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GEOMETRIC INEQUALITIES FOR PLANE CONVEX BODIES

BY
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1. **Introduction.** In what follows we shall mean by a *plane convex body* K a compact convex subset of the Euclidean plane having nonempty interior. We shall denote by $h(K, \theta)$ the supporting function of K restricted to the unit circle. This measures the signed distances from the origin to the supporting line of K with outward normal $(\cos\theta, \sin\theta)$. The right hand and left hand derivatives of $h(K, \theta)$ exist everywhere and are equal except on a countable set. As observed by Blaschke [1], if we define the derivative using a symmetric differential quotient, that is,

$$h'(K,\theta)=\lim_{\varepsilon\to 0}\frac{h(K,\theta+\varepsilon)-h(K,\theta-\varepsilon)}{2\varepsilon}, \tag{1}$$

then $h'(K, \theta)$ exists everywhere and is a function of bounded variation. In any case, defining $h'(K, \theta)$ either as in (1) or in the usual manner, it makes sense to define the *mixed area* $A(P, Q)$ of two plane convex bodies P and Q by

$$A(P,Q)=\frac{1}{2}\int_0^{2\pi} h(P,\theta)h(Q,\theta)-h'(P,\theta)h'(Q,\theta)\,d\theta. \tag{2}$$

Several important properties of the mixed area are evident from the definition (2). For example, we have symmetry in the arguments, that is,

$$A(P,Q)=A(Q,P). \tag{3}$$

Denoting the area of P by $A(P)$, the well-known formula for the area in terms of the supporting function (see Bonnesen and Fenchel [3]) yields,

$$A(P,P)=\frac{1}{2}\int_0^{2\pi} h(P,\theta)^2-h'(P,\theta)^2\,d\theta=A(P). \tag{4}$$

With the usual definitions for the Minkowskian sum of convex bodies and multiplication by nonnegative real numbers, one has for $\alpha, \beta \geq 0$,

$$h(\alpha P+\beta Q,\theta)=\alpha h(P,\theta)+\beta h(Q,\theta). \tag{5}$$

From (2) and (5) one sees that $A(P, Q)$ is linear in each variable under Minkowskian sum and multiplication by nonnegative reals. It follows that for

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$\alpha, \beta \geq 0$,

$$A(\alpha P+\beta Q)=\alpha^2 A(P)+2\alpha\beta A(P, Q)+\beta^2 A(Q). \tag{6}$$

Minkowski's inequality asserts that

$$A(P, Q)^2 \geq A(P)A(Q), \tag{7}$$

with equality holding if and only if P is homothetic to Q, that is $P=x+\lambda Q$ for some point x and some $\lambda>0$. Letting B denote the closed unit disk centered at the origin, as we shall do consistently, one has for the perimeter $L(P)$,

$$L(P)=\int_0^{2\pi} h(P, \theta)\, d\theta=2A(P, B). \tag{8}$$

Thus, taking $Q=B$ in (7) gives the isoperimetric inequality,

$$L^2(P) \geq 4\pi A(P), \tag{9}$$

where equality holds if and only if P is a circular disk.

Hurwitz [11] was the first to apply Fourier series to give a proof of (9). He proceeded by using (4) and (8) to represent $A(P)$ and $L(P)$ in terms of the Fourier coefficients of $h(P, \theta)$. Later Blaschke [1] used the Fourier expansions of supporting functions to give a proof of Minkowski's inequality (7). Heil [10] used this method to prove that if α is a fixed angle and the Steiner point of P is the origin, then

$$\int_0^{2\pi} h(P, \theta)h(P, \theta+\alpha)\, d\theta \geq 2A(P), \tag{10}$$

with equality only when P is a circular disk. (From the form of the integrand in (2), one might at first glance suspect that (10) could be derived directly from Minkowski's inequality; however, as shown in [10], this is not the case.) Chernoff [7] gave a Fourier series proof of (10) in the special case $\alpha=\pi/2$. This case was earlier established by Radziszewski [13] using other methods.

In §2 of this paper we shall use the method of Fourier series to prove the following generalization of Heil's inequality (10). Let P and Q be plane convex bodies, with Steiner points $s(P)$ and $s(Q)$. The Steiner point of a plane convex body K may be defined, in complex notation, by

$$s(K)=\frac{1}{\pi}\int_0^{2\pi} e^{i\theta} h(K, \theta)\, d\theta, \tag{11}$$

and can be shown to coincide with the center of gravity of a mass distribution along the boundary of K with density equal to the curvature at each point. We shall show that

$$\int_0^{2\pi} h(P, \theta)h(Q, \theta)\, d\theta \geq \pi\langle s(P), s(Q)\rangle+\frac{L(Q)}{L(P)}A(P)+\frac{L(P)}{L(Q)}A(Q). \tag{12}$$

The bracketed expression in (12) denotes the usual vector inner product of the

Steiner points. Since for $a, b, \lambda > 0$ we have always $\lambda a + b/\lambda \geq 2\sqrt{(ab)}$, from (12) we obtain

$$\int_0^{2\pi} h(P, \theta)h(Q, \theta)\, s\theta \geq \pi\langle s(P), s(Q)\rangle + 2\sqrt{[A(P)A(Q)]}. \tag{13}$$

If $\langle s(P), s(Q)\rangle = 0$, this reduces to

$$\int_0^{2\pi} h(P, \theta)h(Q, \theta)\, d\theta \geq 2\sqrt{[A(P)A(Q)]}, \tag{14}$$

which in turn gives Heil's inequality (10) when Q is a rotation of P through a fixed angle α. Equality can hold in (12), (13), or (14) only if P and Q are circular disks.

The *width function* of K describes the distance between parallel supporting lines and is given by

$$w(K, \theta) = h(K, \theta) + h(K, \theta + \pi) = h(K, \theta) + h(-K, \theta), \tag{15}$$

where $-K$ is the reflection of K through the origin. Using (13) and the fact that $s(-K) = -s(K)$, one obtains the following generalization of another result of Heil [10],

$$\int_0^{2\pi} w(P, \theta)w(Q, \theta)\, d\theta \geq 8\sqrt{[A(P)A(Q)]}. \tag{16}$$

Equality holds only if P and Q are circular disks. Lutwak [12] has recently proved an interesting n-dimensional generalization of (16). Schneider [14] and the author [4, 5] have given various generalizations of (16) in the case where P and Q are congruent.

In §3 we give a proof of an inequality which may be viewed as a common generalization of (7) and (14). Let E be a fixed plane convex body having the origin as an interior point. For any other plane convex body K the *inradius of K relative to E* is the largest r such that $x + rE \subseteq K$ for some x. This reduces to the usual definition of inradius in case $E = B$. The *kernel of K relative to E*, denoted by K_0, is

$$K_0 = \{x : x + rE \subseteq K\} \tag{17}$$

where r is the inradius of K relative to E. In case $E = B$, this is the locus of the centers of the inscribed circles of K. The kernel is always either a point or a closed line segment. Now suppose P and Q are plane convex bodies such that their respective kernels P_0 and Q_0 relative to E both contain the origin. Let r_1 and r_2 be the inradii of P and Q respectively relative to E. Then we shall prove that

$$\int \frac{h(P, \theta)h(Q, \theta)}{h(E, \theta)}\, ds(E, \theta) \geq \frac{r_2}{r_1} A(P) + \frac{r_1}{r_2} A(Q), \tag{18}$$

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where the integration is with respect to arc length along the boundary of E. More precisely, the integral is taken with respect to the arc length measure induced on the unit circle by E, in the sense given by Fenchel and Jessen [8]; hence the integral in (18) is actually with respect to a certain Borel measure on the unit circle. From (18) we obtain immediately that

$$\int \frac{h(P,\theta)h(Q,\theta)}{h(E,\theta)}\,ds(E,\theta) \geq 2\sqrt{[A(P)A(Q)]}. \tag{19}$$

Assuming P_0 and Q_0 contain the origin, we shall see that equality holds in (19) if and only if $P=r_1E$ and $Q=r_2E$.

The inequality (19) is proved in [6] using the method of inner parallel bodies. The proof we give here follows different lines and is somewhat more elementary.

The inequality (14) is obtained as a special case of (19), although with a different condition on the positioning of P and Q, when we take $E=B$.

If we take $Q=E$ in (19) we obtain Minkowski's inequality, using the fact (see Fenchel and Jessen [8]) that

$$A(P,Q)=\frac{1}{2}\int h(P,Q)\,ds(Q,\theta). \tag{20}$$

One may also view (18) as a generalization of Bonnesen's inequality [3, pp. 112–113]. Letting $J(P,Q)$ denote the integral on the left hand side of (18), where we think of E as being fixed throughout the discussion, we obtain from (18)

$$J(P,Q)^2-4A(P)A(Q) \geq \left[\frac{r_2}{r_1}A(P)-\frac{r_1}{r_2}A(Q)\right]^2 \geq 0, \tag{21}$$

in the same way one obtains the isoperimetric inequality from Bonnesen's inequality. Indeed, for $a, b, c \geq 0$ and $\lambda > 0$, the inequality $b \geq \lambda a + c/\lambda$ is equivalent to $b^2-4ac \geq (\lambda a - c/\lambda)^2$.

2. **Proof of the generalization of Heil's inequality**. As in [10] we expand $h(P,\theta)$ and in $h(Q,\theta)$ in Fourier series

$$h(P,Q)=\sum_{n=-\infty}^{\infty} a_n e^{in\theta}, \qquad h(Q,\theta)=\sum_{n=-\infty}^{\infty} b_n e^{in\theta}, \tag{22}$$

where $a_{-n}=\bar{a}_n$ and $b_{-n}=\bar{b}_n$ since the functions are real valued. Then

$$\int_0^{2\pi} h(P,Q)h(Q,\theta)\,d\theta = 2\pi a_0 b_0 + 2\pi \sum_{n=1}^{\infty} (a_n\bar{b}_n+\bar{a}_n b_n). \tag{23}$$

From the area formula (4) one has

$$A(P)=\pi a_0^2+2\pi\sum_{n=2}^{\infty}(1-n^2)|a_n|^2, \tag{24}$$

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and a corresponding formula for $A(Q)$. We have with a simple rearrangement then,

$$(25)\quad \int_0^{2\pi} h(P,\theta)h(Q,\theta)\,d\theta - A(P) - A(Q) + \pi(a_0-b_0)^2$$
$$= 2\pi(a_1\bar{b}_1+\bar{a}_1b_1) + 2\pi\sum_{n=2}^{\infty}[|a_n+b_n|^2+(n^2-2)(|a_n|^2+|b_n|^2)].$$

From (8) we obtain

$$(26)\quad L(P)=2\pi a_0 \quad\text{and}\quad L(Q)=2\pi b_0,$$

and from (11),

$$(27)\quad s(P)=2\bar{a}_1 \quad\text{and}\quad s(Q)=2\bar{b}_1.$$

Substituting these relations in (25), and observing that the infinite sum on the right hand side is nonnegative, we find

$$(28)\quad \int_0^{2\pi} h(P,\theta)h(Q,\theta)\,d\theta + \frac{(L(P)-L(Q))^2}{4\pi} \geq \pi\langle s(P), s(Q)\rangle + A(P) + A(Q).$$

For convenience, let $L_1=L(P)$, $L_2=L(Q)$, $A_1=A(P)$, and $A_2=A(Q)$. In (28), replace P by $(1/L_1)P$ and Q by $(1/L_2)Q$ and multiply both sides by L_1L_2. This gives

$$(29)\quad \int_0^{2\pi} h(P,\theta)h(Q,\theta)\,d\theta \geq \pi\langle s(P), s(Q)\rangle + \frac{L_2}{L_1}A_1 + \frac{L_1}{L_2}A_2,$$

where we have used the fact that $h(\lambda K, \theta)$, $L(\lambda K)$, and $s(\lambda K)$ are homogeneous of degree 1 in $\lambda>0$ and $A(\lambda K)$ is homogeneous of degree 2. This completes the proof of (12).

Expressing the quantities in (12) in terms of the Fourier coefficients shows that equality holds if and only if $a_n=b_n=0$ for $n\geq 2$, in which case P and Q are both circular disks. If equality holds in (13) then it must hold in (12), so again both P and Q are circular disks. A direct calculation also shows that if P and Q are any circular disks, then equality holds in both (12) and (13).

3. **Proof of the general inequality.** Let r be the inradius of K relative to the fixed plane convex body E. For each λ, $0\leq\lambda\leq r$, the *inner parallel body* K_λ *of* K *relative to* E is defined by

$$(30)\quad K_\lambda = \{x : x+(r-\lambda)E \subseteq K\}.$$

Then K_0 is the kernel of K relative to E and $K_r=K$. The pertinent properties of relative inner parallel bodies that we shall use can be found in Bol [2] and Hadwiger [9, p. 142]. For example, we have

$$(31)\quad K_\lambda + (r-\lambda)E \subseteq K, \qquad 0\leq\lambda\leq r,$$

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from which it follows that

(32) $$h(K_\lambda, \theta)+(r-\lambda)h(E, \theta)\le h(K, \theta), \qquad 0\le\lambda\le r.$$

Integrating (32) over $[0, r]$ with respect to λ yields the fundamental relation

(33) $$2rh(K_\lambda, \theta)\ge 2\int_0^r h(K_\lambda, \theta)\,d\lambda + r^2h(E, \theta).$$

The main inequality (18) will be obtained by taking $K=P$ and $K=Q$ successively in (33), multiplying the resulting inequalities, and integrating with respect to $ds(E, \theta)$. Indeed, if r_1 and r_2 are the inradii of P and Q respectively relative to E, multiplying the inequalities obtained from (33) gives

(34) $$4r_1r_2h(P, \theta)h(Q, \theta)\ge 4h(E, \theta)\times\left[r_2^2\int_0^{r_1} h(P_\lambda, \theta)\,d\lambda + r_1^2\int_0^{r_2} h(Q_\lambda, \theta)\,d\lambda\right]+\Delta,$$

where Δ is the expression

(35) $$\left[2\int_0^{r_1} h(P_\lambda, \theta)\,d\lambda - r_1^2h(E, \theta)\right]\left[2\int_0^{r_2} h(Q_\lambda, \theta)\,d\lambda - r_2^2h(E, \theta)\right].$$

Hence if we can show that (35) is nonnegative, it will follow from (34) that

(36) $$r_1r_2h(P, \theta)h(Q, \theta)\ge h(E, \theta)\left[r_2^2\int_0^{r_1} h(P_\lambda, \theta)\,d\lambda + r_1^2\int_0^{r_2} h(Q_\lambda, \theta)\,d\lambda\right].$$

To see that the expression in (35) is indeed nonnegative, we use the fact that $K_\lambda \supseteq K_0+\lambda E$, $0\le\lambda\le r$, from which follows $h(K_\lambda, \theta)\ge h(K_0, \theta)+\lambda h(E, \theta)$. If the origin belongs to K_0, we have $h(K_0, \theta)\ge 0$ and so obtain $h(K_\lambda, \theta)\ge \lambda h(E, \theta)$. Integrating this over $[0, r]$, with respect to λ, gives finally

(37) $$2\int_0^r h(K_\lambda, \theta)\,d\lambda - r^2h(E, \theta)\ge 0,$$

valid for any plane convex body K having the origin in K_0. Applying (37) with K replaced by P and Q respectively, we see that (35) is nonnegative if the origin belongs to both P_0 and Q_0. Hence, with this restriction, (36) holds.

Using the relation

(38) $$A(K)=2\int_0^r A(K_\lambda, E)\,d\lambda,$$

(see [2] or [6]), we have

(39) $$\int\left\{\int_0^r h(K_\lambda, \theta)\,d\lambda\right\}ds(E, \theta)=\int_0^r\left\{\int h(K_\lambda, \theta)\,ds(E, \theta)\right\}d\lambda = \int_0^r 2A(K_\lambda, E)\,d\lambda = A(K).$$

Since the origin is an interior point of E, we may divide both sides of (36) by $h(E, \theta)$. Then integrating the resulting inequality with respect to $ds(E, \theta)$, and applying (39) with K replaced by P and Q respectively, we obtain

$$(40)\qquad r_1 r_2 \int_0^{2\pi} \frac{h(P,\theta)h(Q,\theta)}{h(E,\theta)}\, ds(E,\theta) \geq r_2^2 A(P) + r_1^2 A(Q),$$

valid when the origin belongs to both P_0 and Q_0. Division by $r_1 r_2$ gives the required inequality (18).

If equality holds in (40) then it must hold in the inequalities leading to (40) and, because of the continuity of the functions involved, we must have equality in (32) for all θ and $0 \leq \lambda \leq r$, with K replaced by P and by Q. In particular, with $\lambda = 0$ we obtain $h(P_0 + r_1 E, \theta) = h(P, \theta)$ and $h(Q_0 + r_2 E, \theta) = h(Q, \theta)$, for all θ, so $P = P_0 + r_1 E$ and $Q = Q_0 + r_2 E$. To avoid strict inequality in (36), the expression (35) must be zero, which implies, by the discussion following (36), that either P_0 or Q_0 is the origin. Thus if equality holds in (18), then $P = P_0 + r_1 E$ and $Q = Q_0 + r_2 E$ and either P_0 or Q_0 is the origin (where we are still operating under the hypothesis that the origin belongs to both P_0 and Q_0). On the other hand, one checks directly that if $P = P_0 + r_1 E$ and $Q = Q_0 + r_2 E$, where one of P_0 or Q_0 is the origin and the other a closed line segment containing the origin, then equality holds in (18).

With our assumption about the kernels, equality holds in (19) if and only if $P = P_0 + r_1 E$ and $Q = Q_0 + r_2 E$, either P_0 or Q_0 is the origin, and additionally $A[(1/r_1)P] = A[(1/r_2)Q]$. If, for instance, P_0 is the origin, this implies $A[(1/r_2)Q_0 + E] = A(E)$, which can happen only if Q_0 is a single point and hence also coincides with the origin. Thus, assuming the origin belongs to both relative kernels, equality holds in (19) if and only if $P = r_1 E$ and $Q = r_2 E$.

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