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A PRESENTATION FOR A GROUP OF INTEGER MATRICES

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ABSTRACT. Let Λ_n be the kernel of the map from $GL(n, \mathbb{Z})$ to $GL(n, \mathbb{Z}_2)$ induced by the quotient map $\mathbb{Z} \to \mathbb{Z}_2$. We give a presentation for Λ_n.

The kernel of the map from $GL(n, \mathbb{Z})$ to $GL(n, \mathbb{Z}_p)$ is of some interest in K-theory. We give here a presentation for this group in the case $p = 2$; we denote the group by Λ_n. Since a general presentation for $GL(n, \mathbb{Z})$ and coset representatives for $GL(n, \mathbb{Z}_2)$ are known (see Coxeter and Moser [1]), a presentation can of course be obtained by using the Reidemeister–Schreier algorithm for any particular value of n. However it is not clear how to do this for general n; and in any case the presentation derived in this way is enormous and essentially useless for deriving properties of the group.

1. **Generators and canonical form.** The group Λ_n consists of $n \times n$ integer matrices of determinant ± 1 with odd entries on the diagonal, even entries off diagonal. Let Γ_n be the subgroup of Λ_n consisting of matrices whose diagonal entries are congruent to 1 modulo 4; also let A_i be the diagonal matrix with entry -1 in the ith position, $+1$ elsewhere, and let P_n be the subgroup of Λ_n generated by $A_1, A_2, \ldots, A_n$. Clearly Λ_n is generated by $\Gamma_n P_n$, the subgroup Γ_n is normal in Λ_n, P_n is an elementary 2-group of rank n, and Λ_n is a semi-direct product of Γ_n and P_n. Finally it should be noted that $A_1 A_2 \cdots A_n = -I$ is the central element of $GL(n, \mathbb{Z})$, and the group $\Lambda_n^+ = \Lambda_n \cap SL(n, \mathbb{Z})$ is the semi-direct product of Γ_n and the subgroup of P_n generated by A_1A_2, $A_1A_3, \ldots, A_1A_n$.

Let B_{ij} denote the matrix which is the identity except for an entry 2 in the (i, j) position. Then Γ_n is generated by $\{B_{ij} : 1 \leq i, j \leq n, i \neq j\}$. Moreover we can write each element of Γ_n as a word in these generators in a canonical way:

THEOREM 1. *Each element of Γ_n has a unique expression as a product*

$$(\beta_{12}\beta_{13} \cdots \beta_{1n})(\beta_{23}\beta_{24} \cdots \beta_{2n}) \cdots (\beta_{n-1n})(\delta_{n-1n} \cdots \delta_{12})$$

where β_{ij} is a reduced word in the free group $\langle B_{ij}, B_{ji}\rangle$ of degree zero in the generator B_{ij}, $i<j$, and δ_{ij} is a power of B_{ij}, $i<j$.

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Proof. Let M be any matrix in Γ_n. Left multiplication of M by B_{12} adds twice the second row of M to the first, and left multiplication by B_{21} adds twice the first row to the second. Hence by left multiplication by elements of $\langle B_{12}, B_{21}\rangle$ we can successively reduce the absolute values of the first two entries in the first column of M. Since the first entry remains odd, there is an element $\beta_{12}\in\langle B_{12}, B_{21}\rangle$ such that $\beta_{12}^{-1}M$ has a zero in the (2, 1) position; furthermore, we may assume that β_{12} has degree zero in B_{12}. Proceeding down the first column, we produce elements $\beta_{1j}\in\langle B_{ij}, B_{j1}\rangle$ $j\le n$ of degree zero in B_{1j}, so that

$$M'=\beta_{1n}^{-1}\beta_{1n-1}^{-1}\cdots\beta_{12}^{-1}M$$

has first column consisting of zeros except in the leading entry, which then must be 1. By right multiplication by suitable powers of $B_{1n},\ldots,B_{12}$ the first row of M' can then be cleared, giving

$$M=(\beta_{12}\beta_{13}\cdots\beta_{1n})M''(\delta_{1n}\cdots\delta_{12})$$

with

$$M''=\begin{pmatrix}1 & 0\\ 0 & M^*\end{pmatrix},$$

M^* an $(n-1)\times(n-1)$ matrix in Γ_{n-1}.

We may assume by induction that

$$M''=(\beta_{23}\cdots\beta_{2n})\cdots\beta_{n-1n}\delta_{n-1n}\cdots\delta_{23}$$

uniquely.

Suppose that M can also be written

$$M=(\gamma_{12}\gamma_{13}\cdots\gamma_{1n})M'''(\varepsilon_{1n}\cdots\varepsilon_{12})$$

where $\gamma_{ij}\varepsilon\langle B_{1j}, B_{j1}\rangle$, γ_{1j} of degree zero in B_{1j}, ε_{1j} being a power of B_{1j} and $M'''\in\langle B_{ij}, B_{ji}: 2\le i<j\le n\rangle$.

Then

$$\gamma_{1n}^{-1}\cdots\gamma_{12}^{-1}\beta_{12}\cdots\beta_{1n}=M'''\varepsilon_{1n}\cdots\varepsilon_{12}\delta_{12}^{-1}\cdots\delta_{1n}^{-1}(M'')^{-1}$$

The right hand side is a matrix of the form

$$\begin{pmatrix}1 & v\\ 0 & M^{**}\end{pmatrix}$$

where v is a $1\times(n-1)$ row of integers. From the form of the left hand side it is then immediate that

$$\gamma_{12}^{-1}\beta_{12}=\left(\begin{array}{cc:c} a & b & \\ & & 0\\ 0 & c & \\ \hdashline & 0 & I\end{array}\right)$$

The only matrices in $\langle B_{12}, B_{21}\rangle$ of this form are powers of B_{12}, and since by assumption the degree of B_{12} in the product is zero (the group $\langle B_{12}, B_{21}\rangle$ is well-known to be free), we have $\beta_{12}=\gamma_{12}$. Similarly $\beta_{1j}=\gamma_{1j}$ for each j.

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Finally, applying the same argument to the resulting equation

$$\varepsilon_{1n}\cdots\varepsilon_{12}\delta_{12}^{-1}\cdots\delta_{1n}^{-1}=(M''')^{-1}M=\begin{pmatrix}1 & 0\\ 0 & \bar{M}\end{pmatrix}$$

we see that $\varepsilon_{1j}=\delta_{1j}$ and $M'''=M''$. The product form for M is therefore unique.

2. **Relations for** Γ_n. It is straightforward to verify that the following relations hold in Γ_n:

(a) $B_{ij}\leftrightarrow B_{ik}$
(b) $B_{ji}\leftrightarrow B_{ki}$
(c) $[B_{ij}, B_{jk}]=B_{ik}^2$
(d) $[B_{ij}B_{kj}, B_{ji}B_{jk}^{-1}]=(B_{ik}B_{ki}^{-1})^2, [B_{ji}B_{jk}, B_{ij}B_{kj}^{-1}]=(B_{ik}B_{ki}^{-1})^2$
(e) $B_{ij}\leftrightarrow B_{kl}$

whenever i, j, k, l are distinct indices. Here "$X\leftrightarrow Y$" means X commutes with Y, and $[X, Y]=XYX^{-1}Y^{-1}$; in (d) one may assume $i<j<k$.

THEOREM 2. *The above form a complete set of relations for* Γ_n.

Proof. That the relations (a), (b), (c), (d) present the group Γ_3 can be verified using the Reidemeister–Schreier algorithm, using for example the presentation for $GL(3,\mathbb{Z})$ given in Coxeter and Moser [1]. The calculation is tedious, but can be done by hand; it has been verified by computer.

(A direct proof that these relations suffice to enable the reduction of any word in Γ_3 to canonical form should be possible, but my attempts at this lead to unsolved problems in number theory involving primes having 2 or -2 as a primitive root; partial results which are obtainable however certainly help greatly in the hand reduction of the enormous Reidemeister–Schreier presentation to the given one.)

We shall assume the inductive hypothesis that Γ_n has the above presentation for $3\le n<N$. In order to prove the result for Γ_N, it suffices to show that every word in the generators B_{ij} can be reduced to the canonical form by means of the relations. Let G_N be the group actually presented by the relations; for any set of integers $\{i_1, i_2, \ldots, i_r\}$ with $i_1<i_2<\cdots<i_r$ let

$\langle i_1, i_2, \ldots, i_r\rangle$ denote the subgroup of G_N generated by all $B_{i_ji_k}$;
$\Delta(i_1, i_2, \ldots, i_r)$ denote the subgroup of G_N generated by those $B_{i_ji_k}$ with $i_j<i_k$;
$[i_1, i_2]$ denote the subgroup of $\langle i_1, i_2\rangle$ consisting of words of degree zero in $B_{i_1i_2}$; (Note $\langle i_1, i_2\rangle$ is free since its image in Γ_N is free).

Then $G_N=\Gamma_N$ if

$$G_N=[1,2][1,3]\cdots[1,N][2,3]\cdots[N-1,N]\Delta(N-1,N)\cdots\Delta(1,2)$$

The right hand side is

$$[1,2][1,3]\cdots[1,N]\langle 2,3,\ldots,N\rangle\Delta(1,N)\Delta(1,N-1)\cdots\Delta(1,2) \qquad (1)$$

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by the induction assumption. (From the diagram

$$\begin{array}{ccccc} G_{N-1} & \longrightarrow & G_N & \longrightarrow & \Gamma_N \\ \| & & & & \| \\ \Gamma_{N-1} & & \xrightarrow{\text{embedding}} & & \Gamma_N \end{array}$$

we may identify $\langle 2, 3, \ldots, N\rangle$ in G_N with G_{N-1}.)

Since the theorem is true for $N=3$, we note that

$$\begin{aligned}\langle i, j, k\rangle &= [1, j][i, k][j, k]\Delta(i, j, k) \quad \text{for all} \quad i<j<k \\ &= [i, k][i, j][j, k]\Delta(i, j, k)\end{aligned}$$

(a) Canonical form for $\Delta(1, 2, \ldots, \mathrm{N})$.

Let Δ_i denote the subgroup of G_N generated by $\{B_{ij} : j>i\}$. Then Δ_i is an abelian group (in fact free abelian of rank $N-i$) and is normal in $\Delta(i, i+1, \ldots, N)$ since

$$\begin{cases} B_{kl}B_{ij}B_{kl}^{-1} = B_{ij} & \text{if} \quad i, j, k, l \text{ are all distinct} \\ & \text{or if} \quad i=k \quad \text{or} \quad j=l \\ B_{jk}B_{ij}B_{jk}^{-1} = B_{ik}^{-2}B_{ij} & \text{if} \quad i<j<k \end{cases}$$

Hence

$$\begin{aligned}\Delta(1, 2, \ldots, N) &= \Delta_{N-1}\Delta_{N-2}\cdots\Delta_1 \\ &= \Delta(N-1, N)\cdots\Delta(1, N)\Delta(1, N-1)\cdots\Delta(1, 2).\end{aligned}$$

(b) Again from these relations we find that the subgroup $\langle B_{ij}, B_{kj}\rangle$ is normal in $\langle B_{ij}, B_{kl}, B_{ik}, B_{ki}\rangle$, and similarly $\langle B_{ij}, B_{ik}\rangle$ is normal in $\langle B_{ij}, B_{ik}, B_{jk}, B_{kj}\rangle$; and further

(i) $\Delta(i, j)[i, k] \subset [i, k]\Delta(k, j)\Delta(i, j)$ for $i<k<j$
(ii) $\Delta(i, j)[j, k] \subset [j, k]\Delta(i, k)\Delta(i, j)$ for $i<j<k$
(iii) $\Delta(i, j)[k, i] \subset [k, i]\Delta(k, j)\Delta(i, j)$ for $k<i<j$
(iv) $\Delta(i, j)[k, j] \subset [k, j]\Delta(i, k)\Delta(i, j)$ for $i<k<j$
(v) $\Delta(i, j)[i, k] \subset [i, k][j, k]\Delta(i, j)$ for $i<j<k$

From these we can quickly deduce

(vi) $\Delta_1 \cdot \langle 2, 3, \ldots, N\rangle \subset \langle 2, 3, \ldots, N\rangle \cdot \Delta_1$
(vii) $\Delta_1 \cdot [1, k] \subset [1, k]\langle 2, 3, \ldots, n\rangle \cdot \Delta_1$

(C) Canonical form for G_N.

Now let w be any word in the generators of G_N of length k say. By induction on k, we may assume that the initial section of w is already in canonical form, so

$$w \in [1, 2]\cdots[1, N]\langle 2, 3, \ldots, N\rangle\Delta_1 \cdot B_{ji}^{\nu} \quad \text{for some } i, j, \nu.$$

By (a) above, if $j<i$ then $\Delta_1 \cdot B_{ji}^{\nu}$ can be put in canonical form, so that w lies in the group (1).

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Thus we may assume that $i<j$; then $B_{ji}^{\nu}\in[i, j]$ and using (b) we see that

$$w\in[1,2][1,3]\cdots[1,N]\langle 2,3,\ldots,N\rangle[i,j]\langle 2,3,\ldots,N\rangle\Delta_1.$$

Case 1. $i>1$

Then $\langle 2,3,\ldots,N\rangle[i,j]\subset\langle 2,3,\ldots,N\rangle$ and we are done.

Case 2. $i=1$, $j=2$

Then

$$\begin{aligned}\langle 2,3,\ldots,N\rangle[1,2]&=[2,3]\cdots[2,N]\langle 3,4,\ldots,N\rangle\Delta_2\cdot[1,2]\\ &\quad\text{(by inductive hypothesis)}\\ &\subset[2,3]\cdots[2,N]\langle 3,4,\ldots,N\rangle[1,2]\Delta(1,2,\ldots,N)\\ &\quad\text{by b(iii)}\\ &\subset[2,3]\cdots[2,N][1,2]\langle 3,4,\ldots,N\rangle\Delta(1,2,\ldots,N)\end{aligned}$$

Now $[2,k][1,2]\subset\langle 1,2,k\rangle\subset[1,k][1,2][2,k]\Delta_2\Delta_1$ hence

$$\begin{aligned}[2,3]\cdots[2,N][1,2]&\subset[2,3]\cdots[2,N-1]\cdot[1,N][1,2][2,N]\Delta_2\Delta_1\\ &\subset[1,N]\cdot[2,3]\cdots[2,N-1][1,2]\cdot\langle 2,3,\ldots,N\rangle\Delta_1\\ &\subset[1,N][2,3]\cdots[2,N-2][1,N-1][1,2]\\ &\quad\times[2,N-1]\Delta_2\Delta_1\langle 2,3,\ldots,N\rangle\Delta_1\\ &\subset[1,N][1,N-1]\cdot[2,3]\cdots[2,N-2]\\ &\quad\times[1,2]\cdot\langle 2,3,\ldots,N\rangle\Delta_1\\ &\;\;\vdots\\ &\subset[1,N][1,N-1]\cdots[1,2]\langle 2,3,\ldots,N\rangle\Delta_1\end{aligned}$$

Finally

$$\begin{aligned}[1,2]\cdots[1,N][1,N-1]\cdots[1,2]&\subset[1,2]\langle 1,3,\ldots,N\rangle[1,2]\\ &\subset[1,2][1,3]\cdots[1,N]\\ &\quad\times\langle 3,4,\ldots,N\rangle\Delta(1,3,4,\ldots,N)\cdot[1,2]\\ &\subset[1,2][1,3]\cdots[1,N][1,2]\langle 2,3,\ldots,N\rangle\Delta_1\\ &\subset[1,2][1,3]\cdots[1,N-1]\cdot[1,2]\\ &\quad\times[1,N]\langle 2,3,\ldots,N\rangle\Delta_1\\ &\subset[1,2][1,3]\cdots[1,N-2]\cdot[1,2][1,N-1]\\ &\quad\times[2,N-1]\Delta(2,N-1)\Delta_1[1,N]\\ &\qquad\qquad\times\langle 2,3,\ldots,N\rangle\Delta_1\\ &\subset[1,2]\cdots[1,N-2][1,2]\cdot[1,N-1][1,N]\\ &\quad\times\langle 2,3,\ldots,N\rangle\Delta_1\quad\text{by b(vii)}\\ &\;\;\vdots\end{aligned}$$

Hence w lies in the group (1).

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Case 3. $i=1, j>2$

Then

$$\begin{aligned}\langle 2,3,\ldots,N\rangle[1,j] &= [2,3]\cdots[2,N]\langle 3,4,\ldots,N\rangle\Delta_2\cdot[1,j]\\ &\subset [2,3]\cdots[2,N]\langle 3,4,\ldots,N\rangle[1,j]\cdot\Delta_1\Delta_2 \quad \text{by b(iii)}\\ &\subset [2,3]\cdots[2,N]\langle 1,3,4,\ldots,N\rangle\Delta_2\Delta_1\\ &\subset [2,3]\cdots[2,N][1,3]\cdots[1,N]\langle 3,4,\ldots,N\rangle\Delta(1,2,\ldots,N)\end{aligned}$$

And

$$\begin{aligned}[2,3]\cdots[2,N][1,j] &= [2,3]\cdots[2,j-1][2,j][1,j][2,j+1]\cdots[2,N]\\ &\subset [2,3]\cdots[2,j-1]\cdot[1,j][1,2]\\ &\quad\times[2,j]\Delta_2\Delta_1\cdot[2,j+1]\cdots[2,N]\\ &\subset [1,j]\cdot[2,3]\cdots[2,j-1][1,2]\langle 2,3,\ldots,N\rangle\Delta_1\end{aligned}$$

Finally

$$\begin{aligned}[1,2]\cdots[1,N][1,j] &\subset [1,2]\cdots[1,N-1]\cdot[1,j][1,N][j,N]\Delta(j,N)\Delta_1\\ &\subset [1,2]\cdots[1,N-2][1,j][1,N-1][j,N-1]\Delta(j,N-1)\Delta_1\\ &\quad\times[1,N][j,N]\Delta(j,N)\Delta_1\\ &\subset [1,2]\cdots[1,N-2][1,j][1,N-1][1,N]\langle 2,3,\ldots,N\rangle\Delta_1\\ &\;\vdots\\ &\subset [1,2]\cdots[1,N]\langle 2,3,\ldots,N\rangle\Delta_1\end{aligned}$$

Hence

$$w\in[1,2]\cdots[1,N]\langle 2,3,\ldots,N\rangle\Delta_1\cdot[2,3]\cdots[2,j-1][1,2]\langle 2,\ldots,N\rangle\Delta_1$$

and then by Case 2, w lies in the group (1). This completes the proof of Theorem 1.

3. **Presentation for** Λ_n. The presentation for Λ_n can now be derived from the presentation for Γ_n and the action of P_n on the generators:

$$A_iB_{jk}A_i=\begin{cases}B_{jk} & \text{if } i,\ j,\ k \text{ are distinct}\\ B_{jk}^{-1} & \text{if } \quad i=j \quad\text{or}\quad i=k\end{cases}$$

Thus we have Λ_n is the group with generators A_i, B_{ij}, $1\le i, j\le n$, $i\ne j$, and relations:

(a) $B_{ij}\leftrightarrow B_{ik}$
(b) $B_{ji}\leftrightarrow B_{ki}$
(c) $[B_{ij}, B_{jk}]=B_{ik}^2$
(d) $[B_{ij}B_{kj}, B_{ji}B_{jk}^{-1})=(B_{ik}B_{ki}^{-1})^2$, $[B_{ji}B_{jk}, B_{ij}B_{kj}^{-1}]=(B_{ik}B_{ki}^{-1})^2$
(e) $B_{ij}\leftrightarrow B_{kl}$

(f) $A_i^2 = 1$
(g) $A_i \leftrightarrow A_j$
(h) $A_i B_{ij} A_i = B_{ij}^{-1}$
(i) $A_i B_{ji} A_i = B_{ji}^{-1}$
(j) $A_i \leftrightarrow B_{jk}$

for distinct indices i, j, k, l. For the group Λ_n^+, as a quotient group of Λ_n, add the relation

(k) $A_1 A_2 \cdots A_n = 1$.

Reference

1. H. S. M. Coxeter and W. O. J. Moser, Generators and relations for discrete groups, Ergebnisse der Mathematik und ihrer Grenzgebiete, Volume 14, Springer 1972.

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