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RESEARCH ARTICLE

The effective Shafarevich conjecture for abelian varieties of GL_2-type

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Abstract

In this article we establish the effective Shafarevich conjecture for abelian varieties over $\mathbb{Q}$ of GL_2-type. The proof combines Faltings' method with Serre's modularity conjecture, isogeny estimates and results from Arakelov theory. Our result opens the way for the effective study of integral points on certain higher dimensional moduli schemes such as, for example, Hilbert modular varieties.

1. Introduction

In this article we combine Faltings' method with Serre's modularity conjecture to establish the effective Shafarevich conjecture for abelian varieties over $\mathbb{Q}$ of GL_2-type.

1.1. Effective Shafarevich conjecture

In 1983, Faltings [21] proved the Shafarevich conjecture for (polarised) abelian varieties over number fields. It is known that an effective version of the Shafarevich conjecture would have many striking Diophantine applications. For example, the following effective Shafarevich conjecture (ES) implies the effective Mordell conjecture for curves over number fields (see Section 6). Let S be a nonempty open subscheme of $\mathrm{Spec}(\mathbb{Z})$, and let $g \geq 1$ be a rational integer. We denote by h_F the usual stable Faltings height; we recall in Section 2 the definition of this height introduced by Faltings.

Conjecture (ES)**.** *There exists an effective constant c, depending only on S and g, such that any abelian scheme A over S of relative dimension g satisfies* $h_F(A) \leq c$.

Coates [11] established (up to a height comparison) the case $g = 1$ of this conjecture. He combined a classical reduction of Shafarevich with the theory of logarithmic forms [1]. However, without introducing substantial new ideas, the current state of the art in the theory of logarithmic forms does not allow proving Conjecture (ES) for higher dimensional abelian varieties. In fact, Conjecture (ES) is widely open when $g \geq 2$. We refer to Section 6 for further discussions of this conjecture and related problems.

Abelian schemes of GL_2-type.
Let A be an abelian scheme over S of relative dimension g. We say[1] that A is of GL_2-type if $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ contains a number field of degree g, where $\mathrm{End}(A)$ denotes the ring of S-group scheme morphisms

[1]Here we follow Ribet [59]; some authors use a more restrictive definition (see Section 5).

$A \to A$. For example, any Hilbert-Blumenthal abelian scheme over S is of GL_2-type and there is a vast literature on abelian schemes of GL_2-type (see Section 5). More generally, we say that A is of product GL_2-type if $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ has a commutative semisimple $\mathbb{Q}$-subalgebra of degree g. We shall motivate our terminology in (5.1) by showing that A is of product GL_2-type if and only if A is isogenous to a product $\prod A_i$ of abelian schemes A_i over S of GL_2-type.

Write $N_S = \prod p$ with the product taken over all rational primes p not in S; we put here $N_S = 1$ if $S = \mathrm{Spec}(\mathbb{Z})$. Now we can state our main result (Theorem 6.2).

Theorem A. *If A is of product* GL_2*-type, then* $h_F(A) \leq (3g)^{144g} N_S^{24}$.

This fully explicit Diophantine inequality establishes Conjecture (ES) for abelian schemes of product GL_2-type and it proves new cases of the effective Shafarevich conjecture for curves (see Corollary 6.4). In addition, Theorem A leads to Corollary 6.5, giving new isogeny estimates for abelian schemes over S of product GL_2-type: Our isogeny estimates are uniform in the sense that they only depend on S and g, which is crucial for Theorem B and for the Diophantine applications in [40]. In Theorem A it is important that we do not make any semistable assumption or assume that A is simple, because these assumptions would be too restrictive for many Diophantine applications of interest.

We next consider the problem of effectively bounding, in terms of S and g, the number of isomorphism classes of abelian schemes over S of relative dimension g. This problem is still widely open[2] in the case when $g \geq 2$. However, Theorem A now allows solving the case of abelian schemes of product GL_2-type: In Theorem 6.6 we obtain the following quantitative finiteness result for the set $M_{\mathrm{GL}_2,g}(S)$ of isomorphism classes of abelian schemes over S of relative dimension g that are of product GL_2-type.

Theorem B. *The cardinality of* $M_{\mathrm{GL}_2,g}(S)$ *is at most* $(14g)^{(9g)^6} N_S^{(18g)^4}$.

We made some effort to obtain an upper bound in Theorem B that is polynomial in terms of N_S for each $g \geq 1$. In the important special case $g = 1$, Brumer-Silverman [9], Poulakis [53] and Ellenberg, Helfgott and Venkatesh [18, 30] already established such polynomial upper bounds: They obtained significantly better exponents of N_S by using completely different methods, based on Diophantine approximation or the theory of logarithmic forms; for an overview see the discussions surrounding [38, Prop 6.4]. However, their methods do not allow dealing with abelian varieties of dimension $g \geq 2$, because they all crucially exploit Weierstrass equations for elliptic curves.

Explicit Diophantine applications.
Theorem A is an effective Diophantine approximation result for S-integral points on certain 'spaces' $M_{\mathrm{GL}_2,g}$ of arbitrarily large dimension. We now discuss some explicit Diophantine applications. Without introducing new ideas it seems difficult to find applications for rational points on curves of genus at least two, because Paršin's construction [51] (see also Jorgenson-Kramer [34]) usually does not have image in $M_{\mathrm{GL}_2,g}$. However, in [38, 42] we showed that there are many other Diophantine problems of interest that can be reduced to the study of $M_{\mathrm{GL}_2,g}(S)$ via 'Paršin constructions' given by forgetful maps of moduli schemes. For example, on combining such Paršin constructions with Conjecture (ES) for $M_{\mathrm{GL}_2,g}(S)$ with $g = 1$, we proved in [38] explicit finiteness results for Diophantine equations inducing integral points on moduli schemes of elliptic curves. Moreover, in the joint work with Matschke [42], we refined the strategy for certain moduli schemes. This led to the construction of new algorithms that allows solving various fundamental Diophantine problems, including S-unit, Mordell, cubic Thue, cubic Thue-Mahler and generalised Ramanujan-Nagell equations. To demonstrate the efficiency and utility of the method, we solved large classes of classical equations and we used the resulting data to formulate and motivate new conjectures.

[2] Deligne [14] solved the analogous (but substantially easier) problem for isogeny classes by using and refining certain parts of Faltings' proof of the Shafarevich conjecture for abelian varieties [21].

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Now, our proof of Conjecture (ES) for $M_{\mathrm{GL}_2,g}(S)$ with $g \geq 2$ opens the way for the effective study of some classes of Diophantine equations that appear to be beyond the reach of the known effective methods. For instance, Theorems A and B are the main tools of the joint project with Kret [40, 41]: Therein we combine the results of the present article with suitable Paršin constructions in order to prove explicit upper bounds for the height and the number of S-integral points on Hilbert modular varieties. To this end, if g is not too large, say $g \leq 100$, then our explicit height bounds in Theorem A are in fact already sufficiently strong for practical computations when combined with efficient sieves (see [42]) that in practice can deal with huge initial bounds.

1.2. *Principal ideas of the proofs*

We continue our notation. Suppose that A is an abelian scheme over S of relative dimension $g \geq 1$ that is of product GL_2-type, with generic fibre $A_{\mathbb{Q}}$. To prove Conjecture (ES) for A, we combine Faltings' method [21] with the following tools:

(i) If $A_{\mathbb{Q}}$ is $\mathbb{Q}$-simple, then it is a quotient $J_1(N) \to A_{\mathbb{Q}}$ of the usual modular Jacobian $J_1(N)$ of some level N. Ribet [59] deduced this statement from Serre's modularity conjecture [43] by using among other things the Tate conjecture for abelian varieties [21].

(ii) Isogeny estimates for abelian varieties over number fields. These estimates were proven by the method of Faltings [21] or by the method of Masser-Wüstholz [46, 47] via the theory of logarithmic forms; see Section 3 for more details.

(iii) Bost's [5] lower bound for h_F in terms of the dimension and Javanpeykar's [31] upper bound for the stable Faltings height of Belyi curves in terms of the Belyi degree and the genus. These results are based on Arakelov theory.

Let N_A be the conductor of A, defined in Subsection 2.2. In the proof we first consider the case when $A_{\mathbb{Q}}$ is $\mathbb{Q}$-simple. A result of Carayol [10] allows controlling the number N in (i). This together with (i)–(iii) leads to an effective bound for $h_F(A)$ in terms of N_A and g and then in terms of N_S and g because A is an abelian scheme over S. To reduce the general case to the case when $A_{\mathbb{Q}}$ is $\mathbb{Q}$-simple, we use inter alia Poincaré's reducibility theorem and again the isogeny estimates in (ii). For the proof of Conjecture (ES) for A, each of the two methods in (ii) is sufficient. However, to obtain the bound in Theorem A we used in (ii) the isogeny estimates based on the method of Masser-Wüstholz, see Remark 1.

We now describe the principal ideas of our proof of Theorem B. Following Faltings [21] we divide our quantitative finiteness proof for $M_{\mathrm{GL}_2,g}(S)$ into the following two parts: (a) finiteness of $M_{\mathrm{GL}_2,g}(S)$ up to isogenies and (b) finiteness of each isogeny class of $M_{\mathrm{GL}_2,g}(S)$. To prove part (a) we use (i) and we show that any $\mathbb{Q}$-simple 'factor' A_i of $A_{\mathbb{Q}}$ is a quotient

$$J_1(\nu) \to A_i,$$

where ν is a rational integer depending only on S and g. To show part (b) we combine Theorem A with an estimate of Masser-Wüstholz [46, 47] for the minimal degree of isogenies of abelian varieties that is based on the theory of logarithmic forms. In fact, we use here the recent version of the Masser-Wüstholz estimate, due to Gaudron-Rémond [25].

Moreover, the just described strategy for Theorems A and B gives effective versions of the Shafarevich conjecture for the class of abelian varieties that are quotients of Jacobians $\mathrm{Pic}^0(X)$ of Belyi curves X with suitably controlled Belyi degree. However, substantially new ideas are required to generalise Theorems A and B to arbitrary number fields K because (i) is not available for a general K. Also, we cannot apply here our reduction to $K = \mathbb{Q}$ used in the proof of Proposition 6.1 because the notion of (product) GL_2-type is not stable under Weil restriction. Nevertheless, for certain abelian varieties of interest that are defined over an arbitrary K, one can reduce via Weil restriction to Theorems A and B; see, for example, [36, Prop 9.9], which is crucial for explicitly bounding in [41] the height and the number of integral points on coarse Hilbert moduli schemes.

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Remarks.

1) The theory of logarithmic forms [1] is a powerful tool that is often indispensable for proving effective Diophantine finiteness results. Right at the beginning of Subsection 6.2.2, we give a proof of Conjecture (ES) for abelian schemes A over S of product GL_2-type that does not rely on the theory of logarithmic forms. This proof uses in (ii) the isogeny estimates [54] based on Faltings' method. However, using in (ii) the isogeny estimates [25, 46] based on the theory of logarithmic forms leads to the bounds in Theorems A and B, which are at least exponentially better in terms of N_S and the relative dimension g of the abelian scheme A over S.
2) The present article is the second part of [36]. In the first part, published in [38], we used the Shimura-Taniyama conjecture [7, 66, 68] to prove Theorems A and B for $g = 1$. In particular, in [36, Cor 6.3] we established the effective bound $h_F(A) \ll_\epsilon N_S^{2+\epsilon}$. To obtain this bound, we first reduced to an explicit height-conductor inequality and then we proved such an inequality (proven independently by Murty-Pasten [50]) by using Frey's approach [23] via the modular degree and congruences. This approach gives better bounds for $g = 1$, but it is not available for $A \in M_{\mathrm{GL}_2,g}(S)$ with $g \geq 2$. In [36], the proof of Theorem B for $g = 1$ uses Mazur's uniform isogeny results. Because such uniform results are not available for $g \geq 2$ and because $A \in M_{\mathrm{GL}_2,g}(S)$ is not necessarily simple, we had to find other arguments (described above): For each g they lead to a bound in Theorem B that is still polynomial in terms of N_S but with a quite large exponent. In general we tried to simplify the form of the explicit bounds appearing in our results. Hence, it would be possible to state our results with explicit bounds that are slightly better. However, bounds of the form $h_F(A) \ll_g \log N_S$ are currently out of reach for any of the above discussed methods, even for $g = 1$.

1.3. Organisation of the article

Outline.

In Section 2 we discuss properties of Faltings heights and of the conductor of abelian varieties over number fields. Then in Section 3 we collect results that control the variation of Faltings heights in an isogeny class. In Section 4 we work out explicit estimates for the stable Faltings heights of Jacobian varieties of certain classical modular curves. Then we prove in Section 5 a height-conductor inequality for abelian varieties over $\mathbb{Q}$ of product GL_2-type. Finally, in Section 6 we establish the effective Shafarevich conjecture (ES) for abelian schemes of product GL_2-type and we deduce some applications. We also added two appendices: First we apply our arguments to prove an asymptotic height-conductor inequality for semistable abelian varieties over $\mathbb{Q}$ with real multiplications, and in the second appendix we simplify the shape of Raynaud's bound [54, Thm 4.4.9].

Notation, conventions and terminology.

Let K be a number field with ring of integers $\mathcal{O}_K$. We identify a nonzero prime ideal of $\mathcal{O}_K$ with the corresponding finite place v of K and vice versa. We write N_v for the number of elements in the residue field of v, we denote by $v(\mathfrak{a})$ the order of v in a fractional ideal $\mathfrak{a}$ of K and we write $v \mid \mathfrak{a}$ (respectively $v \nmid \mathfrak{a}$) if $v(\mathfrak{a}) \neq 0$ (respectively $v(\mathfrak{a}) = 0$). Let A be an abelian variety over K. We say that A is semistable if it has semistable reduction at all finite places of K.

Let S be an arbitrary scheme. We often identify an affine scheme $S = \mathrm{Spec}(R)$ with the ring R. If T and Y are S-schemes, then we denote by $Y(T) = \mathrm{Hom}_S(T, Y)$ the set of S-scheme morphisms from T to Y and we write $Y_T = Y \times_S T$ for the base change of Y from S to T. Further, if A and B are abelian schemes over S, then we denote by $\mathrm{Hom}(A, B)$ the abelian group of S-group scheme morphisms from A to B and we write $\mathrm{End}(A) = \mathrm{Hom}(A, A)$ for the endomorphism ring of A. Following [4], we say that S is a Dedekind scheme if S is a normal noetherian scheme of dimension 0 or 1.

By log we mean the principal value of the natural logarithm and we define the maximum of the empty set and the product taken over the empty set as 1. For any set M, we denote by $|M|$ the (possibly infinite) number of distinct elements of M.

2. Height and conductor of abelian varieties

Let K be a number field and let A be an abelian variety over K. In the first part of this section, we recall the definition of the relative and the stable Faltings height of A, and we review fundamental properties of these heights. In the second part, we define the conductor N_A of A and we recall useful properties of N_A.

2.1. Faltings heights

We begin to define the relative and stable Faltings height of A following [21, p. 354]. If $A = 0$, then we set $h(A) = 0$. We now assume that A has positive dimension $g \geq 1$. Let B be the spectrum of the ring of integers $\mathcal{O}_K$ of K. We denote by $\mathcal{A}$ the Néron model of A over B, with zero section $e : B \to \mathcal{A}$. Let Ω^g be the sheaf of relative differential g-forms of $\mathcal{A}/B$. We now metrise the line bundle $\omega = e^*\Omega^g$ on B. For any embedding $\sigma : K \hookrightarrow \mathbb{C}$, we denote by A^σ the base change of A to $\mathbb{C}$ with respect to σ. We choose a nonzero global section α of ω. Let $\|\alpha_\sigma\|_\sigma$ be the positive real number that satisfies

$$\|\alpha_\sigma\|_\sigma^2 = \left(\frac{i}{2}\right)^g \int_{A^\sigma(\mathbb{C})} \alpha_\sigma \wedge \overline{\alpha_\sigma},$$

where α_σ denotes the holomorphic differential form on A^σ that is induced by α. Then the relative Faltings height $h(A)$ of A is the real number defined by

$$[K : \mathbb{Q}]h(A) = \log |\omega/\alpha\mathcal{O}_K| - \sum \log\|\alpha_\sigma\|_\sigma$$

with the sum taken over all embeddings $\sigma : K \hookrightarrow \mathbb{C}$. The product formula assures that this definition does not depend on the choice of α. The relative Faltings height is compatible with products of abelian varieties: If A' is an abelian variety over K, then $h(A \times_K A') = h(A) + h(A')$. To see the behaviour of h under base change we take a finite field extension L of K. The universal property of Néron models implies

$$h(A_L) \leq h(A). \tag{2.1}$$

This inequality can be strict and thus the height h is in general not stable under base change. To obtain a stable height we may (see [29]) and do take a finite extension L' of K such that $A_{L'}$ is semistable. The stable Faltings height $h_F(A)$ of A is defined as

$$h_F(A) = h(A_{L'}).$$

This definition does not depend on the choice of L', because the formation of the identity components of the corresponding semistable Néron models commutes with the induced base change. In particular, inequality (2.1) becomes an equality when h is replaced by h_F. Further, we define $h_F(0) = 0$. We shall need an effective lower bound for $h_F(A)$ in terms of the dimension g of A. An explicit result of Bost [5] gives

$$-\frac{g}{2}\log(2\pi^2) \leq h_F(A). \tag{2.2}$$

See, for example, [26, Corollaire 8.4] and notice that $h_F(A) = h_B(A) - \frac{g}{2}\log\pi$ where h_B denotes the height, which appears in the statement of [26, Corollaire 8.4].

We shall state several of our results in terms of h_F or h and therefore we now briefly discuss important differences between these heights. From (2.1) we deduce that $h_F(A) \leq h(A)$. Further, as already observed, the height h_F has the advantage over h that it is stable under base change. On the other hand, h_F has in general weaker finiteness properties. For instance, there are only finitely many K-isomorphism classes of elliptic curves over K of bounded h, and h_F is bounded on the infinite set given by the K-isomorphism classes of elliptic curves of any fixed j-invariant in K.

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More generally, let S be a connected Dedekind scheme with field of fractions K. If A is an abelian scheme over S, then we define the stable and relative Faltings height of A by $h_F(A) = h_F(A_K)$ and $h(A) = h(A_K)$, respectively.

2.2. Conductor

We first define the conductor N_A of an arbitrary abelian variety A over any number field K. Let v be a finite place of K. We denote by f_v the usual conductor exponent of A at v; see, for example, [61, Subsection 2.1] for a definition. The conductor N_A of A is defined by

$$N_A = \prod N_v^{f_v} \tag{2.3}$$

with the product taken over all finite places v of K. In particular, $f_v(0) = 0$ and $N_0 = 1$. We now recall some useful properties of f_v and N_A. It holds that $f_v = 0$ if and only if A has good reduction at v. Furthermore, if A' is an abelian variety over K that is K-isogenous to A, then $f_v(A) = f_v(A')$ and thus $N_A = N_{A'}$. Finally, if A' is an abelian variety over K and if $C = A \times_K A'$, then $f_v(C) = f_v(A) + f_v(A')$ and hence $N_C = N_A N_{A'}$.

We shall need an explicit upper bound for f_v in terms of $g = \dim(A)$ and K. Brumer-Kramer [8] obtained such a bound by refining earlier work of Serre [62, Subsection 4.9] and of Lockhart-Rosen-Silverman [45]. To state the main result of [8] we have to introduce some notation. Let p be the residue characteristic of v, let $e_v = v(p)$ be the ramification index of v and let n be the largest integer that satisfies $n \leq 2g/(p-1)$. We define $\lambda_p(n) = \sum i r_i p^i$ for $\sum r_i p^i$ the p-adic expansion of $n = \sum r_i p^i$ with integers $0 \leq r_i \leq p-1$. Then [8, Theorem 6.2] gives

$$f_v \leq 2g + e_v\big(pn + (p-1)\lambda_p(n)\big). \tag{2.4}$$

Furthermore, the examples in [8] show that (2.4) is best possible in a strong sense. In what follows, we shall always combine (2.4) with the upper bound $\lambda_p(n) \leq n\lfloor \frac{\log n}{\log p} \rfloor$ where for any real number x we write $\lfloor x \rfloor$ for the largest integer at most x.

More generally, if S is a connected Dedekind scheme with field of fractions K and if A is an abelian scheme over S, then we define the conductor N_A of A by $N_A = N_{A_K}$.

3. Variation of Faltings heights under isogenies

In this section, we collect results that control the variation of Faltings heights under isogenies. These results are rather direct consequences of theorems in the literature.

Let K be a number field and let A be an abelian variety over K of dimension $g \geq 1$. We denote by $h_F(A)$ the stable Faltings height of A and by $h(A)$ the relative Faltings height of A; see Section 2 for the definitions. The results of Faltings [21, Lemma 5] and Raynaud [54, Corollaire 2.1.4] provide that any K-isogeny $\varphi : A \to A'$ of abelian varieties over K satisfies

$$|h(A) - h(A')| \leq \frac{1}{2}\log\deg(\varphi). \tag{3.1}$$

Let N_A be the conductor of A defined in Subsection 2.2, let D_K be the absolute value of the discriminant of K over $\mathbb{Q}$ and let $d = [K : \mathbb{Q}]$ be the degree of K over $\mathbb{Q}$.

Lemma 3.1. *Suppose A' is an abelian variety defined over K that is K-isogenous to A. Then the following statements hold:*

(i) *There exists an effective constant μ, depending only on g, N_A, d and D_K, such that*

$$|h_F(A) - h_F(A')| \leq \mu.$$

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(ii) *Suppose that* $K = \mathbb{Q}$ *and* A *is semistable. Then any abelian subvariety* C *of* A *satisfies*

$$h(C) \leq h(A) + \frac{g}{2}\log(8\pi^2).$$

The very recent work of Rémond [56] removes in (ii) the following two assumptions: $K = \mathbb{Q}$ and A is semistable. The main ingredients for the proof of this lemma are as follows. Raynaud [54] proved Lemma 3.1 (i) for semistable abelian varieties. His proof relies on refinements of certain arguments in Faltings [21]; these refinements are due to Paršin and Zarhin. To prove (i) we reduce the problem to the semistable case established in [54]. For this reduction we use the semistability criterion of Grothendieck-Raynaud [29], the criterion of Néron-Ogg-Shafarevich [63] and Dedekind's discriminant theorem. On the other hand, we deduce Lemma 3.1 (ii) from Bost's explicit lower bound for h_F in (2.2) and a result of Ullmo-Raynaud given in [67, Proposition 3.3].

Proof of Lemma 3.1. To prove (i) we let $L = K(A[12])$ be the field of definition of the 12-torsion points of A. The semistable reduction criterion [29, Proposition 4.7] shows that A_L is semistable. Let D_L be the absolute value of the discriminant of L over $\mathbb{Q}$ and let l be the relative degree of L over K. We denote by $\mathcal{T}$ the set of finite places of L where A_L has bad reduction. Let ℓ and ℓ' be the smallest rational primes such that any place in $\mathcal{T}$ has residue characteristic different from ℓ and ℓ'. An application of [54, Théorème 4.4.9] (see (6.11) for a simplified bound) with the L-isogenous abelian varieties A_L and A'_L implies

$$|h_F(A_L) - h_F(A'_L)| \leq \mu' \tag{3.2}$$

for μ' an effective constant depending only on $D_L, l, d, |\mathcal{T}|, \ell, \ell'$ and g. We now estimate these quantities effectively in terms of g, N_A, d and D_K. The criterion of Néron-Ogg-Shafarevich [63, Theorem 1] implies that $L = K(A[12])$ is unramified over all finite places v of K such that $v \nmid 12$ and such that A has good reduction at v. Thus [39, Lemma 6.2], which is based on Dedekind's discriminant theorem, gives

$$D_L \leq (D_K N_A)^l (6l^{t+2d})^{ld}$$

for t the number of finite places of K where A has bad reduction. It holds that $|\mathcal{T}| \leq lt$, and it is known that l can be explicitly controlled in terms of g (for example, one can use the embedding $\mathrm{Gal}(L/K) \hookrightarrow \mathrm{Aut}((\mathbb{Z}/12\mathbb{Z})^{2g})$ induced by the action of $\mathrm{Gal}(\bar{K}/K)$ on the 12-torsion points $A[12](\bar{K}) \cong (\mathbb{Z}/12\mathbb{Z})^{2g}$ where $\bar{K}$ is an algebraic closure of K). Further, the explicit prime number theorem in [60] gives effective upper bounds for t, ℓ and ℓ' in terms of N_A. We conclude that μ' is bounded from above by an effective constant μ that depends only on g, N_A, d and D_K. Then (3.2) and the stability of h_F prove (i).

To show (ii) we assume that $K = \mathbb{Q}$ and that A is semistable. Let C be an abelian subvariety of A. Then there exists a short exact sequence

$$0 \to C \to A \to D \to 0$$

of abelian varieties over $\mathbb{Q}$. The semistability of A provides that C and D are semistable as well; see, for example, [4, p. 182]. Therefore [67, Proposition 3.3] implies that $h(C) \leq h(A) - h(D) + g\log 2$ and then the lower bound for $h(D)$ given in (2.2) leads to statement (ii). This completes the proof of Lemma 3.1. □

We point out that Faltings' proof of the Tate conjecture, and its refinement due to Paršin-Zarhin-Raynaud [54], which is applied in Lemma 3.1 (i), both do not use Diophantine approximation or transcendence techniques. On the other hand, Masser-Wüstholz [46] gave a new proof of the Tate conjecture on using their isogeny estimates that rely on the theory of logarithmic forms from transcendence theory. Bost-David (see, for example, [6, p. 121]) showed that these isogeny estimates are fully effective, and completely explicit constants are given by Gaudron-Rémond [25]. For example, [25, Théorème 1.4] combined with (3.1) gives the following result.

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Lemma 3.2. *If A' is an abelian variety over K that is K-isogenous to A, then*

$$|h(A) - h(A')| \leq 2^{10} g^3 \log\big((14g)^{64g^2} d \max(h_F(A), \log d, 1)^2\big). \tag{3.3}$$

A very recent update of (3.3) can be found in the work [27], which is discussed in Remark 5.2 (ii). On calculating the constant μ in Lemma 3.1 (i) explicitly, it turns out that Lemma 3.1 (i) improves (3.3) in some cases and vice versa in other cases; see, for example, (6.11). As already pointed out by Masser-Wüstholz, it is quite difficult to rigorously compare bounds coming from the two different approaches; their comparison in [46, p. 471] contains in particular an interesting discussion of certain features of the two bounds. In addition, we point out that the minimal isogeny degree estimates underlying (3.3), based on the theory of logarithmic forms, have many important Diophantine applications that are completely out of reach for (the methods underlying) Lemma 3.1.

4. Faltings heights of Jacobians of modular curves

In this section, we give explicit upper bounds for the stable Faltings heights of the Jacobians of certain classical modular curves in terms of their level. These upper bounds are based on a result of Javanpeykar given in [31].

We begin to state the result of Javanpeykar. Let X be a smooth, projective and connected curve over $\bar{\mathbb{Q}}$ of genus g, where $\bar{\mathbb{Q}}$ is an algebraic closure of $\mathbb{Q}$. We denote by $\mathbb{P}^1$ the projective line over $\bar{\mathbb{Q}}$ and we let $\mathcal{D}$ be the set of degrees of finite morphisms $X \to \mathbb{P}^1$ that are unramified outside $0, 1, \infty$. Belyi's theorem [2] shows that $\mathcal{D}$ is nonempty. The Belyi degree $\deg_B(X)$ of X is defined by $\deg_B(X) = \min \mathcal{D}$. Let $\mathrm{Pic}^0(X)$ be the Jacobian of X, and let h_F be the stable Faltings height defined in Section 2. We recall that $h_F(0) = 0$ and then Javanpeykar's inequality [31, Theorem 1.1.1] gives

$$h_F(\mathrm{Pic}^0(X)) \leq 13 \cdot 10^6 \deg_B(X)^5 g.$$

We point out that $h_F(\mathrm{Pic}^0(X))$ is well defined, because the height h_F is stable. Next, we introduce some notation and we recall some basic results from the theory of modular curves that can be found, for example, in the books of Shimura [64] or Diamond-Shurman [16]. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. The associated modular curve has a smooth, projective and connected model $X(\Gamma)$ over $\bar{\mathbb{Q}}$. Let g_Γ be the genus of $X(\Gamma)$, and let ϵ_∞ be the number of cusps of $X(\Gamma)$. The inclusion $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ induces a natural projection $X(\Gamma) \to X(1) = X(\mathrm{SL}_2(\mathbb{Z}))$ whose degree d_Γ satisfies

$$g_\Gamma \leq 1 + \frac{d_\Gamma}{12} - \frac{\epsilon_\infty}{2}, \qquad d_\Gamma = \begin{cases} [\mathrm{SL}_2(\mathbb{Z}) : \Gamma] & \text{if -id} \in \Gamma, \\ [\mathrm{SL}_2(\mathbb{Z}) : \Gamma]/2 & \text{if -id} \notin \Gamma, \end{cases} \tag{4.1}$$

where $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$ denotes the index of the subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ and $\mathrm{id} \in \mathrm{SL}_2(\mathbb{Z})$ denotes the identity. Furthermore, the projection $X(\Gamma) \to X(1)$ ramifies at most over the two elliptic points of $X(1)$ or over the cusp of $X(1)$, and it holds that $X(1) \cong \mathbb{P}^1$. Therefore, it follows that $\deg_B(X(\Gamma)) \leq d_\Gamma$ and then the displayed estimate for $h_F(\mathrm{Pic}^0(X))$ implies

$$h_F(J(\Gamma)) \leq 13 \cdot 10^6 d_\Gamma^5 g_\Gamma \tag{4.2}$$

for $J(\Gamma) = \mathrm{Pic}^0(X(\Gamma))$ the Jacobian of $X(\Gamma)$. For any integer $N \geq 1$, we consider the classical congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$ and $\Gamma(N)$ of the group $\mathrm{SL}_2(\mathbb{Z})$ defined in [16, p. 13], and to ease notation we write $J_0(N) = J(\Gamma_0(N))$, $J_1(N) = J(\Gamma_1(N))$ and $J(N) = J(\Gamma(N))$. On combining the above results, we obtain the following lemma.

Lemma 4.1. *If $N \geq 1$ is an integer, then*

$$h_F(J_0(N)) \leq 7 \cdot 10^7 (N \log N)^6, \quad h_F(J_1(N)) \leq 17 \cdot 10^3 N^{12}, \quad h_F(J(N)) \leq 17 \cdot 10^3 N^{18}.$$

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Proof. We recall that $h_F(0) = 0$. Hence, to prove the claimed inequalities, we may and do assume that the Jacobians are nontrivial. Thus, the genus formulas in [16, p. 108] show that we may and do assume that $N \geq 11$ in the case $J_1(N)$ and that $N \geq 3$ in the cases $J_0(N)$ and $J(N)$. On combining (4.2) with [38, (5.1)] and $\prod_{p|N}(1+\frac{1}{p}) \leq 1 + \log N$, we obtain an upper bound for $h_F(J_0(N))$ as stated. For $\Gamma = \Gamma_1(N)$ or $\Gamma = \Gamma(N)$ there exist standard formulas that express d_Γ, $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$ and ϵ_∞ in terms of N; see, for example, [16, §3.9]. These formulas together with (4.1) and (4.2) imply upper bounds for $h_F(J_1(N))$ and $h_F(J(N))$ as claimed in Lemma 4.1.

We now include some additional details of our computations that we used here. Write $\mathrm{rad}(m)$ for the radical of $m \in \mathbb{Z}_{\geq 1}$. In the case when $\Gamma = \Gamma(N)$ and $N \geq 3$, one can use, for instance, the following formulas, which are given in [16, p.106 and p.107]:

$$d_\Gamma = \tfrac{1}{2}N^3 \prod (1 - p^{-2}) \quad \text{and} \quad \epsilon_\infty = \tfrac{d_\Gamma}{N} = \tfrac{1}{2}\tfrac{N^2}{\mathrm{rad}(N)^2}\prod (p^2 - 1)$$

with the products taken over all rational primes p dividing N. In our situation where $N \geq 3$, these formulas imply that $d_\Gamma \leq \frac{1}{2}N^3$ and $\epsilon_\infty \geq 2$. In the case $\Gamma = \Gamma_1(N)$ and $N \geq 5$, one can use, for example, the following formulas in [16, p. 107]:

$$d_\Gamma = \tfrac{d_{\Gamma(N)}}{N} \quad \text{and} \quad \epsilon_\infty = \tfrac{1}{2}\sum \Phi(d)\Phi(N/d)$$

with the sum taken over all $d \in \mathbb{Z}_{\geq 1}$ dividing N. Here Φ is the multiplicative (Euler totient) function given by $\Phi(p^k) = p^{k-1}(p-1)$ for p a prime and $k \in \mathbb{Z}_{\geq 1}$. In our situation where $N \geq 11$, the first formula together with the above formula for $d_{\Gamma(N)}$ implies that $d_\Gamma \leq \frac{1}{2}N^2$ and the second formula shows that $\epsilon_\infty \geq 2$ because $\Phi(N) \geq 4$. Indeed, $\Phi(N) \geq 4$ if N has a prime factor $p \geq 5$, and if $N = 2^a3^b$ with $a, b \in \mathbb{Z}_{\geq 0}$, then we compute that $\Phi(N) \geq 4$ when $b = 0$ (thus $a \geq 4$), when $b = 1$ (thus $a \geq 2$) and when $b \geq 2$. □

To conclude this section we discuss results in the literature that are related to Lemma 4.1. We begin with a theorem of Ullmo and we put $g = g_{\Gamma_0(N)}$. If $N \geq 1$ is a square-free integer, then [67, Théorème 1.2] gives the asymptotic upper bound

$$h_F(J_0(N)) \leq \frac{g}{2}\log N + o(g \log N). \tag{4.3}$$

Further, if $N \geq 1$ is a square-free integer, with $2 \nmid N$ and $3 \nmid N$, then Jorgenson-Kramer provide in [33, Theorem 6.2] the asymptotic formula

$$h_F(J_0(N)) = \frac{g}{3}\log N + o(g \log N). \tag{4.4}$$

On combining (4.1) with the above displayed results, one can asymptotically improve the bounds for $h_F(J_0(N))$ given in Lemma 4.1 for a special class of integers $N \geq 1$. However, our proofs of the Diophantine results in the following sections require bounds for all integers $N \geq 1$ and thus the above discussed results of Ullmo and Jorgenson-Kramer are not sufficiently general for our purpose.

5. Abelian varieties of product GL_2-type

In the first part, we define and discuss abelian schemes of product GL_2-type. Then we state inequalities relating the stable Faltings height and the conductor of abelian varieties over $\mathbb{Q}$ of product GL_2-type. In the second part, we prove the height-conductor inequalities.

Let S be a connected Dedekind scheme, with field of fractions K a number field. Let A be an abelian scheme over S of relative dimension $g \geq 1$. We say that A is of GL_2-type if there exists a number field F of degree $[F : \mathbb{Q}] = g$ together with an embedding

$$F \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

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The terminology GL_2-type comes from the following property: If A is of GL_2-type and if $V_\ell(A) = T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ denotes the rational ℓ-adic Tate module associated to the generic fibre of A, then $V_\ell(A)$ is a free $F \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$-module of rank 2 and thus the action on $V_\ell(A)$ of the absolute Galois group of K defines a representation with values in

$$\mathrm{GL}_2(F \otimes_{\mathbb{Q}} \mathbb{Q}_\ell).$$

Abelian varieties of GL_2-type were studied by several authors.[3] For example, we mention the fundamental contributions of Ribet [57, 59]. Further, we remark that elliptic curves and rational points on Hilbert modular varieties provide natural examples of abelian varieties of GL_2-type, and there exists a vast literature on special classes (e.g., Hilbert-Blumenthal type) of abelian varieties of GL_2-type; see, for instance, van der Geer [28].

More generally, we say that A is of product GL_2-type if A is isogenous to a product $\prod A_i$ of nonzero abelian schemes A_i over S such that each A_i is of GL_2-type. If A is isogenous to $\prod A_i$ and each $\mathrm{End}(A_i) \otimes_{\mathbb{Z}} \mathbb{Q}$ contains a number field F_i of degree equal to the relative dimension g_i of A_i over S, then $E = \prod F_i$ is a commutative semisimple $\mathbb{Q}$-subalgebra of $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ with $[E : \mathbb{Q}] = g$. In the case when $K = \mathbb{Q}$, we obtain in addition the following: The abelian scheme A is of product GL_2-type if and only if there exists a commutative semisimple $\mathbb{Q}$-subalgebra

$$E \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q} \tag{5.1}$$

with $[E : \mathbb{Q}] = g$. To prove this statement, we suppose that $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ has a commutative semisimple $\mathbb{Q}$-subalgebra E with $[E : \mathbb{Q}] = g$. Wedderburn's theorem gives an isomorphism of $\mathbb{Q}$-algebras $E \cong \prod F_i$ where each F_i is a number field. Then, on using the idempotents of E, we obtain that A is isogenous to a product $\prod A_i$ of nonzero abelian schemes A_i over S such that F_i embeds into $\mathrm{End}(A_i) \otimes_{\mathbb{Z}} \mathbb{Q}$. We now use that $K = \mathbb{Q}$. By functoriality, F_i acts on the $\mathbb{Q}$-vector space $\mathrm{Lie}(A_i \times_S \mathbb{Q})$, which has dimension g_i. It follows that $[F_i : \mathbb{Q}] \leq g_i$. This together with the equalities $\sum g_i = g = \sum [F_i : \mathbb{Q}]$ implies that the strict inequality $[F_i : \mathbb{Q}] < g_i$ is impossible. We deduce that $[F_i : \mathbb{Q}] = g_i$, which means that each A_i is of GL_2-type. Thus, A is of product GL_2-type as desired.

5.1. Height and conductor

Let A be an abelian variety over $\mathbb{Q}$ of dimension $g \geq 1$. We denote by $h_F(A)$ the stable Faltings height of A, and we denote by N_A the conductor of A. See Section 2 for the definitions of $h_F(A)$ and N_A. We obtain the following result.

Theorem 5.1. *If A is of product GL_2-type, then the following statements hold:*

(i) *There is an effective constant k, depending only on g, N_A, such that $h_F(A) \leq k$.*
(ii) *It holds $h_F(A) \leq (3N_A)^{12}$.*

Here (i) is a direct consequence of (ii). An advantage of first stating (i) separately is that our proof of (i) is different and this will allow us in Subsection 6.2.2 to give via (i) an interesting proof of Conjecture (ES) for A of product GL_2-type. However, the proofs of Theorem 5.1 (i) and (ii) are in principle the same. The main difference is that in (i) we use the isogeny result in Lemma 3.1 (i) based on 'essentially algebraic' methods, and in (ii) we apply the isogeny result (3.3) based on the theory of logarithmic forms. On calculating explicitly the constant k in our proof of (i), it turns out (see Remark 5.2) that the resulting bound for $h_F(A)$ is at least exponential in terms of N_A and g. The idea underlying the proof of (i), to apply isogeny results based on essentially algebraic methods for proving height-conductor inequalities, can be useful in certain situations for improving bounds; see, for example,

[3] Some authors use a more restrictive definition. For example, they assume in addition simplicity or they make extra assumptions on the Lie algebra. Assuming simplicity would be too restrictive for our purpose (e.g., it would significantly weaken our results, ruling out interesting Diophantine applications). On the other hand, we do not make extra assumptions on the Lie algebra because either we do not need them or these assumptions are automatically satisfied in the situations under consideration.

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Proposition 6.9 and the discussion before its proof. Further, we show in Appendix A that one can refine the proof of (i) or the proof of (ii) to asymptotically improve the bound in (ii) for semistable abelian varieties with real multiplications.

5.2. *Proof of Theorem 5.1*

We first collect useful properties of abelian varieties over $\mathbb{Q}$ of GL_2-type. Then we combine these properties with results obtained in Sections 2, 3 and 4.

5.2.1. Preliminaries

To prove Theorem 5.1 we use Serre's modularity conjecture [62, (3.2.4)?]. Building on the work of many mathematicians, Khare-Wintenberger [43] recently proved Serre's modularity conjecture. Furthermore, Ribet generalised the arguments of Serre [62, Théorème 5] and he showed in [59, Theorem 4.4] that Serre's modularity conjecure has the following consequence. Suppose that A is an abelian variety over $\mathbb{Q}$ of GL_2-type. If A is $\mathbb{Q}$-simple, then there exists an integer $N \geq 1$ together with a surjective morphism

$$J_1(N) \to A \tag{5.2}$$

of abelian varieties over $\mathbb{Q}$. Here $J_1(N)$ denotes the usual modular Jacobian, defined, for example, in Section 4. We note that Serre and Ribet used the Tate conjecture [21] to prove the implication 'Serre's modularity conjecture $\Rightarrow$ (5.2)'.

We now collect additional results that will be used in the proof of Theorem 5.1. Assume that A is an abelian variety over $\mathbb{Q}$ of GL_2-type. Then there exists a $\mathbb{Q}$-simple abelian variety B over $\mathbb{Q}$ of GL_2-type, an integer $n \geq 1$ and a $\mathbb{Q}$-isogeny

$$A \to B^n \tag{5.3}$$

for B^n the n-fold product of B. This result was established by Ribet in the course of his proof of [59, Theorem 2.1]. Further, we shall use the following important property of abelian varieties of GL_2-type. Suppose that A and A' are $\mathbb{Q}$-isogenous abelian varieties over $\mathbb{Q}$. Then A is of GL_2-type if and only if A' is of GL_2-type. Indeed, this follows directly from $\dim(A) = \dim(A')$ and $\mathrm{End}(A)\otimes_{\mathbb{Z}}\mathbb{Q} \cong \mathrm{End}(A')\otimes_{\mathbb{Z}}\mathbb{Q}$.

For any abelian variety A over $\mathbb{Q}$, we denote by N_A the conductor of A. Let $N \geq 1$ be an integer and consider the classical congruence subgroup $\Gamma_1(N) \subset \mathrm{SL}_2(\mathbb{Z})$. For any normalised newform $f \in S_2(\Gamma_1(N))$, we let $A_f = J_1(N)/I_f J_1(N)$ be the abelian variety over $\mathbb{Q}$ associated to f by Shimura's construction; see, for example, [16] for the definitions. The abelian variety A_f is $\mathbb{Q}$-simple and the $\mathbb{Q}$-simple 'factors' of $J_1(N)$ are unique up to $\mathbb{Q}$-isogenies. Thus, [58, Proposition 2.3] implies that any $\mathbb{Q}$-simple quotient of $J_1(N)$ is $\mathbb{Q}$-isogenous to A_f for some normalised newform $f \in S_2(\Gamma_1(M))$ of level M with $M \mid N$. Further, a result of Carayol in [10] gives for each $M \geq 1$ that any normalised newform $f \in S_2(\Gamma_1(M))$ satisfies $N_{A_f} = M^{\dim(A_f)}$. Suppose now that A is the $\mathbb{Q}$-simple abelian variety over $\mathbb{Q}$ of GL_2-type, which appears in (5.2). Then on combining the above observations, we see that one can choose the number N in (5.2) such that

$$N_A = N^{\dim(A)}. \tag{5.4}$$

We are now ready to prove the height-conductor inequalities.

5.2.2. Proofs

We continue our notation. For any abelian variety A over $\mathbb{Q}$, we denote by $h_F(A)$ its stable Faltings height. In our proofs we shall obtain a result that is more technical than Theorem 5.1 and that we now state because it will be used again in Section 6.

Theorem 5.1*. *Let A be an abelian variety over $\mathbb{Q}$ of dimension $g \geq 1$. Suppose that A is of product GL_2-type. Then there exist positive integers N_i and e_i, together with $\mathbb{Q}$-simple abelian subvarieties A_i*

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of $J_1(N_i)$ of dimension g_i, such that A is $\mathbb{Q}$-isogenous to $\prod A_i^{e_i}$ and such that $N_i^{g_i} = N_{A_i}$. Furthermore, it holds that $N_A = \prod N_i^{e_i g_i}$ and

$$h_F(A) \leq \sum e_i\big(18 \cdot 10^3 N_i^{12} + (8g)^6 \log N_i\big), \quad N_i \geq 11.$$

As already mentioned, our proofs of Theorem 5.1 (i) and (ii) are essentially the same. We now describe the proof of (ii), which is divided into the following two parts. In the first part, we use (5.2) and (5.4) to show that any abelian variety A as in Theorem 5.1 is $\mathbb{Q}$-isogenous to a product $\prod A_i^{e_i}$, where $e_i \geq 1$ is an integer and A_i is an abelian subvariety of $J_1(N_i)$ for $N_i \geq 1$ an integer dividing N_{A_i}. In the second part, we then combine results from Sections 2 and 3 to deduce an upper bound for $h_F(A)$ in terms of $h_F(J_1(N_i))$, $\dim(J_1(N_i))$ and $g = \dim(A)$, then in terms of N_i and g by invoking results from Section 4 and finally in terms of N_A and g, because each $N_i \mid N_{A_i}$ divides N_A by Subsection 2.2.

Proof of Theorems 5.1 and 5.1.* We take an abelian variety A over $\mathbb{Q}$ of dimension $g \geq 1$ and we assume that A is of product GL_2-type.

1. Poincaré's reducibility theorem gives positive integers e_i together with $\mathbb{Q}$-simple abelian varieties A_i over $\mathbb{Q}$ such that A is $\mathbb{Q}$-isogenous to the product $\prod A_i^{e_i}$. We write g_i for the dimension of A_i. The $\mathbb{Q}$-simple 'factors' A_i of A are unique up to $\mathbb{Q}$-isogeny, and by assumption A is $\mathbb{Q}$-isogenous to a product of abelian varieties over $\mathbb{Q}$ of GL_2-type. Therefore, (5.3) implies that A_i is $\mathbb{Q}$-isogenous to an abelian variety over $\mathbb{Q}$ of GL_2-type and thus A_i is of GL_2-type as well. Then the results collected in (5.2) and (5.4) provide a positive integer N_i with $N_i^{g_i} = N_{A_i}$ together with a surjective morphism

$$J_i = J_1(N_i) \to A_i \tag{5.5}$$

of abelian varieties over $\mathbb{Q}$. Let B_i be the identity component of the kernel of $J_i \to A_i$. It is an abelian subvariety of J_i. Then Poincaré's reducibility theorem gives a complementary abelian subvariety A_i' of J_i together with a $\mathbb{Q}$-isogeny $A_i' \times_{\mathbb{Q}} B_i \to J_i$ induced by addition. We next verify that A_i and A_i' are $\mathbb{Q}$-isogenous. The kernel of the surjective morphism $J_i \to A_i$ is a $\mathbb{Q}$-subgroup scheme of J_i whose dimension coincides with $\dim(B_i)$. Hence, the dimension formula implies that the dimensions of A_i and A_i' coincide. Let $A_i' \to A_i$ be the morphism obtained by composing the natural inclusion $A_i' \hookrightarrow A_i' \times_{\mathbb{Q}} B_i$ with the $\mathbb{Q}$-isogeny $A_i' \times_{\mathbb{Q}} B_i \to J_i$ and then with the morphism $J_i \to A_i$. We recall that the surjective morphism $A_i' \times_{\mathbb{Q}} B_i \to J_i$ is induced by addition, and B_i is the identity component of the kernel of $J_i \to A_i$. Therefore, we see that the morphism $A_i' \to A_i$ is surjective, and thus it is a $\mathbb{Q}$-isogeny because $\dim(A_i') = \dim(A_i)$. Hence, after replacing A_i by A_i', we may and do assume that A_i is an abelian subvariety of J_i and that there exists a $\mathbb{Q}$-isogeny

$$A_i \times_{\mathbb{Q}} B_i \to J_i. \tag{5.6}$$

2. We now begin to estimate the heights. For any abelian variety B over $\mathbb{Q}$, we denote by v_B the maximal variation of the stable Faltings height h_F in the $\mathbb{Q}$-isogeny class of B; that is, $v_B = \sup|h_F(B) - h_F(B')|$ with the supremum taken over all abelian varieties B' over $\mathbb{Q}$ that are $\mathbb{Q}$-isogenous to B. The abelian variety $A' = \prod A_i^{e_i}$ satisfies

$$h_F(A') = \sum e_i h_F(A_i), \quad \text{and} \quad h_F(A) \leq v_{A'} + h_F(A') \tag{5.7}$$

because A is an abelian variety over $\mathbb{Q}$ that is $\mathbb{Q}$-isogenous to A'. Write $n_i = \dim(J_i)$ and define $J_i' = A_i \times_{\mathbb{Q}} B_i$. It holds that $h_F(A_i) = h_F(J_i') - h_F(B_i)$, and it follows from (5.6) that $h_F(J_i')$ is at most $v_{J_i} + h_F(J_i)$. Therefore, the lower bound for $h_F(B_i)$ in (2.2) implies

$$h_F(A_i) \leq v_{J_i} + h_F(J_i) + \tfrac{n_i}{2}\log(2\pi^2). \tag{5.8}$$

Here we used that the dimension of B_i is at most $\dim(J_i') = n_i$ and that the lower bound for $h_F(B_i)$ in (2.2) holds in addition for $B_i = 0$ because $h_F(0) = 0$. To control n_i in terms of N_i, we consider the modular curve $X_1(N_i) = X(\Gamma_1(N_i))$ over $\mathbb{Q}$ defined in Section 4. We recall that $J_i = J_1(N_i)$ is the Jacobian of $X_1(N_i)$ and hence the genus of $X_1(N_i)$ coincides with the dimension n_i of J_i. Therefore, (4.1) together with [16, p. 107] implies

$$n_i \leq \tfrac{1}{24} N_i^2. \tag{5.9}$$

To bound N_{J_i} in terms of N_i, we use the classical result of Igusa, which says that $X_1(N_i)$ has good reduction at all primes $p \nmid N_i$. In particular, the Jacobian $J_i = \mathrm{Pic}^0(X_1(N_i))$ has good reduction at all primes $p \nmid N_i$. It follows that all prime factors of N_{J_i} divide N_i. Then (2.4) gives an effective bound for N_{J_i} in terms of n_i and N_i, which together with (5.9) shows that there exists an effective constant c_i, depending only on N_i, such that

$$N_{J_i} \leq c_i. \tag{5.10}$$

We recall that $A' = \prod A_i^{e_i}$ is an abelian variety over $\mathbb{Q}$ that is $\mathbb{Q}$-isogenous to A. Hence, we get that $N_A = N_{A'}$ and then the equality $N_{A_i} = N_i^{g_i}$ in statement (5.5) gives

$$N_A = \prod N_{A_i}^{e_i} = \prod N_i^{e_i g_i}. \tag{5.11}$$

In addition, the dimension formula gives that $g = \sum e_i g_i$. In particular, we obtain $e_i \leq g$.

We now prove (i). Lemma 3.1 (i) gives an effective upper bound for $v_{A'}$ in terms of $\dim(A') = g$ and $N_{A'} = N_A$ and for v_{J_i} in terms of n_i and N_{J_i}. On combining these upper bounds with (5.7) and (5.8), we obtain an effective estimate for $h_F(A)$ in terms of g, N_A, n_i, $h_F(J_i)$ and N_{J_i}; then in terms of g, N_A and N_i by (5.9), Lemma 4.1 and (5.10) and finally in terms of g and N_A by (5.11). This completes the proof of (i).

To show (ii) we use the inequality (3.3), which holds with h_F in place of h because the degree of an isogeny is stable. This inequality gives an upper bound for $v_{A'}$ in terms of $\dim(A') = g$ and $h_F(A')$ and for v_{J_i} in terms of n_i and $h_F(J_i)$. On combining these upper bounds with (5.7) and (5.8), we obtain an estimate for $h_F(A)$ in terms of g, e_i, n_i and $h_F(J_i)$. Then (5.9) together with the upper bound for $h_F(J_i)$ in Lemma 4.1 leads to an estimate for $h_F(A)$ in terms of g, e_i and N_i. More precisely, on computing the bounds explicitly (see below for some possible intermediate steps), one obtains, for example, the bound in Theorem 5.1* and then

$$h_F(A) \leq \tfrac{1}{2}(3N_A^*)^{12} + (8g)^6 \log N_A^*, \quad N_A^* = \prod N_i^{e_i}. \tag{5.12}$$

To simplify the bound we used here that $N_i > 10$. Indeed, in the cases $N_i \leq 10$, the modular curve $X_1(N_i)$ has genus zero and $J_i = \mathrm{Pic}^0(X_1(N_i)) = 0$, which is not possible because A_i is a nonzero abelian subvariety of J_i. Finally, it follows from (5.11) that N_A^* divides N_A and that $g \leq \frac{\log N_A}{\log 11}$, and then (5.12) implies (ii).

We now include some details of our computations described above in which we may and do assume that $N_i \geq 11$: For example, one can use the inequalities

$$a)\, v_{J_i} \leq N_i^{12},\ b)\, h_F(A_i) \leq 18 \cdot 10^3 N_i^{12},\ c) \sum e_i N_i^{12} \leq (N_A^*)^{12},\ d)\, v_{A'} \leq (8g)^6 \log(N_A^*)$$

as intermediate steps in order to compute the bound in Theorem 5.1*, which follows from (5.7), b) and d). Here *a*) follows by combining (3.3), (5.9) and Lemma 4.1. Then b) follows by combining (5.8), (5.9), a) and Lemma 4.1, and c) holds because $x + y \leq xy$ for $x, y \in \mathbb{R}_{\geq 2}$. Finally, d) follows by combining (3.3), (5.7), b) and c). □

Remark 5.2. (i) Without introducing new ideas, the proof of (i) gives a bound for $h_F(A)$ that is at least exponential in terms of N_A and g. Indeed, this follows from (6.11), which shows that Lemma 3.1 (i) gives a bound for v_{J_i} and $v_{A'}$ that is at least exponential in terms of $n_i = \dim(J_i)$ and $g = \dim(A)$, respectively. The bound in (6.11) was obtained by simplifying the main result of [54]. However, any bound for v_{J_i} and $v_{A'}$ coming from [54] has to be at least exponential in terms of n_i and g, respectively, because the crucial result [54, 4.3.7] involves a certain quantity $M \geq \binom{2n}{n} \geq 2^n$ for $n \in \{n_i, g\}$.

(ii) The above proof shows that the (main) terms N_A^{12} and N_i^{12} in our bounds in Theorems 5.1 and 5.1* come from the inequality $h_F(J_1(N)) \leq 17 \cdot 10^3 N^e$ in Lemma 4.1 for $e = 12$. This inequality relies on [31], which can be refined in our special situation where $\mathrm{Pic}^0(X) = J_1(N)$. In case one can reduce here the exponent $e = 12$ to $e < 10$, one obtains further improvements of our bounds by using in addition two very recent results: A generalisation of Lemma 3.1 (ii) to arbitrary abelian varieties, which is contained in Rémond's work [56], and a refinement of (3.3) due to Gaudron-Rémond [27]. We would like to thank the referee for informing us about these recent works.

6. Effective Shafarevich conjecture

In the first part of this section, we discuss several aspects of the effective Shafarevich conjecture. In the second part, we give our explicit version of the effective Shafarevich conjecture for abelian varieties of product GL_2-type and we deduce some applications. In the third part, we prove the results of Section 6.

Let S be a nonempty open subscheme of $\mathrm{Spec}(\mathbb{Z})$ and let $g \geq 1$ be an integer. We denote by $h_F(A)$ the stable Faltings height of an abelian scheme A over S. See Section 2 for the definition. We now recall the effective Shafarevich conjecture.

Conjecture (ES)**.** *There exists an effective constant c, depending only on S and g, such that any abelian scheme A over S of relative dimension g satisfies $h_F(A) \leq c$.*

As already mentioned, Conjecture (ES) would have striking applications to classical Diophantine problems. For example, the following proposition gives that Conjecture (ES) implies the effective Mordell conjecture for curves over number fields.

Proposition 6.1. *Suppose that Conjecture (ES) holds. If X is a smooth, projective and geometrically connected curve of genus at least 2, defined over an arbitrary number field, then one can determine in principle all rational points of X.*

Let K be a number field. In what follows, by a curve over K we always mean a smooth, projective and geometrically connected curve over K. For any curve X over K, we denote by $h_F(X)$ the stable Faltings height of the Jacobian $\mathrm{Pic}^0(X)$ of X. In the first part of the proof of Proposition 6.1, we will show that Conjecture (ES) implies in particular the following 'classical' effective Shafarevich conjecture $(ES)^*$ for curves over K.

Conjecture $(ES)^*$**.** *Let T be a finite set of places of K. There exists an effective constant c, depending only on K, T and g, such that any curve X over K of genus g, with good reduction outside T, satisfies*

$$h_F(X) \leq c$$

In the second part of the proof of Proposition 6.1, we apply Rémond's effective Kodaira construction in [55], which gives that $(ES)^*$ implies the effective Mordell conjecture.

We now discuss several aspects of Conjectures (ES) and $(ES)^*$. First, we mention that Conjecture (ES), which implies $(ES)^*$, is a priori considerably stronger than $(ES)^*$. For example, if X is a curve over K, then Conjecture (ES) would also allow controlling the finite places of K where the reductions of X and $\mathrm{Pic}^0(X)$ are different; see the discussion at the end of Section 6 for more details. Furthermore, it is shown in [40] that already special cases of Conjecture (ES), such as, for example, Theorem 6.2, have direct applications to the effective study of Diophantine equations. On the other hand, one needs to prove Conjecture $(ES)^*$ in quite general situations to get effective Diophantine applications. For example, de Jong-Rémond [32] established Conjecture $(ES)^*$ for curves over K that are geometrically cyclic covers of prime degree of the projective line $\mathbb{P}^1_K$: They combined the method introduced by Paršin [52] with the theory of logarithmic forms; see also the proof of [39, Thm 3.2] for some refinements and

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[19, 24, 35, 37] for stronger results for (hyper)elliptic curves via different methods. However, the case of cyclic covers is not general enough to directly deduce applications for rational points via the known constructions of Kodaira or Paršin [51].

The proof of Proposition 6.1 shows moreover that Conjecture (ES) is in fact equivalent to the following (a priori more general) conjecture: For any nonempty open subscheme S of the spectrum of the ring of integers of K, there exists an effective constant c, depending only on K, S and g, such that any abelian scheme A over S of relative dimension g satisfies $h_F(A) \le c$. To prove the equivalence one uses inter alia the Weil restriction. We refer to the proof of Proposition 6.1 for details. Further, we mention that one can, of course, formulate Conjecture (ES) more classically in terms of $\mathbb{Q}$-isomorphism classes of abelian varieties over $\mathbb{Q}$ of dimension g, with good reduction outside a finite set of rational prime numbers. However, our formulation of Conjecture (ES) in terms of abelian schemes is more convenient for the effective study of integral points on moduli schemes.

Finally, we remark that a geometric analogue of Conjecture (ES) was established by Faltings [20]; see also Deligne [15, p. 14] for some refinements.

6.1. Abelian schemes of product GL_2*-type*

We continue the notation of the previous section. Let S be a nonempty open subscheme of $\mathrm{Spec}(\mathbb{Z})$ and let $g \ge 1$ be an integer. We write $N_S = \prod p$ with the product taken over all rational prime numbers p that are not in S. The following theorem establishes the effective Shafarevich conjecture (ES) for all abelian schemes of product GL_2-type.

Theorem 6.2. *Let A be an abelian scheme over S of relative dimension g. If A is of product* GL_2*-type, then*

$$h_F(A) \le (3g)^{144g} N_S^{24}.$$

We point out that the bound in Theorem 6.2 is polynomial in terms of N_S. In course of the proof of Theorem 6.2 we shall obtain the more precise inequality (6.7), which improves in particular the estimate of Theorem 6.2 and which is polynomial in terms of the relative dimension g of A. Moreover, it is possible to refine (6.7) in special cases. For example, we obtain the following result for semistable abelian varieties of product GL_2-type.

Proposition 6.3. *Let A be an abelian scheme over S of relative dimension g. If A is of product* GL_2*-type and if the generic fibre of A is semistable, then*

$$h_F(A) \le g(3N_S)^{12} + (6g)^7 \log N_S.$$

Next, we deduce from Theorem 6.2 new cases of the 'classical' effective Shafarevich conjecture $(ES)^*$. We say that a curve X over $\mathbb{Q}$ is of product GL_2-type if the Jacobian $\mathrm{Pic}^0(X)$ of X is of product GL_2-type. There exist many curves over $\mathbb{Q}$ of genus ≥ 2 that are of product GL_2-type; see, for example, the articles in [13]. Let T be a finite set of rational prime numbers and write $N_T = \prod p$ with the product taken over all $p \in T$.

Corollary 6.4. *Let X be a curve over $\mathbb{Q}$ of genus g that is of product* GL_2*-type. If* $\mathrm{Pic}^0(X)$ *has good reduction outside T, then*

$$h_F(X) \le (3g)^{144g} N_T^{24}.$$

Proof. The Néron model of $\mathrm{Pic}^0(X)$ over $S = \mathrm{Spec}(\mathbb{Z}) - T$ is an abelian scheme, because $\mathrm{Pic}^0(X)$ has good reduction outside T. Therefore, Theorem 6.2 implies Corollary 6.4. □

If X is a curve over $\mathbb{Q}$ with good reduction at a rational prime p, then $\mathrm{Pic}^0(X)$ has good reduction at p. This shows that Corollary 6.4 establishes in particular the classical effective Shafarevich conjecture $(ES)^*$ for all curves over $\mathbb{Q}$ of product GL_2-type.

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We now derive new isogeny estimates for abelian varieties over $\mathbb{Q}$ of product GL_2-type. Masser-Wüstholz bounded in [46, 47] the minimal degree of isogenies of abelian varieties. On combining Theorem 6.2 with the recent version of the Masser-Wüstholz results due to Gaudron-Rémond [25], we obtain the following corollary.

Corollary 6.5. *Suppose that A and B are isogenous abelian schemes over S of relative dimension g. If A or B is of product GL_2-type, then the following statements hold:*

(i) *There exist isogenies $A \to B$ and $B \to A$ of degree at most $(14g)^{(12g)^5} N_S^{(37g)^3}$.*
(ii) *In particular, it holds that $|h_F(A) - h_F(B)| \leq (30g)^3 \log N_S + (9g)^6$.*

The proof of this result shows, moreover, that Corollary 6.5 (ii) holds with h_F replaced by the relative Faltings height h. Further, we point out that the above isogeny estimates are independent of A and B. As already mentioned, this is absolutely crucial for Theorem 6.6 and for certain Diophantine applications such as, for example, [40, 41]. On calculating the constant of Lemma 3.1 (i) explicitly, we see that the bound in Corollary 6.5 (ii) is better in terms of N_S and g than Lemma 3.1 (i). On the other hand, Lemma 3.1 (i) is considerably more general than Corollary 6.5 (ii).

Recall that $M_{\mathrm{GL}_2,g}(S)$ denotes the set of isomorphism classes of abelian schemes over S of relative dimension g that are of product GL_2-type. Corollary 6.5 (i) is one of the main ingredients for the proof of the following quantitative finiteness result for $M_{\mathrm{GL}_2,g}(S)$.

Theorem 6.6. *The cardinality of $M_{\mathrm{GL}_2,g}(S)$ is at most $(14g)^{(9g)^6} N_S^{(18g)^4}$.*

To state some consequences of (the proof of) Theorem 6.6 for $\mathbb{Q}$-isomorphism classes of abelian varieties over $\mathbb{Q}$, we recall that T denotes an arbitrary finite set of rational prime numbers and we let $N_T = \prod_{p \in T} p$ be as above. We obtain the following corollary.

Corollary 6.7. *Let A be an abelian variety over $\mathbb{Q}$ of dimension g. We assume that A has the following properties: (a) A is of product GL_2-type and (b) A has good reduction outside T. Then the following statements hold:*

(i) *Up to $\mathbb{Q}$-isomorphisms, there exist at most $(14g)^{(9g)^6} N_T^{(18g)^4}$ abelian varieties over $\mathbb{Q}$ that are $\mathbb{Q}$-isogenous to our given A.*
(ii) *Up to $\mathbb{Q}$-isogenies, there exist at most $(3g)^{32g^2} N_T^{4g}$ abelian varieties over $\mathbb{Q}$ of dimension g that have properties (a) and (b).*

We remark that it is possible to prove a considerably more general version of Corollary 6.7 (ii) by refining Faltings' proof of [21, Satz 5] with an effective Čebotarev density theorem; see, for example, Deligne [14]. However, the resulting unconditional bound for the number of isogeny classes would be worse than the estimate in Corollary 6.7 (ii).

6.2. Proof of the results of Section 6

In the first part of this section, we collect some useful results for abelian schemes. In the second part, we first show Theorem 6.2, Proposition 6.3 and Corollary 6.5; then we prove Theorem 6.6 and Corollary 6.7 and finally we give the proof of Proposition 6.1.

6.2.1. Preliminaries

Let S be a connected Dedekind scheme, with field of fractions K. We begin to prove some properties of morphisms of abelian schemes over S that we shall use later on. Suppose that A and B are abelian schemes over S with generic fibres A_K and B_K, respectively. Then base change from S to K induces an isomorphism of abelian groups

$$\mathrm{Hom}(A, B) \cong \mathrm{Hom}(A_K, B_K). \tag{6.1}$$

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We now verify (6.1). Any abelian scheme over S is the Néron model of its generic fibre; see, for example, [4, p. 15]. Thus, the Néron mapping property gives that any K-scheme morphism $\varphi_K : A_K \to B_K$ extends to a unique S-scheme morphism $\varphi : A \to B$. In addition, if φ_K is a K-group scheme morphism, then φ is a S-group scheme morphism. Therefore, we see that base change from S to K induces a bijection of sets $\mathrm{Hom}(A, B) \cong \mathrm{Hom}(A_K, B_K)$. Finally, base change properties of group schemes show that this bijection is in fact a homomorphism of abelian groups and hence we conclude (6.1). Furthermore, base change from S to K induces an isomorphism of rings

$$\mathrm{End}(A) \cong \mathrm{End}(A_K). \tag{6.2}$$

Indeed, if $A = B$ then the isomorphism of abelian groups in (6.1) is an isomorphism of rings, because base change from S to K is a covariant functor from S-schemes to K-schemes.

We shall use the following property of semistable abelian varieties. Let A and B be abelian varieties over K and let v be a closed point of S. If there is a surjective morphism

$$A \to B \tag{6.3}$$

of abelian varieties over K and if A has semistable reduction at v, then B has semistable reduction at v. We now verify this statement. Let C be the reduced underlying scheme of the identity component of the kernel of $A \to B$. There exists an abelian variety B' over K that is K-isogenous to B and that fits into an exact sequence $0 \to C \to A \to B' \to 0$ of abelian varieties over K. Therefore, the semistability of A at v together with [4, p. 182] gives that B' has semistable reduction at v, and then [4, p. 180] shows that B is semistable at v because B and B' are K-isogenous. This proves the assertion in (6.3).

Next, we give Lemma 6.8, which will allow us later on to control the conductor of certain abelian varieties. In this lemma, we assume that K is a number field with ring of integers $\mathcal{O}_K$ and we assume that S is a nonempty open subscheme of $\mathrm{Spec}(\mathcal{O}_K)$. We write $d = [K : \mathbb{Q}]$ for the degree of K over $\mathbb{Q}$ and we define $N_S = \prod N_v$ with the product taken over all $v \in \mathrm{Spec}(\mathcal{O}_K) - S$. Let $g \geq 1$ be an integer and let $\rho = \rho(S, g)$ be the number of rational primes p such that $p \leq 2g + 1$ and such that there exists $v \in \mathrm{Spec}(\mathcal{O}_K) - S$ with $v \mid p$. If A is an abelian scheme over S, then we denote by N_A the conductor of A defined in Subsection 2.2. The following global result in Lemma 6.8 (i) uses inter alia the local conductor estimates of Brumer-Kramer [8] stated in (2.4).

Lemma 6.8. *Suppose that A is an abelian scheme over S of relative dimension g. Then the following statements hold:*

(i) *There exists a positive integer ν, depending only on K, S and g, such that $N_A \mid \nu$ and such that*

$$\nu \leq (2g + 1)^{6gd\rho} N_S^{2g}.$$

(ii) *If the generic fibre of A is semistable, then $N_A \mid N_S^g$.*

Proof. The generic fibre A_K of A has good reduction at all closed points of S, because A is an abelian scheme over S. Therefore, N_A takes the form

$$N_A = \prod N_v^{f_v}$$

with the product taken over all $v \in \mathrm{Spec}(\mathcal{O}_K) - S$, where $f_v = \varepsilon_v + \delta_v$ for ε_v and δ_v the tame and the wild conductor of A_K at v, respectively; see, for example, [61, Subsection 2.1].

We now prove (i). Let v be a closed point of $\mathrm{Spec}(\mathcal{O}_K)$. We see that $\varepsilon_v \leq \dim V_\ell(A) = 2g$ for $V_\ell(A)$ the rational ℓ-adic Tate module of A_K. If the residue characteristic p of v satisfies $p > 2g + 1$, then $\delta_v = 0$ and hence $f_v = \varepsilon_v \leq 2g$. We denote by b_v the right-hand side of the inequality of Brumer-Kramer stated in (2.4). This b_v is an integer that depends only on K, S, g and that satisfies $f_v \leq b_v$. Then we observe that N_A divides

$$\nu = N_S^{2g} \prod N_v^{(b_v - 2g)}$$

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with the product taken over all $v \in \mathrm{Spec}(\mathcal{O}_K) - S$ of residue characteristic at most $2g+1$. On using the definition of b_v via (2.4), we deduce an upper bound for b_v that then leads to an estimate for ν as claimed in (i). Before we include the details of the computation that we used here, we mention that one can in fact compute the precise value of ν for any explicitly given K, S and g. Let $p = p_v$ be the residue characteristic of v, write $\mathfrak{e}_v = v(p)$ for the ramification index of v and put $n = \lfloor \frac{2g}{p-1} \rfloor$. In our computation, we used

$$\tfrac{1}{\mathfrak{e}_v}(b_v - 2g) = pn + (p-1)\lambda_p(n) \le 4g + 2g\lfloor \tfrac{\log(2g)}{\log p} \rfloor,$$

which follows from $\lambda_p(n) \le n\lfloor \frac{\log n}{\log p} \rfloor$ and $\frac{p}{p-1} \le 2$. Further, any $v \in \mathrm{Spec}(\mathcal{O}_K) - S$ with $p_v \le 2g+1$ lies in the set T of all $v \in \mathrm{Spec}(\mathcal{O}_K)$ with $p_v \mid N_S$ and $p_v \le 2g+1$, and it holds that $\sum_{v\in T} \mathfrak{e}_v \mathfrak{f}_v = d\rho$ and $N_v = p^{\mathfrak{f}_v}$ where $\mathfrak{f}_v$ denotes the residue degree of v. Therefore, the above displayed results show that $\log(\nu/N_S^{2g}) \le 2g\sum_{v\in T} \mathfrak{e}_v \mathfrak{f}_v \log(p_v^2 2g)$ is at most $6gd\rho\log(2g+1)$, because any $v \in T$ satisfies $p_v \le 2g+1$.

To show (ii) we take again a closed point v of $\mathrm{Spec}(\mathcal{O}_K)$. Let $\mathcal{A}_v$ be the fibre at v of the Néron model of A_K over $\mathrm{Spec}(\mathcal{O}_K)$. The identity component $\mathcal{A}_v^0$ of $\mathcal{A}_v$ is an extension of an abelian variety C_v by the product of a torus part T_v with a unipotent part U_v. Let t_v, u_v and a_v be the dimensions of T_v, U_v and C_v, respectively. It holds that $\dim(\mathcal{A}_v^0) = \dim(\mathcal{A}_v) = g$ and then the dimension formula gives $g = (t_v + u_v) + a_v$. Further, it is known that $\varepsilon_v = t_v + 2u_v$; see, for example, [29, p. 364]. Our additional assumption in (ii), that A_K is semistable, implies that $u_v = 0$ and $\delta_v = 0$. Therefore, we deduce that $f_v = \varepsilon_v = t_v$, and this together with $t_v \le t_v + u_v + a_v = g$ leads to $f_v \le g$. Then the displayed formula for N_A shows that $N_A \mid N_S^g$, which proves (ii). This completes the proof of Lemma 6.8. □

We are now ready to prove the results of Section 6.

6.2.2. Proofs

Theorem 5.1 (i) and Lemma 6.8 directly imply Conjecture (ES) for abelian schemes of product GL_2-type. This proof via Theorem 5.1 (i) and Lemma 6.8 has an interesting feature discussed in the first remark after Subsection 1.2. To prove our other results, we continue the notation of the previous section. In addition, we assume that $K = \mathbb{Q}$ and that S is a nonempty open subscheme of $\mathrm{Spec}(\mathbb{Z})$. Let N_S and $g \ge 1$ be as above. Further, we assume that A is an abelian scheme over S of relative dimension g that is of product GL_2-type and we denote by $h_F(A)$ its stable Faltings height.

The principal ideas of the proof of Theorem 6.2 are as follows. Theorem 5.1 (ii) together with Lemma 6.8 implies directly Conjecture (ES) for A, with an inequality of the form $h_F(A) \le c(g)N_S^{24g}$ for $c(g)$ a constant depending only on g. However, to obtain the better bound $h_F(A) \le c(g)N_S^{24}$ and to improve the dependence on g of $c(g)$, we go into the proof of Theorem 5.1 and we apply therein Lemma 6.8 with the ‘simple factors’ of A.

Proof of Theorem 6.2. 1. By assumption, A is isogenous to a product of abelian schemes over S that are all of GL_2-type. Then (6.2) gives that the generic fibres of these abelian schemes are all of GL_2-type as well. It follows that the generic fibre $A_\mathbb{Q}$ of A is $\mathbb{Q}$-isogenous to a product of abelian varieties over $\mathbb{Q}$ of GL_2-type. In other words, the abelian variety $A_\mathbb{Q}$ is of product GL_2-type and thus satisfies all of the assumptions of Theorem 5.1.

2. We now go into the proof of Theorem 5.1. Therein we showed Theorem 5.1*, which provides positive integers N_i and e_i, together with $\mathbb{Q}$-simple abelian varieties A_i over $\mathbb{Q}$ of dimension g_i, such that $A_\mathbb{Q}$ is $\mathbb{Q}$-isogenous to $\prod A_i^{e_i}$ and such that

$$N_i^{g_i} = N_{A_i}. \tag{6.4}$$

Here N_{A_i} denotes the conductor of A_i. Furthermore, Theorem 5.1* gives

$$h_F(A) \le \sum e_i\big(18 \cdot 10^3 N_i^{12} + (8g)^6 \log N_i\big). \tag{6.5}$$

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3. Next, we estimate the numbers N_i in terms of g and S. There exists a surjective morphism $A_{\mathbb{Q}} \to A_i$ of abelian varieties over $\mathbb{Q}$, and the abelian variety $A_{\mathbb{Q}}$ has good reduction at all closed points of S because it extends to an abelian scheme over S. Therefore, [63, Corollary 2] provides that A_i has good reduction at all closed points of S, and this shows that the Néron model $\mathcal{A}_i$ of A_i over S is an abelian scheme over S. Then an application of Lemma 6.8 with the abelian scheme $\mathcal{A}_i$ over S of conductor N_{A_i} gives that $N_{A_i} \le (2g_i+1)^{6g_i\rho_i} N_S^{2g_i}$, where $\rho_i = \rho(S, g_i)$ denotes the number of rational primes $p \notin S$ with $p \le 2g_i + 1$. Further, because $A_{\mathbb{Q}}$ is $\mathbb{Q}$-isogenous to $\prod A_i^{e_i}$, we obtain

$$g = \sum e_i g_i. \tag{6.6}$$

It follows that $g_i \le g$, and this leads to $\rho_i \le \rho = \rho(S, g)$. Then the above upper bound for N_{A_i} together with (6.4) proves that $N_i \le (2g+1)^{6\rho} N_S^2$.

4. We observe that $\rho \le 2g$ and (6.6) implies that $\sum e_i \le g$. Thus, on combining (6.5) with the above estimate for N_i, we deduce an inequality as claimed by Theorem 6.2. To simplify the form of the final result, we may and do assume here that $g \ge 2$. Indeed, in the case $g = 1$, the above arguments simplify and they give a bound that is even better than Theorem 6.2 for $g = 1$; the reason is that any elliptic curve over $\mathbb{Q}$ of conductor N is a quotient of the modular Jacobian $J_0(N)$ by [7] and Lemma 4.1 is better for $h_F(J_0(N))$ than for $h_F(J_1(N))$. This completes the proof of Theorem 6.2. □

We recall that $\rho = \rho(S, g)$ denotes the number of rational primes $p \notin S$ with $p \le 2g + 1$. The proof of Theorem 6.2 also gives the following more precise result: If A is an abelian scheme over S of relative dimension g and if A is of product GL_2-type, then

$$h_F(A) \le g\big(18 \cdot 10^3 \nu_0^{12} + (8g)^6 \log \nu_0\big), \quad \nu_0 = (2g+1)^{6\rho} N_S^2. \tag{6.7}$$

Let s be the number of rational primes that are not in S. It follows that $\rho \le s < \infty$ and then we see that (6.7) is polynomial in terms of g, because s depends only on S. Furthermore, on looking, for example, at products of elliptic curves over S, we see that any upper bound for $h_F(A)$ has to be at least linear in terms of g. This shows that the polynomial dependence on g of (6.7) is not too far from the optimal dependence on g. On the other hand, Lemma 6.8 implies that Frey's height conjecture [22, p. 39] would give an upper bound for $h_F(A)$ that is linear in terms of $\log N_S$, whereas (6.7) depends polynomially on N_S. We remark that an (effective) estimate for $h_F(A)$ that is linear in terms of $\log N_S$ would imply some (effective) version of the abc-conjecture.

In the following proof of Proposition 6.3, we use the arguments of Theorem 6.2 and we replace therein Lemma 6.8 (i) by Lemma 6.8 (ii).

Proof of Proposition 6.3. We freely use the notations and definitions of the proof of Theorem 6.2. In addition, we assume that $A_{\mathbb{Q}}$ is semistable. Therefore, (6.3) implies that A_i is semistable, because there exists a surjective morphism $A_{\mathbb{Q}} \to A_i$ of abelian varieties over $\mathbb{Q}$. We showed that A_i extends to an abelian scheme $\mathcal{A}_i$ over S. Thus, an application of Lemma 6.8 (ii) with the abelian scheme $\mathcal{A}_i$ over S of conductor N_{A_i} and relative dimension g_i gives that $N_{A_i} \mid N_S^{g_i}$. Hence, the equality $N_i^{g_i} = N_{A_i}$ in (6.4) implies that $N_i \le N_S$ and then (6.6) together with the upper bound for $h_F(A)$ in (6.5) leads to an inequality as claimed. This completes the proof of Proposition 6.3. □

To prove Corollary 6.5 we combine Theorem 6.2 with the recent version of the Masser-Wüstholz results [46, 47] due to Gaudron-Rémond [25].

Proof of Corollary 6.5. We suppose that A and B are isogenous abelian schemes over S of relative dimension g. Let $A_{\mathbb{Q}}$ and $B_{\mathbb{Q}}$ be the generic fibres of A and B, respectively. By assumption, A or B is of product GL_2-type. Thus, both are of product GL_2-type.

To show (i) we observe that the constant $\kappa(A_{\mathbb{Q}})$ in [25] depends only on g and $h_F(A)$. Let κ be the constant that one obtains by replacing the number $h_F(A)$ with $(3g)^{144g} N_S^{24}$ in the definition of $\kappa(A_{\mathbb{Q}})$; notice that κ depends only on N_S and g. An application of Theorem 6.2 with A shows that $\kappa(A_{\mathbb{Q}}) \le \kappa$.

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The abelian varieties $A_{\mathbb{Q}}$ and $B_{\mathbb{Q}}$ are $\mathbb{Q}$-isogenous. Therefore, [25, Théorème 1.4] gives $\mathbb{Q}$-isogenies $\varphi_{\mathbb{Q}} : A_{\mathbb{Q}} \to B_{\mathbb{Q}}$ and $\psi_{\mathbb{Q}} : B_{\mathbb{Q}} \to A_{\mathbb{Q}}$ of degree at most $\kappa(A_{\mathbb{Q}}) \le \kappa$. As in the proof of (6.1) we see that $\varphi_{\mathbb{Q}}$ and $\psi_{\mathbb{Q}}$ extend to S-group scheme morphisms $\varphi : A \to B$ and $\psi : B \to A$, respectively. Furthermore, it follows from [4] that φ and ψ are isogenies because A and B are abelian (and thus semi-abelian) schemes over S. Hence, we conclude (i).

It remains to prove (ii). We showed in (i) that there is a $\mathbb{Q}$-isogeny $\varphi_{\mathbb{Q}} : A_{\mathbb{Q}} \to B_{\mathbb{Q}}$ with $\deg(\varphi_{\mathbb{Q}}) \le \kappa$, and (3.1) gives that $|h(A_{\mathbb{Q}}) - h(B_{\mathbb{Q}})| \le \frac{1}{2}\log\deg(\varphi_{\mathbb{Q}})$ for h the relative Faltings height. Hence, we deduce a version of (ii) involving h. To prove the version involving h_F we use [29]. It provides a number field L such that A_L and B_L are semistable, and thus $h_F(A) = h(A_L)$ and $h_F(B) = h(B_L)$. If $\varphi_L : A_L \to B_L$ is the base change of $\varphi_{\mathbb{Q}}$, then (3.1) gives that $|h(A_L) - h(B_L)| \le \frac{1}{2}\log\deg(\varphi_L)$. Therefore, $\deg(\varphi_L) = \deg(\varphi_{\mathbb{Q}}) \le \kappa$ leads to (ii). This completes the proof of Corollary 6.5. □

We refer to the introduction for an outline of the strategy used in the following proof.

Proof of Theorem 6.6. We recall that $M_{\mathrm{GL}_2,g}(S)$ denotes the set of isomorphism classes of abelian schemes over S of relative dimension g that are of product GL_2-type. To bound $|M_{\mathrm{GL}_2,g}(S)|$ we may and do assume that the set $M_{\mathrm{GL}_2,g}(S)$ is not empty.

1. We denote by $M_{\mathrm{GL}_2,g}(S)_{\mathbb{Q}}$ the set of $\mathbb{Q}$-isomorphism classes of abelian varieties over $\mathbb{Q}$ of dimension g that extend to an abelian scheme over S and that are of product GL_2-type. Base change from S to $\mathbb{Q}$ induces a canonical bijection

$$M_{\mathrm{GL}_2,g}(S) \cong M_{\mathrm{GL}_2,g}(S)_{\mathbb{Q}}.$$

To verify this statement we observe that $M_{\mathrm{GL}_2,g}(S)$ coincides with the set of S-scheme isomorphism classes generated by abelian schemes over S of relative dimension g that are of product GL_2-type. Further, it follows from (6.2) that the generic fibre $A_{\mathbb{Q}}$ of any $[A] \in M_{\mathrm{GL}_2,g}(S)$ is of product GL_2-type. Thus, base change from S to $\mathbb{Q}$ induces a map $M_{\mathrm{GL}_2,g}(S) \to M_{\mathrm{GL}_2,g}(S)_{\mathbb{Q}}$, which is surjective by (6.2) and [4, p. 180]. The abelian scheme A is the Néron model of $A_{\mathbb{Q}}$ over S and then the Néron mapping property shows that $M_{\mathrm{GL}_2,g}(S) \to M_{\mathrm{GL}_2,g}(S)_{\mathbb{Q}}$ is injective. We conclude that $M_{\mathrm{GL}_2,g}(S) \cong M_{\mathrm{GL}_2,g}(S)_{\mathbb{Q}}$.

2. Next, we estimate the number of distinct $\mathbb{Q}$-isogeny classes of abelian varieties over $\mathbb{Q}$ generated by $M_{\mathrm{GL}_2,g}(S)_{\mathbb{Q}}$. Let $[A] \in M_{\mathrm{GL}_2,g}(S)_{\mathbb{Q}}$. Theorem 5.1* gives positive integers N_i and e_i, together with $\mathbb{Q}$-simple abelian varieties A_i over $\mathbb{Q}$ of dimension g_i and of conductor $N_{A_i} = N_i^{g_i}$, such that A is $\mathbb{Q}$-isogenous to $\prod A_i^{e_i}$ and such that A_i is an abelian subvariety of $J_1(N_i)$. Here $J_1(N)$ denotes the usual modular Jacobian of level $N \in \mathbb{Z}_{\ge 1}$ defined in Section 4. The abelian variety A extends to an abelian scheme over S, because $[A] \in M_{\mathrm{GL}_2,g}(S)_{\mathbb{Q}}$. Thus, each A_i extends to an abelian scheme over S and then the arguments of the proof of Lemma 6.8 together with $g_i \le g$ lead to $N_{A_i} \mid \nu^{g_i}$ for

$$\nu = N_S^2 \prod p^{c_p}, \quad c_p = 4 + 2\lfloor \log(2g)/\log p \rfloor.$$

Here the product is taken over all rational primes $p \notin S$ with $p \le 2g + 1$, and for any real number x we recall that $\lfloor x \rfloor$ denotes the largest integer at most x. We warn the reader that the displayed number ν is related to the number appearing in Lemma 6.8 (i), but these numbers are not necessarily the same. It follows that $N_i \mid \nu$ because $N_i^{g_i} = N_{A_i}$, and this implies that $J_1(N_i)$ is a $\mathbb{Q}$-quotient of $J_1(\nu)$. On using that A_i is an abelian subvariety of $J_1(N_i)$, we then see that there exists a surjective morphism of abelian varieties over $\mathbb{Q}$

$$J_1(\nu) \to A_i.$$

Hence, Poincare's reducibility theorem shows that each A_i is $\mathbb{Q}$-isogenous to a $\mathbb{Q}$-simple 'factor' of $J_1(\nu)$. Furthermore, the dimension of $J_1(\nu)$ coincides with the genus g_ν of the modular curve $X_1(\nu) = X(\Gamma_1(\nu))$ defined in Section 4, and the abelian variety $J_1(\nu)$ (respectively A) has at most g_ν (respectively g) $\mathbb{Q}$-simple factors up to $\mathbb{Q}$-isogenies. Therefore, there exists a set of at most $g \cdot g_\nu^g$ distinct abelian varieties over $\mathbb{Q}$ such that any $[A] \in M_{\mathrm{GL}_2,g}(S)_{\mathbb{Q}}$ is $\mathbb{Q}$-isogenous to some abelian variety in this set. In other

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words, the abelian varieties in $M_{\mathrm{GL}_2,g}(S)_{\mathbb{Q}}$ generate at most $g\cdot g_\nu^g$ distinct $\mathbb{Q}$-isogeny classes of abelian varieties over $\mathbb{Q}$.

As pointed out by the referee, one can remove here the factor g by using in addition the following combinatorial counting result: For any given integers $l,m,c_1,\ldots,c_m$ in $\mathbb{Z}_{\geq 1}$ and $n=l+m-1$, there exist at most $\binom{n}{l}\leq m^l$ distinct $e\in\mathbb{Z}^m_{\geq 0}$ with $\sum e_ic_i=l$. This is then applied with $l=g$, m the number of $\mathbb{Q}$-simple factors C_i of $J_1(\nu)$ up to $\mathbb{Q}$-isogenies and $c_i=\dim(C_i)$. The combinatorial counting result can be verified as follows. We denote by Σ the set of all $e\in\mathbb{Z}^m_{\geq 0}$ with $\sum e_ic_i=l$. For any $e\in\Sigma$, we define $\iota(e)\in\{0,1\}^n$ as follows: The first e_1c_1-entries of $\iota(e)$ are 0 followed by an entry that is 1; then the next e_2c_2-entries are again 0 followed by an entry that is 1 and so on until all n entries of $\iota(e)$ are defined. Then $e\mapsto\iota(e)$ defines an injective map from Σ into the set W of all $w\in\{0,1\}^n$ such that w has precisely l distinct entries that are 0. Therefore, we conclude that $|\Sigma|\leq|W|=\binom{n}{l}$ as desired.

3. To bound the size of each $\mathbb{Q}$-isogeny class we take an arbitrary $[A]\in M_{\mathrm{GL}_2,g}(S)_{\mathbb{Q}}$. We denote by $\mathcal{C}$ the set of $\mathbb{Q}$-isomorphism classes of abelian varieties over $\mathbb{Q}$ that are $\mathbb{Q}$-isogenous to A. Let κ be the constant that appears in the proof of Corollary 6.5. If $[B]\in\mathcal{C}$, then the proof of Corollary 6.5 provides a $\mathbb{Q}$-isogeny $\varphi: A\to B$ of degree at most κ. Furthermore, the quotient of A by the kernel of φ is an abelian variety over $\mathbb{Q}$ that is $\mathbb{Q}$-isomorphic to B. On combining the above observations, we see that $|\mathcal{C}|$ is bounded from above by the number of subgroups of A^t of order at most κ, where A^t is the group of geometric torsion points of A. It holds that $A^t\cong(\mathbb{Q}/\mathbb{Z})^{2g}$, and [46, Lemma 6.1] gives that $(\mathbb{Q}/\mathbb{Z})^{2g}$ has at most κ^{2g} subgroups of order at most κ. This implies that $|\mathcal{C}|\leq\kappa^{2g}$.

4. The results obtained in 1–3 imply that $|M_{\mathrm{GL}_2,g}(S)|\leq g(g_\nu\kappa^2)^g$, and (4.1) together with [16, p. 107] proves that $g_\nu\leq\frac{1}{24}\nu^2$. Therefore, the definitions of κ and ν lead to an upper bound for $|M_{\mathrm{GL}_2,g}(S)|$ as claimed in Theorem 6.6. □

The arguments used in the proof of Theorem 6.6 also give Corollary 6.7.

Proof of Corollary 6.7. We observe that part 3 of the proof of Theorem 6.6 implies (i), and we notice that (ii) follows from part 2 of the proof of Theorem 6.6. □

It remains to prove Proposition 6.1. In the first part of the proof we show that Conjecture (ES) implies Conjecture $(ES)^*$, and in the second part we use the effective version of the Kodaira construction due to Rémond [55].

Proof of Proposition 6.1. We recall some notation. Let K be a number field of degree $d=[K:\mathbb{Q}]$, with ring of integers $\mathcal{O}_K$. We denote by D_K the absolute value of the discriminant of K over $\mathbb{Q}$. Let h_F be the stable Faltings height and let T be a finite set of places of K. We write $N_T=\prod N_v$ with the product taken over all finite places $v\in T$. Let X be a smooth, projective and geometrically connected curve over K of genus $g\geq 1$.

1. To prove that Conjecture (ES) implies $(ES)^*$ we assume that Conjecture (ES) holds. In addition, we suppose that the Jacobian $J_K=\mathrm{Pic}^0(X)$ of X has good reduction outside T. The Weil restriction $A_{\mathbb{Q}}=\mathrm{Res}_{K/\mathbb{Q}}(J_K)$ of J_K is an abelian variety over $\mathbb{Q}$ of dimension $n=dg$, which is geometrically isomorphic to $\prod J_K^\sigma$. Here the product is taken over all embeddings σ from K into an algebraic closure of K, and J_K^σ is the base change of J_K with respect to σ. The Galois invariance $h_F(J_K)=h_F(J_K^\sigma)$ implies that $h_F(A_{\mathbb{Q}})=dh_F(J_K)$. Let S be the open subscheme of $\mathrm{Spec}(\mathbb{Z})$ formed by the generic point together with the closed points where $A_{\mathbb{Q}}$ has good reduction. The Néron model A of $A_{\mathbb{Q}}$ over S is an abelian scheme. Therefore, an application of Conjecture (ES) with A, S and n gives an effective constant c, depending only on S and n, such that

$$dh_F(J_K)=h_F(A_{\mathbb{Q}})\leq c. \tag{6.8}$$

We write $\mathcal{D}=\{D_K,d,g,N_T\}$ and we now construct an effective constant c', depending only on $\mathcal{D}$, such that $c\leq c'$. The finite places in T form a closed subset of $\mathrm{Spec}(\mathcal{O}_K)$, whose complement S' has the structure of an open subscheme of $\mathrm{Spec}(\mathcal{O}_K)$. The Néron model J of J_K over S' is an abelian scheme, because J_K has good reduction outside T. We denote by N_J and N_A the conductors of J_K and

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$A_{\mathbb{Q}}$, respectively. A result of Milne [49, Proposition 1] gives that $N_A = N_J D_K^{2g}$, and an application of Lemma 6.8 (i) with the abelian scheme J over S' of relative dimension g implies that $N_J \leq \Omega D_K^{-2g}$ for $\Omega = (3g)^{12g^2 d}(N_T D_K)^{2g}$. We deduce that $N_A \leq \Omega$, and this leads to

$$N_S \leq \Omega,$$

because N_S divides N_A by the construction of S. Here for any open subscheme U of $\mathrm{Spec}(\mathbb{Z})$ we write $N_U = \prod p$ with the product taken over all rational primes $p \notin U$. It follows that $S \in \mathcal{U}$ for $\mathcal{U}$ the set of open subschemes U of $\mathrm{Spec}(\mathbb{Z})$ with $N_U \leq \Omega$. An application of Conjecture (ES) with $U \in \mathcal{U}$ and n gives an effective constant $c_U \geq 1$, depending only on U and n. We define $c' = \max c_U$ with the maximum taken over all $U \in \mathcal{U}$. If $\mathcal{D}$ is given, then the set $\mathcal{U}$ can be determined effectively. Thus, we see that c' is an effective constant, depending only on $\mathcal{D}$. On using that $S \in \mathcal{U}$, we obtain that $c \leq c'$ and then (6.8) gives

$$h_F(J_K) \leq c'.$$

In other words, we proved that Conjecture (ES) would give an effective constant c', depending only on $\mathcal{D}$, with the following property: If J_K has good reduction outside T, then $h_F(J_K) \leq c'$. Further, if X has good reduction at a finite place v of K, then J_K has good reduction at v. Therefore, we conclude that Conjecture (ES) implies $(ES)^*$.

2. It follows from part 1 that Conjecture (ES) implies $(ES)^*$. Furthermore, [55] gives that Conjecture $(ES)^*$ implies that the set of rational points of X can be determined effectively if $g \geq 2$. This completes the proof of Proposition 6.1. □

We remark that the above proof of Proposition 6.1 assumes the validity of Conjecture (ES) in quite general situations. In particular, it is a priori not possible to use the above arguments in order to deduce special cases of the effective Mordell conjecture from special cases of Conjecture (ES) such as, for example, Theorem 6.2. To 'transfer' special cases between these conjectures, an effective version of Paršin's construction [51] would be more useful than Kodaira's construction, which is used in the proof of Proposition 6.1.

We mention that the implication $(ES)^* \Rightarrow (ES)$ remains an interesting open problem, which is nontrivial because (ES) is a priori considerably stronger than $(ES)^*$. To discuss parts of the additional information contained in Conjecture (ES), we consider an arbitrary hyperelliptic curve X of genus $g \geq 2$ over a number field K. Let T be the set of finite places of K where $\mathrm{Pic}^0(X)$ has bad reduction. Suppose that v is a finite place of K where X has bad reduction but $\mathrm{Pic}^0(X)$ has good reduction; the minimal regular model of X over $\mathrm{Spec}(\mathcal{O}_v)$ is then automatically semistable for $\mathcal{O}_v$ the local ring at v. Then on combining the arguments of [39, Proposition 5.1 (i)] with part 1 of the proof of Proposition 6.1, we see that already very special cases of Conjecture (ES) would give an effective estimate for N_v in terms of K, g and T. We note that Levin [44] proved that such an effective estimate for N_v would solve the following classical problem: Give an effective version of Siegel's theorem for arbitrary hyperelliptic curves of genus $g \geq 2$ defined over a number field K. In fact, the latter problem is already open for $g = 2$ and $K = \mathbb{Q}$.

Appendix A: A height-conductor inequality for semistable abelian varieties over $\mathbb{Q}$ with real multiplications

This appendix generalises the asymptotic bound [36, Prop 8.2] for semistable elliptic curves to semistable abelian varieties with real multiplications. In particular, for such abelian varieties we show that one can refine the proof of Theorem 5.1 (i) or (ii) in order to asymptotically improve the bound in Theorem 5.1 (ii).

Continue the notation of Sections 4 and 5. Let A be an abelian variety over $\mathbb{Q}$ of dimension $g \geq 1$. We say that A is with real multiplications if $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ has a $\mathbb{Q}$-subalgebra of dimension g that is the product of totally real number fields. This generalises the definition of Ribet [57] and Serre [62], who

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studied the case when A is $\mathbb{Q}$-simple. If A is with real multiplications, then (5.1) gives that A is of product GL_2-type. Write $\mathrm{rad}(m)$ for the radical of $m \in \mathbb{Z}_{\geq 1}$ and define $f(x) = x\log(x)\log\log x$ for $x \in \mathbb{R}_{>1}$.

Proposition 6.9. *Suppose that A is of product* GL_2*-type. Assume, moreover, that A is with real multiplications and that A is semistable. Then the following statements hold:*

(i) *Let* $\gamma = 0.5772\ldots$ *be Euler's constant and put* $n = \mathrm{rad}(N_A)$*. If* $n \to \infty$*, then*

$$h_F(A) \leq \frac{e^{\gamma}}{u\pi^2} g f(n) + o_g(f(n)). \tag{6.9}$$

Here $u = 6$ *if* $\gcd(n, 6) = 1$ *and* $u = 4$ *otherwise. Further,* $o_g(f(n)) = c(g)o(f(n))$ *where* $c(g)$ *is a constant depending only g.*

(ii) *It holds that* $h_F(A) \ll_\epsilon N_A^{1+\epsilon}$.

The bound in (ii) and the estimate (6.9), which we obtained in a more precise form in (6.10), both asymptotically improve the bound in Theorem 5.1 (ii). On the other hand, Theorem 5.1 (ii) makes no additional semistable or real multiplication assumptions, and this is crucial for many Diophantine applications (e.g., [40]). In view of this, it would be interesting to remove in Proposition 6.9 the semistable and real multiplication assumptions. Unfortunately, in the proof we use both assumptions in order to work with $X_0(N) = X(\Gamma_0(N))$ with N square-free: This allows us to apply results of Ullmo [67] and Jorgenson-Kramer [33], whose proofs crucially exploit that the modular curve is $X_0(N)$ with N square-free. To remove any of these two assumptions, one has to overcome various technical (and also conceptual) difficulties; see, for example, the discussions in Coleman-Edixhoven [12, Sect 5], the proof of Edixhoven-de Jong [17] dealing with the case $A = J_1(pq)$ for p, q distinct rational primes or Mayer's result [48] on $h_F(J_1(N))$ for some square-free N.

To prove Proposition 6.9 (i), we refine the proof of Theorem 5.1 (i) for semistable abelian varieties with real multiplications by applying Lemma 3.1 (ii) and the bounds for $h_F(J_0(N))$ in (4.3) and (4.4) obtained by Ullmo [67] and Jorgenson-Kramer [33], respectively. Here one can replace Lemma 3.1 (i) by (3.3). This has the advantage that one obtains a constant $c(g)$ in $o_g(f(n))$ that is polynomial in terms of g and thus allows deducing Proposition 6.9 (ii). We shall see below that Lemma 3.1 (ii) based on 'essentially algegraic' methods is crucial for obtaining the bounds in Proposition 6.9 (i) and (ii), and in view of (6.11) the result (3.3) based on the theory of logarithmic forms is crucial for obtaining the bound in Proposition 6.9 (ii).

Proof of Proposition 6.9. As in the statement of the proposition, we assume that A is with real multiplications and that A is semistable. In [36, Prop 8.2] we proved Proposition 6.9 in the case $g = 1$. However, we now show that the proof works for any $g \geq 2$ when combined with our arguments of Theorem 5.1. The details are as follows.

We first prove (i). Our assumption that A is with real multiplications assures that $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ contains a $\mathbb{Q}$-subalgebra $E = \prod F_i$ with $[E : \mathbb{Q}] = g$, where each F_i is a totally real number field. On using the idempotents of E, we obtain abelian varieties B_i over $\mathbb{Q}$ such that F_i is contained in $\mathrm{End}(B_i) \otimes_{\mathbb{Z}} \mathbb{Q}$ and such that A is $\mathbb{Q}$-isogenous to the product $\prod B_i$. Then, as in the proof of (5.1), we see that a Lie algebra argument together with $[E : \mathbb{Q}] = g$ implies that $[F_i : \mathbb{Q}] = \dim(B_i)$ and hence each B_i is of GL_2-type. Then (5.3) gives an integer e_i together with a $\mathbb{Q}$-simple abelian variety A_i over $\mathbb{Q}$ of GL_2-type such that B_i is $\mathbb{Q}$-isogenous to $A_i^{e_i}$. Moreover, Ribet's arguments in the proof of [59, Thm 2.1] show that $\mathrm{End}(A_i) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a number field of degree $\dim(A_i)$ that is a subfield of F_i and hence is totally real. We conclude that each abelian variety A_i is with real multiplications in the sense of Serre [62, Subsect 4.7]. Hence, Serre's modularity conjecture [43] combined with [62, Thm 5] provides a positive integer N_i with $N_i^{g_i} = N_{A_i}$ together with a surjective morphism $J_0(N_i) \to A_i$ of abelian varieties over $\mathbb{Q}$, where $g_i = \dim(A_i)$. On repeating word by word the (general) arguments given below (5.5), we deduce the following: After possibly replacing A_i by a $\mathbb{Q}$-isogenous abelian variety over $\mathbb{Q}$, we may and do assume that A_i is an abelian subvariety of $J_i = J_0(N_i)$. To estimate the heights, we recall that $\nu_{A'}$

denotes the maximal variation of the stable Faltings height h_F in the isogeny class of the abelian variety $A' = \prod A_i^{e_i}$ over $\mathbb{Q}$. It holds that

$$h_F(A') = \sum e_i h_F(A_i) \quad \text{and} \quad h_F(A) \le v_{A'} + h_F(A')$$

because A is an abelian variety over $\mathbb{Q}$ that is $\mathbb{Q}$-isogenous to A'. Our assumption that A is semistable implies that the $\mathbb{Q}$-isogenous A' is semistable as well. In particular, the factors A_i of A' are all semistable and thus the level N_i is square-free. Indeed, the square-free claim follows, for example, from Lemma 6.8 (ii), which gives that the conductor $N_{A_i} = N_i^{g_i}$ of the semistable abelian variety A_i divides $(\prod p)^{g_i}$ with the product taken over all bad reduction primes of A_i. Now, because N_i is square-free, we obtain that J_i is semistable and then an application of Lemma 3.1 (ii) with the abelian subvariety A_i of J_i gives

$$h_F(A_i) \le h_F(J_i) + \tfrac{n_i}{2} \log(8\pi^2),$$

where $n_i = \dim(J_i)$. To control the variation $v_{A'}$, we use again our assumption that A is semistable and we exploit that the $\mathbb{Q}$-isogenous A' has good reduction outside the set T of rational primes dividing $N_A = N_{A'}$. Then, on applying the explicit bound (3.2) for $|h_F(A) - h_F(A')|$ of Raynaud [54, Thm 4.4.9] with A', T and on using the prime number theorem, we deduce that

$$v_{A'} \le \kappa n,$$

where κ is a constant depending only on g. The genus of the modular curve $X_0(N_i)$ coincides with the dimension n_i of its Jacobian $J_i = J_0(N_i)$. Therefore, we obtain that $n_i \le \frac{e^\gamma}{2\pi^2} N_i \log\log(N_i) + o(N_i)$; see, for example, the proof of [36, Prop 8.2]. Then, on applying Ullmo's asymptotic bound (4.3) or the asymptotic formula (4.4) of Jorgenson-Kramer when 6 is coprime to n, we see that the above displayed results lead to

$$h_F(A) \le \kappa n + \tfrac{e^\gamma}{u\pi^2} \sum e_i f(N_i) + o(f(N_i)) \le \tfrac{e^\gamma}{u\pi^2} \sum e_i f(N_{A_i}^{1/g_i}) + o_g(f(n)). \tag{6.10}$$

Here we used that $N_A = N_{A'} = \prod N_{A_i}^{e_i}$ and $N_{A_i} = N_i^{g_i}$, which imply $N_A = \prod N_i^{e_i g_i}$ and the square-free number N_i divides $n = \mathrm{rad}(N_A)$. Then (6.10) proves (i).

We now show (ii). At the end of the proof of Theorem 5.1 (ii), we obtained in particular that $g \le \log N_A$. Thus, the bound (ii) follows by replacing in the proof of (i) the estimate $v_{A'} \le \kappa n$ by the estimate $v_{A'} \le (8g)^6 \log N_A$, which is a consequence of d) obtained in course of the proof of Theorem 5.1 (ii). This completes the proof of Proposition 6.9. ◻

Let A be as in Proposition 6.9. Depending on the decomposition (in the $\mathbb{Q}$-isogeny category) of A into $\mathbb{Q}$-simple abelian varieties $A \sim \prod A_i^{e_i}$, one can remove the factor g in the main term of (6.9). For example, if all $e_i = 1$ and if the $\mathbb{Q}$-simple factors A_i of A have pairwise coprime conductor, then (6.10) implies that (6.9) holds with $gf(n)$ replaced by $f(n)$. In general, one can always replace in (6.9) the quantity $gf(n)$ by $n\log(N_A)\log\log n$. Indeed, this follows directly from (6.10), because $\prod N_i^{e_i}$ divides $N_A = \prod N_i^{e_i g_i}$.

Appendix B: A simplified version of Raynaud's bound for the variation of the Faltings height under isogenies

In this appendix we simplify the shape of Raynaud's bound [54, Thm 4.4.9], which is based on Faltings' method [21] and refinements of Paršin and Zarhin.

We continue the notation of Section 3 and we denote by h_F the stable Faltings height. Let K be a number field of degree d and discriminant D_K. Further, let A be an abelian variety defined over K of dimension $g \ge 1$, and let ℓ, ℓ' be two distinct rational primes that both do not divide the conductor N_A. Now, we can state the following.

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Proposition 6.10 ([54]). *Suppose that A is semistable. Then any abelian variety A' defined over K, which is K-isogenous to A, satisfies*

$$|h_F(A) - h_F(A')| \le 8g^3 t(\ell^{2m} + m\log\ell' + 4gd^2\log D_K), \quad m = dg\binom{2g}{g}^d. \tag{6.11}$$

Here $t = \max(1, |T|)$ for T the set of finite places of K where A has bad reduction.

We point out that one cannot replace t by $|T|$ in (6.11), because $|h_F(A) - h_F(A')| = 0$ is wrong for certain A/K with $|T| = 0$ as explained in (6.12). In particular, this shows that there is a minor inaccuracy in the explicit shape of Raynaud's original bound in [54, Thm 4.4.9], which we shall correct in Lemma 6.12.

B.1 Proof of Proposition 6.10

We continue our notation and we take A, A' as in Proposition 6.10. To prove the bound (6.11), we go into Raynaud's work [54] using his terminology. In particular, in what follows the numbering $x.y.z$ refers to $x.y.z$ in [54], isogeny means K-isogeny, $\mathcal{O}$ denotes the ring of integers of K and $\bar{F}$ denotes an algebraic closure of a field F. Further, T denotes the set of finite places of K where A has bad reduction and $g = \dim(A)$.

Let p be a rational prime. We recall that the notion of a belle p-isogeny was introduced in Définition 4.1.1 and that the nonnegative rational integers Δ_p are defined in 4.4.7. The following more precise version of Corollaire 4.4.8 will be used below.

Lemma 6.11 ([54]). *Any p-isogeny $A \to A'$ factors into a product*

$$\varphi_1 \cdots \varphi_k, \quad k \le \begin{cases} 2g & \text{if } |T| = 0, \\ 4g^2|T| & \text{if } |T| \ge 1 \end{cases}$$

of isogenies φ_i such that $\deg(\varphi_i) \le p^{4g\Delta_p}$ or such that φ_i factors into the product of a belle p-isogeny and two isogenies that both have degree at most $p^{2g\Delta_p}$.

Proof. Lemme 4.4.3 factors any p-isogeny into a product $\varphi_1 \cdots \varphi_r$ of isogenies φ_i whose extensions φ_i^0 satisfy condition 1) $M_i(\bar{K}) \cong (\mathbb{Z}/p^{n_i}\mathbb{Z})^{h_i}$ of 4.4.7, where $r \le 2g$ and M_i is the kernel of the extension φ_i^0 of φ_i to the open subschemes of the Néron models with connected fibres. In the case $|T| \ge 1$, Lemme 4.4.4 shows that after possibly further factoring each φ_i into a product of at most $2h_i|T| \le 2g|T|$ isogenies, we may and do assume that each φ_i^0 satisfies not only condition 1) but also condition 2) of 4.4.7: If $v \in T$, then $(\hat{M}_i)_v(\bar{K}_v)$ is a direct factor of $M_i(\bar{K}_v)$ where K_v is the completion of K at v with ring of integers $\mathcal{O}_v$ and $(\hat{M}_i)_v$ is the finite part [54, (1.2)] of $M_i \otimes_{\mathcal{O}} \mathcal{O}_v$. In the case $|T| = 0$, condition 2) of 4.4.7 is automatically satisfied because T is empty.

We now show that each isogeny φ_i has the desired properties. In the case when $n_i < 2\Delta_p$, we obtain that $\deg(\varphi_i) < p^{4g\Delta_p}$ because $h_i \le 2g$. Suppose now that $n_i \ge 2\Delta_p$. Then we may apply the arguments of 4.4.7 because φ_i^0 satisfies conditions 1) and 2). These arguments factor φ_i into three isogenies: The middle isogeny is a belle p-isogeny and the extensions of the other two isogenies both satisfy condition 1) $M(\bar{K}) \cong (\mathbb{Z}/p^n\mathbb{Z})^{h_i}$ of 4.4.7 with $n = \Delta_p = n_i - (n_i - \Delta_p)$. Therefore, each of these other two isogenies has degree at most $p^{2g\Delta_p}$ and then we conclude Lemma 6.11. □

We take two distinct rational primes ℓ and ℓ' that both do not divide the conductor N_A. Then A has good reduction at each finite place of K dividing ℓ or ℓ'. Recall that for each rational prime $p \ne \ell$ the nonnegative integer $n_p = n_p(\ell)$ is defined in [54, p. 227] with respect to ℓ and put $n_{\ell,\ell'} = n_\ell(\ell')$. Now, we define

$$a = \sum 2\Delta_p \log p, \quad b = \sum n_p \log p, \quad c = (2\Delta_\ell + n_{\ell,\ell'})\log\ell$$

with both sums taken over all $p \ne \ell$ in $S \cup R \cup \{2\}$, where S is the set of rational primes $p \ne \ell$ with $n_p \ge 1$ and R is the set of rational primes that ramify in K. The following result, involving $t = \max(1, |T|)$,

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gives a more precise version (and corrects minor inaccuracies in the explicit shape) of the bound for the variation in Théorème 4.4.9.

Lemma 6.12 ([54]). *Define* $\alpha = 4g^3t(a+b+c)$. *Then it holds that*

$$|h_F(A) - h_F(A')| \le \begin{cases} \alpha/2g & \text{if } |T| = 0, \\ \alpha & \text{if } |T| \ge 1. \end{cases}$$

Proof. We begin by noting that Raynaud works in [54] with a different normalisation of the metric involved in the definition of the stable Faltings height h_F. However, the logarithm of the normalisation factor cancels out in the difference $h_F(A) - h_F(A')$.

In the case $|T| = 0$, the explicit shape of the bound in Théorème 4.4.9 is not correct; the mistake is not serious and lies in the application of Corollaire 4.4.8, which is done in the paragraph of the proof of Théorème 4.4.9 starting with ‘D’après 4.4.7’. However, on replacing this paragraph by the version of Corollaire 4.4.8 given in Lemma 6.11, Raynaud’s arguments give the bound $\alpha/2g$ for $|T| = 0$ as claimed in Lemma 6.12.

In the case $|T| \ge 1$, the arguments of Théorème 4.4.9 give the bound α with the following two caveats. Firstly, as far as we can see, a factor 2 was forgotten in the last step of the proof where one puts everything together; this factor 2 is now taken into account in our α. Secondly, we were not able to verify the proof of Corollaire 4.4.8 nor how 4.4.7 is applied in the proof of Théorème 4.4.9; the problem is that the arguments in 4.4.7 can only be applied to p-isogenies whose extensions (to the connected Néron models) satisfy conditions 1) and 2) such that the integer n in 1) is at least $2\Delta_p$. However, on using instead the version of Corollaire 4.4.8 given in Lemma 6.11, the arguments of Théorème 4.4.9 go through and give the bound α for $|T| \ge 1$ as claimed in Lemma 6.12. □

We are now ready to prove the simplified bound in Proposition 6.10.

Proof of Proposition 6.10. Lemma 6.12 gives that $|h_F(A) - h_F(A')| \le 4g^3t(a+b+c)$ and we now explicitly estimate from above the three quantities a, b and c.

We first bound a. Let p be a rational prime and let v be a finite place of K with residue characteristic p. We denote by e_v the ramification index of v. It holds that $v(n) = e_v \mathrm{ord}_p(n)$ for each $n \in \mathbb{Z}$ and the different ideal $\mathfrak{D}_{\mathcal{O}/\mathbb{Z}}$ satisfies $v(\mathfrak{D}_{\mathcal{O}/\mathbb{Z}}) < e_v + v(e_v)$ by Dedekind’s different theorem [3, B.2.11]. Therefore, the number δ_v appearing in the definition of Δ_p in 4.4.7 is bounded by $\delta_v \le \frac{1}{p-1} + 1 + \mathrm{ord}_p(e_v)$, and hence we obtain that $\delta_v \le 1 + e_v \le 2d$ because $\mathrm{ord}_p(e_v) \le e_v - 1$ and $e_v \le d$ where $d = [K : \mathbb{Q}]$. This leads to

$$\Delta_p \le \begin{cases} 4gd^2 & \text{if } p \in R \cup \{2\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then, on using that the set R is the set of rational prime divisors of the discriminant D_K of K by Dedekind’s discriminant theorem [3, B.2.12], we obtain

$$a + 2\Delta_\ell \log \ell = \sum_{p \in R \cup \{2\}} 2\Delta_p \log p \le 8gd^2 \log D_K + \tfrac{1}{2}\ell^{2m}, \quad m = dg\binom{2g}{g}^d.$$

Here we used that $8gd^2 \log 2 \le \frac{1}{2}\ell^{2m}$, which follows from $16gd^2 = 2^r \le \ell^r \le \ell^{2m}$, where $r = 4 + \log(gd^2)/\log 2$ is bounded by $r \le 2gd(2^g)^d \le 2m$.

Next, we bound b and $c - 2\Delta_\ell \log \ell$. We take again a rational prime p. An application of Corollaire 4.3.5 with n_p for $p \in S$ and $n_{\ell,\ell'} = n_\ell(\ell')$ shows for $p \ne \ell$ that

$$n_p \log p \le \begin{cases} 2m \log \ell & \text{if } p \in S, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad c - 2\Delta_\ell \log \ell = n_{\ell,\ell'} \log \ell \le 2m \log \ell'.$$

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Here $m/gd = \binom{2g}{g}^d \geq \binom{2g}{g}^n$ for $n = \max(N, N')$ with N and N' the number of finite places of K of residue characteristic ℓ and ℓ', respectively, and $n_p = 0$ for $p \notin S \cup \{\ell\}$ because S is by definition the set of all primes $p \neq \ell$ with $n_p \geq 1$. Now, we deduce

$$b = \sum_{p \in S} n_p \log p \leq |S| 2m \log \ell \leq \tfrac{3}{2}\ell^{2m}$$

because $|S| \leq \frac{3}{2}\ell^{2m}/(2m \log \ell)$. This bound for $|S|$ can be obtained as follows: Corollaire 4.3.5 gives that any $p \in S$ satisfies $p \leq p^{n_p} \leq \ell^{2m}$ because $n_p \geq 1$ for $p \in S$, and therefore $|S|$ is bounded by the number $\pi(x)$ of rational primes at most $x = \ell^{2m}$ that satisfies $\pi(x) \leq \frac{3}{2}x/\log x$ by the explicit prime number theorem in [60, Cor 1].

Finally, on combining the above displayed bounds, we deduce that $a + b + c$ is at most $2\ell^{2m} + 2m \log \ell' + 8gd^2 \log D_K$ and this implies Proposition 6.10. □

B.2 Comments about $|h_F(A) - h_F(A')| = 0$

We continue our notation and we take A, A' as in Proposition 6.10. There are results in [54, §4.3] proving $|h_F(A) - h_F(A')| = 0$ if A and A' satisfy certain (strong) extra assumptions. However, $|h_F(A) - h_F(A')| = 0$ is wrong for certain A/K with $|T| = 0$. To give an example, let E be an elliptic curve over a number field k such that $\mathrm{End}(E_{\bar{k}}) = \mathbb{Z}$ and such that E has everywhere good reduction. In particular, E is semistable. Then for each rational prime p, an application of Ullmo-Szpiro [65, Thm 1.1. (3)] with E and $n = p^2$ gives

$$h_F(E') - h_F(E) = \log p(1 - \tfrac{1}{p}) + O(1)$$

where $E' = E_{k(P)}/G$ for G the group (viewed as a constant group scheme over $k(P)$) generated by a point $P \in E(\bar{k})$ of order p^2 and where $O(1)$ only depends on E/k. Thus, after choosing the rational prime p large enough, we obtain

$$|h_F(A) - h_F(A')| > 0 \tag{6.12}$$

for $A = E_{k(P)}$, $A' = E'$ and $K = k(P)$, because the stable Faltings height h_F is invariant under any finite field extension. Notice that the abelian variety A' is K-isogenous to A and it holds that $|T| = 0$ because $A = E_{k(P)}$ has everywhere good reduction.

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