell\cdot i})$$

and the result then follows from Lemma 2.3.

Next, we wish to extend Theorem 3.1 to a result for the modulus 4 (thanks to insights gained from (2.3)). We will stop short of writing down a characterisation modulo 4. Instead, we will settle for proving the following theorem which provides an infinite family of Ramanujan-like congruences modulo 4 for these functions.

https://doi.org/10.1017/S0004972724001023 Published online by Cambridge University Press

THEOREM 3.2. *Let p be an odd prime, and let r, $1 \le r \le p-1$, be a quadratic nonresidue modulo p. Then, for all $n \ge 0$, $\overline{R}_p^*(pn+r) \equiv 0 \pmod{4}$.*

PROOF. Beginning with (2.3),

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{R}_p^*(n)q^n &= f_p \prod_{i\ge 0} \varphi(q^{2^i})^{2^i} \\ &\equiv f_p\varphi(q) \pmod{4} \\ &= f_p\Big(1 + 2\sum_{k=1}^{\infty} q^{k^2}\Big).\end{aligned}$$

Because f_p is a function of q^p, and because we are interested in arguments which are arithmetic progressions of the form $pn+r$ where $1 \le r \le p-1$, we see that we simply need to consider whether we can represent $pn+r$ as

$$pn + r = k^2.$$

If this is possible, then we know that $r \equiv k^2 \pmod{p}$. However, r is assumed to be a quadratic nonresidue modulo p. Therefore, there are no such solutions, and this implies that, for all $n \ge 0$, $\overline{R}_p^*(pn+r) \equiv 0 \pmod{4}$. □

Clearly, the above yields infinitely many congruences modulo 4 satisfied by functions within this family, the 'first' of which is that, for all $n \ge 0$,

$$\overline{R}_3^*(3n+2) \equiv 0 \pmod{4}. \tag{3.1}$$

4. Returning to $\overline{R}_3^*(n)$

We next turn our attention specifically to the function $\overline{R}_3^*(n)$. We begin by providing an alternative proof of Theorem 1.2 which will motivate much of the work in the rest of this section.

ALTERNATIVE PROOF OF THEOREM 1.2. We begin with (2.1) and work modulo 3:

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{R}_3^*(n)q^n &= \frac{f_3f_2}{f_1^2} \\ &\equiv \frac{f_1^3f_2}{f_1^2} \pmod{3} \quad \text{thanks to Theorem 2.6} \\ &= f_1f_2 \\ &= \frac{f_6f_9^4}{f_3f_{18}^2} - qf_9f_{18} - 2q^2\frac{f_3f_{18}^4}{f_6f_9^2} \quad \text{thanks to Lemma 2.5.}\end{aligned}$$

https://doi.org/10.1017/S0004972724001023 Published online by Cambridge University Press

Thanks to this 3-dissection of the generating function of $\overline{R}_3^*(n) \pmod 3$, we can immediately see that

$$\sum_{n=0}^{\infty} \overline{R}_3^*(3n+1)q^n \equiv 2f_3f_6 \pmod 3.$$

Because this last expression is a function of q^3, we know that, for all $n \geq 0$,

$$\overline{R}_3^*(3(3n+1)+1) = \overline{R}_3^*(9n+4) \equiv 0 \pmod 3$$

and

$$\overline{R}_3^*(3(3n+2)+1) = \overline{R}_3^*(9n+7) \equiv 0 \pmod 3.$$

We note that the proof above is similar in nature to the proof of Theorem 1.2 in [1]. Even so, we share the above for the sake of completeness, and also so that we might provide a proof of the following new 'internal congruence' satisfied by $\overline{R}_3^*(n)$.

LEMMA 4.1. *For all $n \geq 0$, $\overline{R}_3^*(9n+1) \equiv 2\overline{R}_3^*(n) \pmod 3$.*

PROOF. From the proof of Theorem 1.2 provided above, we see that

$$\sum_{n=0}^{\infty} \overline{R}_3^*(3n+1)q^n \equiv 2f_3f_6 \pmod 3.$$

Because the right-hand side of this congruence is a function of q^3, we know that the generating function for $\overline{R}_3^*(9n+1)$ satisfies

$$\sum_{n=0}^{\infty} \overline{R}_3^*(9n+1)q^{3n} \equiv 2f_3f_6 \pmod 3$$

or

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{R}_3^*(9n+1)q^n &\equiv 2f_1f_2 \pmod 3\\ &= 2\sum_{n=0}^{\infty} \overline{R}_3^*(n)q^n\end{aligned}$$

thanks to the proof of Theorem 1.2 above. The result follows immediately. □

Lemma 4.1 now provides the machinery necessary to prove the following new congruence family.

THEOREM 4.2. *For all $n \geq 0$ and all $\alpha \geq 1$,*

$$\overline{R}_3^*\Big(9^\alpha n + \frac{33 \cdot 9^{\alpha-1} - 1}{8}\Big) \equiv 0 \pmod 3$$

https://doi.org/10.1017/S0004972724001023 Published online by Cambridge University Press

and

$$\overline{R}_3^*\Big(9^{\alpha}n + \frac{57\cdot 9^{\alpha-1}-1}{8}\Big) \equiv 0 \pmod 3.$$

PROOF. We prove this result by induction on α. The basis case ($\alpha = 1$) states that

$$\overline{R}_3^*(9n+4) \equiv \overline{R}_3^*(9n+7) \equiv 0 \pmod 3.$$

These congruences were proven in Theorem 1.2, so the basis case holds.

Next, we prove the induction step holds for each case. Assume that, for all $n \geq 0$ and some $\alpha \geq 1$, we have

$$\overline{R}_3^*\Big(9^{\alpha}n + \frac{33\cdot 9^{\alpha-1}-1}{8}\Big) \equiv 0 \pmod 3.$$

We wish to prove that

$$\overline{R}_3^*\Big(9^{\alpha+1}n + \frac{33\cdot 9^{\alpha}-1}{8}\Big) \equiv 0 \pmod 3.$$

Note that

$$\begin{aligned} 9^{\alpha+1}n + \frac{33\cdot 9^{\alpha}-1}{8} &= 9^{\alpha+1}n + \frac{33\cdot 9^{\alpha}-(9-8)}{8} \\ &= 9^{\alpha+1}n + \frac{33\cdot 9^{\alpha}-9}{8} + 1 \\ &= 9\Big(9^{\alpha}n + \frac{33\cdot 9^{\alpha-1}-1}{8}\Big) + 1. \end{aligned}$$

Therefore, with Lemma 4.1 in hand,

$$\begin{aligned} \overline{R}_3^*\Big(9^{\alpha+1}n + \frac{33\cdot 9^{\alpha}-1}{8}\Big) &= \overline{R}_3^*\Big(9\Big(9^{\alpha}n + \frac{33\cdot 9^{\alpha-1}-1}{8}\Big) + 1\Big) \\ &\equiv 2\overline{R}_3^*\Big(9^{\alpha}n + \frac{33\cdot 9^{\alpha-1}-1}{8}\Big) \pmod 3 \\ &\equiv 0 \pmod 3 \end{aligned}$$

thanks to the induction hypothesis. Similarly, assume that, for all $n \geq 0$ and some $\alpha \geq 1$, we have

$$\overline{R}_3^*\Big(9^{\alpha}n + \frac{57\cdot 9^{\alpha-1}-1}{8}\Big) \equiv 0 \pmod 3.$$

We wish to prove that

$$\overline{R}_3^*\Big(9^{\alpha+1}n + \frac{57\cdot 9^{\alpha}-1}{8}\Big) \equiv 0 \pmod 3.$$

https://doi.org/10.1017/S0004972724001023 Published online by Cambridge University Press

Note that

$$\begin{aligned}
9^{\alpha+1}n + \frac{57 \cdot 9^{\alpha} - 1}{8} &= 9^{\alpha+1}n + \frac{57 \cdot 9^{\alpha} - (9-8)}{8} \\
&= 9^{\alpha+1}n + \frac{57 \cdot 9^{\alpha} - 9}{8} + 1 \\
&= 9\Big(9^{\alpha}n + \frac{57 \cdot 9^{\alpha-1} - 1}{8}\Big) + 1.
\end{aligned}$$

Therefore, with Lemma 4.1 in hand,

$$\begin{aligned}
\overline{R}_3^*\Big(9^{\alpha+1}n + \frac{57 \cdot 9^{\alpha} - 1}{8}\Big) &= \overline{R}_3^*\Big(9\Big(9^{\alpha}n + \frac{57 \cdot 9^{\alpha-1} - 1}{8}\Big) + 1\Big) \\
&\equiv 2\overline{R}_3^*\Big(9^{\alpha}n + \frac{57 \cdot 9^{\alpha-1} - 1}{8}\Big) \pmod 3 \\
&\equiv 0 \pmod 3
\end{aligned}$$

thanks to the induction hypothesis. This completes the proof. □

We close this section by providing an infinite family of congruences modulo 4 satisfied by $\overline{R}_3^*$. In order to do so, we first prove the following lemma which is extremely helpful.

LEMMA 4.3. *We have*

$$\sum_{n=0}^{\infty} \overline{R}_3^*(3n+1)q^n \equiv 2f_3^3 \pmod 4.$$

PROOF. In [1], the authors prove that

$$\sum_{n=0}^{\infty} \overline{R}_3^*(3n+1)q^n = 2\frac{f_2^3 f_3^3}{f_1^6}.$$

Thanks to Theorem 2.6, we know $f_2^3 \equiv f_1^6 \pmod 2$. Thus, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{R}_3^*(3n+1)q^n &= 2\frac{f_2^3 f_3^3}{f_1^6} \\
&\equiv 2\frac{f_1^6 f_3^3}{f_1^6} \pmod 4 \\
&= 2f_3^3.
\end{aligned}$$

□

We have already seen in (3.1) that, for all $n \geq 0$,

$$\overline{R}_3^*(3n+2) \equiv 0 \pmod 4.$$

In addition to this single congruence, we can prove two additional 'isolated' congruences.

https://doi.org/10.1017/S0004972724001023 Published online by Cambridge University Press

THEOREM 4.4. *For all $n \geq 0$,*

$$\overline{R}_3^*(9n+4) \equiv \overline{R}_3^*(9n+7) \equiv 0 \pmod 4.$$

PROOF. Thanks to Lemma 4.3,

$$\sum_{n=0}^{\infty} \overline{R}_3^*(3n+1)q^n \equiv 2f_3^3 \pmod 4.$$

Clearly, $2f_3^3$ is a function of q^3, so the corresponding series expansion will contain only powers of q with exponents that are multiples of 3. Therefore, for all $n \geq 0$,

$$\overline{R}_3^*(3(3n+1)+1) = \overline{R}_3^*(9n+4) \equiv 0 \pmod 4$$

and

$$\overline{R}_3^*(3(3n+2)+1) = \overline{R}_3^*(9n+7) \equiv 0 \pmod 4. \qquad \square$$

We now provide the following infinite family of congruences.

THEOREM 4.5. *Let $p \geq 5$ be prime and let r, $1 \leq r \leq p-1$, be such that* $\mathrm{inv}(3,p) \cdot 8 \cdot r + 1$ *is a quadratic nonresidue modulo p where* $\mathrm{inv}(3,p)$ *is the inverse of 3 modulo p. Then, for all $n \geq 0$, $\overline{R}_3^*(3(pn+r)+1) \equiv 0 \pmod 4$.*

PROOF. Thanks to Lemmas 4.3 and 2.4,

$$\sum_{n=0}^{\infty} \overline{R}_3^*(3n+1)q^n \equiv 2\sum_{j=0}^{\infty}(-1)^j(2j+1)q^{3j(j+1)/2} \pmod 4.$$

Therefore, if we wish to consider values of the form $\overline{R}_3^*(3(pn+r)+1)$, then we need to know whether we can write $pn+r = 3j(j+1)/2$ for some nonnegative integer j. If we can show that no such representations exist, then the theorem is proved. Note that, if a representation of the form $pn+r = 3j(j+1)/2$ exists, then $r \equiv 3j(j+1)/2 \pmod p$. Since $p \geq 3$, this is equivalent to $\mathrm{inv}(3,p) \cdot r \equiv j(j+1)/2 \pmod p$. We complete the square to obtain $\mathrm{inv}(3,p) \cdot 8 \cdot r + 1 \equiv (2j+1)^2 \pmod p$. However, we have assumed that $\mathrm{inv}(3,p) \cdot 8 \cdot r + 1$ is a quadratic nonresidue modulo p. Therefore, no such representation is possible, and this completes the proof. $\square$

Two sets of comments are in order in light of the above theorem. First, it is clear that, for any prime $p \geq 5$,

$$\mathrm{inv}(3,p) = \begin{cases} \dfrac{2p+1}{3} & \text{if } p \equiv 1 \pmod 3, \\ \dfrac{p+1}{3} & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

So from a computational standpoint, for any given prime $p \geq 5$, it is very straightforward to determine the values r which provide the congruences mentioned in the theorem.

https://doi.org/10.1017/S0004972724001023 Published online by Cambridge University Press

Secondly, we see that Theorem 4.5 provides $\frac{1}{2}(p-1)$ Ramanujan-like congruences modulo 4 for any prime $p \geq 5$. So, for example, it is easy to determine that, for all $n \geq 0$,

$$\overline{R}_3^*(3(5n+1)+1) = \overline{R}_3^*(15n+4) \equiv 0 \pmod{4},$$

and

$$\overline{R}_3^*(3(5n+2)+1) = \overline{R}_3^*(15n+7) \equiv 0 \pmod{4}.$$

5. Closing thoughts

We close this work with two comments for potential future work.

(1) In light of (2.3), it is possible to obtain congruences satisfied by $\overline{R}_\ell^*(n)$ for moduli which are higher powers of 2. See, for example, the work of Munagi and Sellers [23] for such a result modulo 8.

(2) Some of the results above, particularly Theorem 3.2, can be generalised to the functions $\overline{R}_{p^t}^*(n)$ for $t > 1$.

The interested reader may choose to pursue these items further.

References

[1] A. M. Alanazi, B. M. Alenazi, W. J. Keith and A. O. Munagi, 'Refining overpartitions by properties of nonoverlined parts', *Contrib. Discrete Math.* **17**(2) (2022), 96–111.

[2] A. M. Alanazi, A. O. Munagi and J. A. Sellers, 'An infinite family of congruences for l-regular overpartitions', *Integers* **16** (2016), Article no. A37.

[3] G. E. Andrews, *The Theory of Partitions*, Encyclopedia of Mathematics and its Applications, 2 (Addison–Wesley Publishing Co., Reading, MA, 1976).

[4] G. E. Andrews and K. Eriksson, *Integer Partitions* (Cambridge University Press, Cambridge, 2004).

[5] H. S. S. Bharadwaj, B. Hemanthkumar and M. S. M. Naika, 'On 3- and 9-regular over partitions modulo powers of 3', *Colloq. Math.* **154**(1) (2018), 121–130.

[6] K. Bringmann, J. Dousse, J. Lovejoy and K. Mahlburg, 'Overpartitions with restricted odd differences', *Electron. J. Combin.* **22**(3) (2015), Article no. 3.17.

[7] W. Y. C. Chen, Q.-H. Hou, L. H. Sun and L. Zhang, 'Ramanujan-type congruences for overpartitions modulo 16', *Ramanujan J.* **40**(2) (2016), 311–322.

[8] W. Y. C. Chen, L. H. Sun, R.-H. Wang and L. Zhang, 'Ramanujan-type congruences for overpartitions modulo 5', *J. Number Theory* **148** (2015), 62–72.

[9] W. Y. C. Chen and E. X. W. Xia, 'Proof of a conjecture of Hirschhorn and Sellers on overpartitions', *Acta Arith.* **163**(1) (2014), 59–69.

[10] J. Chetry and N. Saikia, 'New infinite families of congruences modulo 3, 5 and 7 for overpartition function', *J. Ramanujan Math. Soc.* **34**(4) (2019), 417–426.

[11] S. Corteel and J. Lovejoy, 'Overpartitions', *Trans. Amer. Math. Soc.* **356**(4) (2004), 1623–1635.

[12] R. da Silva and J. A. Sellers, 'Infinitely many congruences for k-regular partitions with designated summands', *Bull. Braz. Math. Soc. (N.S.)* **51**(2) (2020), 357–370.

[13] D. Q. J. Dou and B. L. S. Lin, 'New Ramanujan type congruences modulo 5 for overpartitions', *Ramanujan J.* **44**(2) (2017), 401–410.

[14] M. D. Hirschhorn, *The Power of q: A Personal Journey* (Springer, Cham, 2017).

[15] M. D. Hirschhorn and J. A. Sellers, 'Arithmetic properties for overpartitions', *J. Combin. Math. Combin. Comput.* **53** (2005), 65–73.

[16] M. D. Hirschhorn and J. A. Sellers, 'Arithmetic properties of overpartitions into odd parts', *Ann. Comb.* **10**(3) (2006), 353–367.

[17] M. D. Hirschhorn and J. A. Sellers, 'A congruence modulo 3 for partitions into distinct non-multiples of four', *J. Integer Seq.* **17** (2014), Article no. 14.9.6.

[18] M. D. Hirschhorn and J. A. Sellers, 'Congruences for overpartitions with restricted odd differences', *Ramanujan J.* **53**(1) (2020), 167–180.

[19] A. Jindal and N. K. Meher, 'New congruences for ℓ-regular overpartitions', *J. Korean Math. Soc.* **59**(5) (2022), 945–962.

[20] B. Kim, 'The overpartition function modulo 128', *Integers* **8** (2008), Article no. A38.

[21] B. L. S. Lin, J. Liu, A. Y. Z. Wang and J. Xiao, 'Infinite families of congruences for overpartitions with restricted odd differences', *Bull. Aust. Math. Soc.* **102**(1) (2020), 59–66.

[22] M. Merca, 'On the Ramanujan-type congruences modulo 8 for the overpartitions into odd parts', *Quaest. Math.* **45**(10) (2022), 1567–1574.

[23] A. O. Munagi and J. A. Sellers, 'Refining overlined parts in overpartitions via residue classes: Bijections, generating functions, and congruences', *Util. Math.* **95** (2014), 33–49.

[24] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences* (2024), published electronically at https://oeis.org.

[25] D. Ranganatha, 'On some new congruences for ℓ-regular overpartitions', *Palest. J. Math.* **7**(1) (2018), 345–362.

[26] C. Ray, 'Divisibility and distribution of 2ℓ-regular overpartitions', *J. Ramanujan Math. Soc.* **36**(4) (2021), 331–337.

[27] C. Ray and R. Barman, 'Infinite families of congruences for k-regular overpartitions', *Int. J. Number Theory* **14**(1) (2018), 19–29.

[28] N. Saikia and C. Boruah, 'Congruences for ℓ-regular overpartition for $\ell \in \{5, 6, 8\}$', *Indian J. Pure Appl. Math.* **48**(2) (2017), 295–308.

[29] J. A. Sellers, 'An elementary proof of a conjecture of Saikia on congruences for t-colored overpartitions', *Bol. Soc. Mat. Mex. (3)* **30**(1) (2024), Article no. 2.

[30] E. Y. Y. Shen, 'Arithmetic properties of ℓ-regular overpartitions', *Int. J. Number Theory* **12**(3) (2016), 841–852.

[31] C. Shivashankar and D. S. Gireesh, 'Congruences for ℓ-regular overpartitions into odd parts', *Bol. Soc. Mat. Mex. (3)* **28**(1) (2022), Article no. 23.

[32] D. Tang, 'Congruence properties modulo powers of 2 for overpartitions and overpartition pairs', *Int. J. Number Theory* **20**(2) (2024), 357–380.

[33] E. X. W. Xia, 'Some new infinite families of congruences modulo 3 for overpartitions into odd parts', *Colloq. Math.* **142**(2) (2016), 255–266.

[34] E. X. W. Xia, 'Congruences modulo 9 and 27 for overpartitions', *Ramanujan J.* **42**(2) (2017), 301–323.

[35] X. Yang, S.-P. Cui and B. L. S. Lin, 'Overpartition function modulo powers of 2', *Ramanujan J.* **44**(1) (2017), 89–104.

[36] O. X. M. Yao, 'Congruences modulo 64 and 1024 for overpartitions', *Ramanujan J.* **46**(1) (2018), 1–18.

[37] O. X. M. Yao and E. X. W. Xia, 'New Ramanujan-like congruences modulo powers of 2 and 3 for overpartitions', *J. Number Theory* **133**(6) (2013), 1932–1949.

[38] L. Zhang, 'New congruences for overpartitions modulo 3 and 9', *Rocky Mountain J. Math.* **51**(6) (2021), 2269–2273.

[39] T. Y. Zhao and L. J. Jin, 'New congruences modulo 5 for overpartitions', *Colloq. Math.* **145**(2) (2016), 285–290.

JAMES A. SELLERS, Department of Mathematics and Statistics,
University of Minnesota Duluth, Duluth, MN 55812, USA
e-mail: jsellers@d.umn.edu

https://doi.org/10.1017/S0004972724001023 Published online by Cambridge University Press