SUM FUNCTIONS OF k-ARY AND SEMI-k-ARY DIVISORS

D. SURYANARAYANA and V. SIVA RAMA PRASAD

(Received May 1970; revised 20 July 1970)

Communicated by E. S. Barnes

1. Introduction

It is known that a divisor $d > 0$ of the positive integer n is called unitary [3; § 1] if $d\delta = n$ and $(d,\delta) = 1$, where (d,δ) denotes the g.c.d. of d and δ. We write $d \parallel n$. Let k be a fixed positive integer. A positive integer n is called k-free if n is not divisible by the kth power of any prime. Let $\mathscr{V}_k(n)$ be the characteristic function of the set of k-free integers (i.e. $\mathscr{V}_k(n) = 1$ or 0 according as n is k-free or not). For integers a, b, not both zero, let us denote by $(a,b)_k$ and $(a,b)_k^*$, the greatest kth power divisor of a and b and the greatest kth power divisor of a which is a unitary divisor of b respectively. It is clear that $(a,b)_1 = (a,b)$ and $(a,b)_1^* = (a,b)^*$, the greatest divisor of a which is a unitary divisor of b [3; §1]. We say that a divisor $d > 0$ of the positive integer n is *k-ary* [9; § 1] or *semi-k-ary* according as $d\delta = n$ and $(d,\delta)_k = 1$ or $d\delta = n$ and $(d,\delta)_k^* = 1$. It is clear that a 1-ary divisor is a unitary divisor and a semi-1-ary divisor is a semi-unitary divisor [1; § 1]. It may be noted that if d is a k-ary divisor of n then n/d (the complementary divisor to d of n) is also a k-ary divisor of n, but if d is a semi-k-ary divisor of n, then its complementary divisor need not be a semi-k-ary divisor of n.

For any pair of arithmetical functions $f(n)$ and $g(n)$, we define

$$F_k^*(n) = \sum_{\substack{d\delta = n \\ (d,\delta)_k = 1}} f(d)g(\delta) \tag{1.1}$$

$$F_k^{s*}(n) = \sum_{\substack{d\delta = n \\ (d,\delta)_k^* = 1}} f(d)g(\delta), \tag{1.2}$$

where the summation in (1.1) and (1.2) are over the k-ary and semi-k-ary divisors of n respectively.

For any r such that $|r| \geqq 1$, let

$$\sigma_{r,k}^*(n);\ \sigma_{r,k}^{*\prime}(n);\ \sigma_{r,k}^{s*}(n);\ \sigma_{r,k}^{s*\prime}(n), \tag{1.3}$$

denote respectively the sum of the rth powers of the k-ary divisors, k-ary $(k+1)$-

https://doi.org/10.1017/S1446788700012908 Published online by Cambridge University Press

free divisors, semi-k-ary divisors and semi-k-ary $(k+1)$-free divisors of n.

For $|r| \geqq 1$, let

(1.4) $$\sigma_{r,k}^{*c}(n);\ \sigma_{r,k}^{*\prime c}(n);\ \sigma_{r,k}^{s*c}(n);\ \sigma_{r,k}^{s*c}\ (n),$$

denote respectively the sum of the rth powers of the k-ary divisors of n whose complementary divisors are $(k+1)$-free, k-ary $(k+1)$-free divisors of n whose complementary divisors are $(k+1)$-free, semi-k-ary divisors of n whose complementary divisors are $(k+1)$-free, semi-k-ary $(k+1)$-free divisors of n whose complementary divisors are $(k+1)$-free.

Again, for $|r| \geqq 1$, let

(1.5) $$\sigma_{r,k}^{cs*}(n);\ \sigma_{r,k}^{\prime cs*}(n);\ \sigma_{r,k}^{cs*\prime}(n)\ ;\ \sigma_{r,k}^{\prime cs*\prime}\ (n),$$

denote respectively the sum of the rth powers of the complementary semi-k-ary divisors of n, $(k+1)$-free complementary semi-k-ary divisors of n, complementary semi-k-ary $(k+1)$-free divisors of n, $(k+1)$-free complementary semi-k-ary $(k+1)$-free divisor of n.

Any complementary k-ary divisor of n is also a k-ary divisor of n, so that the four functions corresponding to (1.5) in case of k-ary divisors reduce to the first two functions of (1.3) and the first two functions of (1.4) respectively.

Further, if δ is $(k+1)$-free, then $(d,\delta)_k^* = 1$ if and only if $(d,\delta)_k = 1$. Hence the first and third functions of (1.4) are equal and the second and fourth functions of (1.4) are equal. Also, the second function of (1.5) is equal to the second function of (1.3) and the fourth function of (1.5) is equal to the second function of (1.4). Hence we discuss only the following functions together with the functions in (1.3):

(1.6) $$\sigma_{r,k}^{*c}(n);\ \sigma_{r,k}^{*\prime c}(n);\ \ \sigma_{r,k}^{cs*}(n);\ \sigma_{r,k}^{cs*\prime}\ (n).$$

In this paper (see §4) we obtain the average orders of magnitude of the functions $F_k^*(n)$ and $F_k^{s*}(n)$ defined in (1.1) and (1.2) in the following cases and deduce the average orders of magnitude of the functions considered in (1.3) and (1.6):

(i) $g(n)$ is bounded and $f(n) = n^r$, $\mathscr{V}_{k+1}(n)n^r$, $r \geqq 1$.

(ii) $g(n) = 1$, $\mathscr{V}_{k+1}(n)$ and $f(n) = h(n)/n^r$, where $h(n)$ is bounded, $r \geqq 1$,

(iii) $f(n)$ is bounded and $g(n) = n^r$, $r \geqq 1$.

(iv) $f(n) = 1$, $\mathscr{V}_{k+1}(n)$ and $g(n) = h(n)/n^r$, where $h(n)$ is bounded, $r \geqq 1$.

2. Preliminaries

Let $\phi(n)$ denote the Euler totient function, $J_k(n)$ denote the Jordan totient function (cf. [4], p. 147; also [2]) and $\Phi_k(n)$ denote the Klee's function (cf. [6] and [7]). Let $\mu(n)$ denote the Möbius function and $\mu^*(n)$ be defined by $\mu^*(1) = 1$ and $\mu^*(n) = (-1)^{w(n)}$, where $w(n)$ denotes the number of distinct prime factors of n. The function $\mu^*(n)$ has been discussed by Cohen [3]. The following known arithmetical forms are needed in our later discussion:

$$(2.1)\qquad \phi(n) = n \sum_{d\mid n} \frac{\mu(d)}{d} = n \prod_{p\mid n} \left(1 - \frac{1}{p}\right),$$

$$(2.2)\qquad J_k(n) = n^k \sum_{d\mid n} \frac{\mu(d)}{d^k} = n^k \prod_{p\mid n} \left(1 - \frac{1}{p^k}\right),$$

$$(2.3)\qquad \Phi_k(n) = n \sum_{d^k\mid n} \frac{\mu(d)}{d^k} = n \prod_{p^k\mid n} \left(1 - \frac{1}{p^k}\right),$$

the products being extended over all prime divisors p of n.

Let $\phi_k^*(n)$ denote the number of integers h in the set $\{1, 2, \cdots, n\}$ such that $(h, n)_k^* = 1$. It has been proved in [10; Theorem 2.5] that

$$(2.4)\qquad \phi_k^*(n) = n \sum_{d^k\| n} \frac{\mu^*(d)}{d^k} = n \prod_{\substack{p^\alpha \| n \\ k/\alpha}} \left(1 - \frac{1}{p^\alpha}\right).$$

LEMMA 2.1.

$$(2.5)\qquad \sum_{m\leqq x} m^r = \frac{x^{r+1}}{r+1} + O(x^r), \quad \text{if } r \geqq 0,\ x \geqq 1.$$

$$(2.6)\qquad \sum_{m\leqq x} \frac{1}{m^r} = O(1), \qquad \text{if } r > 1,\ x \geqq 1,$$

$$= O(\log x), \qquad \text{if } r = 1,\ x \geqq 2.$$

$$(2.7)\qquad \sum_{m > x} \frac{1}{m^r} = O\left(\frac{1}{x^{r-1}}\right), \qquad \text{if } r > 1,\ x \geqq 1.$$

This is well-known.

LEMMA 2.2. (*cf.* [11], *lemma* 2.3) *For* $s > 1$,

$$(2.8)\qquad \sum_{\substack{m=1 \\ (m,n)=1}}^{\infty} \frac{\mu(m)}{m^s} = \frac{n^s}{\zeta(s) J_s(n)},$$

where $J_s(n) = n^s \prod_{p\mid n} (1 - 1/p^s)$ *and* $\zeta(s)$ *is the Riemann zeta function.*

LEMMA 2.3. (*cf.* [8], *theorem* 1). *For* $s > 2$,

$$(2.9)\qquad \sum_{n=1}^{\infty} \frac{\Phi_k(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(ks)}.$$

LEMMA 2.4. *For* $s > 2$,

$$(2.10)\qquad \sum_{n=1}^{\infty} \frac{\mathscr{V}_{k+1}(n)\Phi_k(n)}{n^s} = \zeta(s-1) \prod_{p} \left\{1 - \frac{1}{p^{(k+1)(s-1)}} - \frac{1 - p^{-s+1}}{p^{ks}}\right\},$$

the product being extended over all primes p.

PROOF. The series is absolutely convergent for $s > 2$, since $\Phi_k(n) \leqq n$ by (2.3) and the general term is multiplicative. Hence the series can be expanded into an infinite product of Euler type (cf. [5], Theorem 286) so that we have

$$\sum_{n=1}^{\infty} \frac{\mathscr{V}_{k+1}(n)\Phi_k(n)}{n^s} = \prod_p \left\{ \sum_{i=0}^{\infty} \frac{\mathscr{V}_{k+1}(p^i)\Phi_k(p^i)}{p^{is}} \right\}.$$

By making use of (2.3), we get the lemma after simplification.

LEMMA 2.5. *For* $s > 2$,

$$\sum_{n=1}^{\infty} \frac{\phi_k{}^*(n)}{n^s} = \zeta(s-1)\zeta(ks) \prod_p \left\{ 1 - \frac{2}{p^{ks}} + \frac{1}{p^{(k+1)s-1}} \right\}. \tag{2.11}$$

PROOF. This can be proved in the same way as lemma 2.4, by expressing the series into an infinite product of Euler type and then making use of (2.4).

LEMMA 2.6.

$$\sum_{\substack{d^k | m \\ d^k | n}} = \begin{cases} 1 & \text{if } (m,n)_k = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.12}$$

PROOF. We note that $(a, b)_k = 1$ if and only if (a, b) is k-free and

$$\mathscr{V}_k(n) = \sum_{d^k | n} \mu(d) \quad \text{so that} \tag{2.13}$$

$$\sum_{d^k | (m,n)} \mu(d) = \mathscr{V}_k((m,n)) = 1 \text{ or } 0 \text{ according as } (m,n)_k = 1 \text{ or } > 1,$$

which proves the lemma.

LEMMA 2.7. (cf. [1], lemma 2.3(ii)). *For* $r \geqq 0$, $x \geqq 1$, $n \geqq 1$,

$$\phi_r(x,n) \equiv \sum_{\substack{m \leqq x \\ (m,n)=1}} m^r = \frac{x^{r+1}}{r+1} \cdot \frac{\phi(n)}{n} + O(x^r \vartheta(n)), \tag{2.14}$$

uniformly, where $\vartheta(n)$ *is the number of square-free divisors of* n.

LEMMA 2.8. *For* $r \geqq 0$, $x \geqq 1$,

$$\sum_{\substack{m \leqq x \\ (m,n)=1}} \mathscr{V}_{k+1}(m) m^r = \frac{x^{r+1}}{r+1} \cdot \frac{\alpha_k(n)}{\zeta(k+1)} + O(x^{r+1'/(k+1)}\vartheta(n)), \tag{2.15}$$

uniformly, where

$$\alpha_k(n) = \frac{n^k \phi(n)}{J_{k+1}(n)}. \tag{2.16}$$

PROOF. By (2.13) and (2.14), we have

https://doi.org/10.1017/S1446788700012908 Published online by Cambridge University Press

$$\sum_{\substack{m \leqq x \\ (m,n)=1}} \mathscr{V}_{k+1}(m)m^r = \sum_{\substack{m \leqq x \\ (m,n)=1}} m^r \sum_{t^{k+1}|m} \mu(t) = \sum_{\substack{t^{k+1}s \leqq x \\ (t^{k+1}s,n)=1}} t^{(k+1)r} s^r \mu(t)$$

$$= \sum_{\substack{t \leqq x^{1/k+1} \\ (t,n)=1}} t^{(k+1)r} \mu(k) \sum_{\substack{s \leqq x/t^{k+1} \\ (s,n)=1}} s^r$$

$$= \sum_{\substack{t \leqq x^{1/k+1} \\ (t,n)=1}} \mu(t) t^{(k+1)r} \left\{ \frac{1}{r+1} \left(\frac{x}{t^{k+1}} \right)^{r+1} \frac{\phi(n)}{n} + O\left(x^r \frac{\vartheta(n)}{t^{(k+1)r}} \right) \right\}$$

$$= \frac{x^{r+1}}{r+1} \cdot \frac{\phi(n)}{n} \sum_{\substack{t \leqq x^{1/k+1} \\ (t,n)=1}} \frac{\mu(t)}{t^{k+1}} + O\left(x^r \vartheta(n) \sum_{t \leqq x^{1/k+1}} 1 \right)$$

$$= \frac{x^{r+1}}{r+1} \cdot \frac{\phi(n)}{n} \sum_{\substack{t=1 \\ (t\ n)=1}}^{\infty} \frac{\mu(t)}{t^{k+1}} + O\left(x^{r+1} \sum_{t > x^{1/k+1}} t^{-(k+1)} \right)$$

$$+ O\left(x^{r+1/(k+1)} \vartheta(n) \right).$$

The first O-term is $O(x^{r+1/(k+1)})$ by (2.7). Hence the lemma follows, by (2.8).

LEMMA 2.9.

$$\sum_{\substack{d^k | m \\ d^k \| n}} \mu^*(d) = \begin{cases} 1 \text{ if } (m,n)_k^* = 1 \\ 0 \text{ otherwise.} \end{cases} \tag{2.17}$$

PROOF. Suppose $(m,n)_k^* = 1$. Then 1 is the only kth power divisor of m which is a unitary divisor of n so that the left hand side of (2.17) is $\mu^*(1) = 1$.

Suppose $(m,n)_k^* > 1$. Then there exists a kth power divisor > 1 of m which is a unitary divisor of n. Let $p_1^{k\alpha_1} \cdots p_r^{k\alpha_r}$ be the canonical representation of the greatest kth power divisor of m which is a unitary divisor of n. Then the left hand side of (2.17) is equal to

$$\prod_{i=1}^{r} \{\mu^*(1) + \mu^*(p_i^{\alpha_i})\} = 0.$$

Hence the lemma follows.

3. Auxiliary lemmas

In this section, we prove some more lemmas which are needed in our later discussion.

LEMMA 3.1. *For* $r \geqq 0$, $x \geqq 1$,

$$(3.1)\qquad \phi_{r,k}(x,n) \equiv \sum_{\substack{m \leqq x \\ (m,n)_k = 1}} m^r = \frac{x^{r+1}}{r+1} \cdot \frac{\Phi_k(n)}{n} + O(x^r \vartheta_k(n)),$$

$$(3.2)\qquad \phi^*_{r,k}(x,n) \equiv \sum_{\substack{m \leqq x \\ (m,n)^*_k = 1}} m^r = \frac{x^{r+1}}{r+1} \frac{\phi^*_k(n)}{n} + O(x^r \tau^*_k(n)),$$

uniformly, where $\vartheta_k(n) = \sum_{d^k | n} \mu^2(d)$ *and* $\tau^*_k(n) = \sum_{d^k \| n} 1$.

Proof. We prove (3.2) and (3.1) can be proved similarly. By (2.17), (2.5) and (2.4),

$$\phi^*_{r,k}(x,n) = \sum_{\substack{m \leqq x \\ (m,n)^*_k = 1}} m^r = \sum_{m \leqq x} \sum_{\substack{d^k | m \\ d^k \| n}} \mu^*(d) = \sum_{\substack{d^k \delta \leqq x \\ d^k \| n}} \mu^*(d) d^{kr} \delta^r$$

$$= \sum_{d^k \| n} \mu^*(d) d^{kr} \sum_{\delta \leqq x/d^k} \delta^r$$

$$= \sum_{d^k \| n} \mu^*(d) d^{kr} \left\{ \frac{1}{r+1} \left(\frac{x}{d^k} \right)^{r+1} + O\left(\frac{x^r}{d^{kr}} \right) \right\}$$

$$= \frac{x^{r+1}}{r+1} \sum_{d^k \| n} \frac{\mu^*(d)}{d^k} + O\left(x^r \sum_{d^k \| n} 1 \right)$$

$$= \frac{x^{r+1}}{r+1} \cdot \frac{\phi^*_k(n)}{n} + O(x^r \tau^*_k(n)).$$

Thus (3.2) is proved. To prove (3.1) we use (2.12) and (2.3) instead of (2.17) and (2.4).

Remark 3.1. The functions $\vartheta_k(n)$ and $\tau^*_k(n)$ can be replaced by $\tau_k(n) = \sum_{d^k | n} 1$ in (3.1) and (3.2) since $\vartheta_k(n) \leqq \tau_k(n)$ and $\tau^*_k(n) \leqq \tau_k(n)$.

Lemma 3.2. *For* $r \geqq 0$, $x \geqq 1$,

$$(3.3)\qquad \phi'_{r,k}(x,n) \equiv \sum_{\substack{m \leqq x \\ (m,n)_k = 1}} \mathscr{V}_{k+1}(m) m^r = \frac{x^{r+1}}{r+1} \cdot \frac{\beta_k(n)}{\zeta(k+1)} + O(x^{r+1/(k+1)} \theta_k(n)),$$

$$(3.4)\qquad \phi^{*\prime}_{r,k}(x,n) \equiv \sum_{\substack{m \leqq x \\ (m,n)^*_k = 1}} \mathscr{V}_{k+1}(m) m^r = \frac{x^{r+1}}{r+1} \frac{\beta^*_k(n)}{\zeta(k+1)} + O(x^{r+1/(k+1)} \theta^*_k(n)),$$

uniformly, where

$$(3.5)\qquad \beta_k(n) = \sum_{d^k | n} \frac{\mu(d) \alpha_k(d)}{d^k}$$

$$(3.6)\qquad \theta_k(n) = \sum_{d^k | n} \frac{\mu^2(d) \vartheta(d)}{d^{k/(k+1)}}$$

https://doi.org/10.1017/S1446788700012908 Published online by Cambridge University Press

$$\beta_k^*(n) = \sum_{d^k \| n} \frac{\mu^2(d)\mu^*(d)\alpha_k(d)}{d^k} = \sum_{d^k \| n} \frac{\mu(d)\alpha_k(d)}{d^k} \tag{3.7}$$

$$\theta_k^*(n) = \sum_{d^k \| n} \frac{\mu^2(d)\vartheta(d)}{d^{k/(k+1)}}. \tag{3.8}$$

PROOF. Here also, we give the proof of (3.4) and omit the proof of (3.3) as it can be proved similarly.

By (2.17) and (2.15), we have

$$\begin{aligned}
\phi_{r,k}^{*\prime}(x,n) &= \sum_{m \leq x} \mathscr{V}_{k+1}(m) m^r \sum_{\substack{d^k | m \\ d^k \| n}} \mu^*(d) \\
&= \sum_{\substack{d^k\delta \leq x \\ d^k \| n}} \mathscr{V}_{k+1}(d^k\delta) d^{kr}\delta^r \mu^*(d) \\
&= \sum_{\substack{d^k\delta \leq x \\ d^k \| n \\ (d,\delta)=1}} \mathscr{V}_{k+1}(d^k)\mathscr{V}_{k+1}(\delta) d^{kr}\delta^r \mu^*(d) \\
&= \sum_{d^k \| n} \mu^2(d)\mu^*(d) d^{kr} \sum_{\substack{\delta \leq x/d^k \\ (\delta,d)=1}} \mathscr{V}_{k+1}(\delta)\delta^r \\
&= \sum_{d^k \| n} \mu^2(d)\mu^*(d) d^{kr} \left\{ \frac{1}{r+1} \left(\frac{x}{d^k}\right)^{r+1} \frac{\alpha_k(d)}{\zeta(k+1)} + O\left(\frac{x^{r+1/(k+1)}\vartheta(d)}{d^{kr+k/(k+1)}}\right)\right\} \\
&= \frac{x^{r+1}}{(r+1)\zeta(k+1)} \sum_{d^k \| n} \frac{\mu^2(d)\mu^*(d)\alpha_k(d)}{d^k} + O\left(x^{r+1/(k+1)} \sum_{d^k \| n} \frac{\mu^2(d)\vartheta(d)}{d^{k/(k+1)}}\right).
\end{aligned}$$

Thus (3.4) is proved. To prove (3.3) we use (2.12) instead of (2.17).

REMARK 3.2. We note that $\beta_k(n)$ and $\theta_k(n)$ are multiplicative (follow from lemma 2.4 of [9]) and $\beta_k^*(n)$ and $\theta_k^*(n)$ are also multiplicative (follow from theorem 2.4 of [10]). Their evaluations are given by the following:

$$\beta_k(n) = \prod_{p^k | n} \left(\frac{p(p^k-1)}{p^{k+1}-1}\right) \tag{3.9}$$

$$\theta_k(n) = \prod_{p^k | n} \left(1 + \frac{2}{p^{k/(k+1)}}\right) \tag{3.10}$$

$$\beta_k^*(n) = \prod_{p^k \| n} \left(\frac{p(p^k+1)}{p^{k+1}-1}\right) \tag{3.11}$$

$$\theta_k^*(n) = \prod_{p^k \| n} \left(1 + \frac{2}{p^{k/(k+1)}}\right). \tag{3.12}$$

REMARK 3.3. $\theta_k(n) = O(n^\varepsilon)$ and $\theta_k^*(n) = O(n^\varepsilon)$ for every $\varepsilon > 0$. These can be proved by making use of theorem 316 of [5].

LEMMA 3.3. *For* $s > 1$,

$$\sum_{n=1}^{\infty} \frac{\beta_k(n)}{n^s} = \zeta(s) \prod_p \left(1 - \frac{p-1}{p^{ks}(p^{k+1}-1)}\right) \tag{3.13}$$

$$\sum_{n=1}^{\infty} \frac{\beta_k^*(n)}{n^s} = \zeta(s) \prod_p \left(1 - \frac{(p-1)(1-p^{-s})}{p^{k+1}-1}\right) \tag{3.14}$$

$$\sum_{n=1}^{\infty} \frac{\mathscr{V}_{k+1}(n)\beta_k(n)}{n^s} = \zeta(s) \prod_p \left(1 - \frac{1}{p^{(k+1)s}} - \frac{(p-1)(1-p^{-s})}{p^{ks}(p^{k+1}-1)}\right), \tag{3.15}$$

the products being extended over all primes.

Proof. By (3.9) and (3.11), $\beta_k(n) \leqq 1$ and $\beta_k^*(n) \leqq 1$ so that the three series are absolutely convergent for $s > 1$. Since $\beta_k(n)$ and $\beta_k^*(n)$ are multiplicative, the series can be expanded into infinite products of Euler type (cf. [5] theorem 286). By making use of (3.9) and (3.11) we get the lemma after simplification.

Lemma 3.4. *For* $r \geqq 0$, $x \geqq 1$,

$$\psi_{r,k}^*(x,n) \equiv \sum_{\substack{m \leqq x \\ (n,m)_k^* = 1}} m^r = \frac{x^{r+1}}{r+1} \frac{\psi_k^*(n)}{n} + O(x^r I_k(n)), \tag{3.16}$$

uniformly, where

$$\psi_k^*(n) = n \sum_{d^k | n} \frac{\mu^*(d)\phi(d)}{d^{k+1}} \tag{3.17}$$

$$I_k(n) = \sum_{d^k | n} \vartheta(d). \tag{3.18}$$

Proof. Interchanging m and n in lemma 2.9, we have

$$\sum_{\substack{d^k \| m \\ d^k | n}} \mu^*(d) = \begin{cases} 1 & \text{if } (n,m)_k^* = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\psi_{r,k}^*(x,n) = \sum_{m \leqq x} m^r \sum_{\substack{d^k \| m \\ d^k | n}} \mu^*(d) = \sum_{\substack{d^k\delta \leqq x \\ d^k | n \\ (d,\delta)=1}} \mu^* d^{kr} \delta^r$$

$$= \sum_{d^k | n} \mu^*(d) d^{kr} \sum_{\substack{\delta \leqq x/d^k \\ (\delta,d)=1}} \delta^r,$$

so that by (2.14),

$$\psi_{r,k}^*(x,n) = \sum_{d^k | n} \mu^*(d) d^{kr} \left\{\frac{1}{r+1}\left(\frac{x}{d^k}\right)^{r+1} \frac{\phi(d)}{d} + O\left(x^r \frac{\vartheta(d)}{d^{kr}}\right)\right\}$$

$$= \frac{x^{r+1}}{r+1} \sum_{d^k | n} \frac{\mu^*(d)\phi(d)}{d^{k+1}} + O\left(x^r \sum_{d^k, n} \vartheta(d)\right).$$

Hence (3.16) follows.

https://doi.org/10.1017/S1446788700012908 Published online by Cambridge University Press

REMARK 3.4. We note that $\psi_k^*(n)$ is multiplicative (follows from lemma 2.4 of [9]) and its evaluation is given by

$$\psi_k^*(n) = n \prod_{p^k \mid n} \left\{1 - \frac{(p-1)(1-p^{-tk})}{p(p^k-1)}\right\}, \tag{3.20}$$

where $t = [\alpha/k]$, α being the multiplicity of p in n.

LEMMA 3.5. *For* $s > 2$,

$$\sum_{n=1}^{\infty} \frac{\psi_k^*(n)}{n^s} = \zeta(s-1) \prod_p \left(1 - \frac{p-1}{p(p^{ks}-1)}\right) \tag{3.21}$$

$$\sum_{n=1}^{\infty} \frac{\mathscr{V}_{k+1}\psi_k^*(n)}{n^s} = \zeta(s-1) \prod_p \left\{1 - \frac{1}{p^{(k+1)(s-1)}} - \frac{(p-1)(1-p^{-s+1})}{p^{ks+1}}\right\}. \tag{3.22}$$

PROOF. By (3.20), $\psi_k^*(n) \leqq n$, so that the series are absolutely convergent for $s > 2$ and the general terms of the series are multiplicative. By expanding the series into an infinite product of Euler type the lemma follows from theorem 286 of [5].

LEMMA 3.6. *For* $r \geqq 1$, $x \geqq 2$,

$$E_{r,k}(x) \equiv \sum_{n \leqq x} \frac{\tau_k(d)}{d^r} = \begin{cases} O(\log^2 x) & \text{if } r = 1,\ k = 1 \\ O(\log x) & \text{if } r = 1,\ k > 1 \\ O(1) & \text{if } r > 1, \end{cases} \tag{3.23}$$

where $\tau_k(n)$ *is as defined in Remark* 3.1.

PROOF. We have

$$E_{r,k}(x) = \sum_{n \leqq x} \frac{1}{n^r} \sum_{d^k \mid n} 1 = \sum_{d^k\delta \leqq x} \frac{1}{d^{kr}\delta^r}$$

$$= \sum_{d \leqq x^{1/k}} \frac{1}{d^{kr}} \sum_{\delta \leqq x/d^k} \frac{1}{\delta^r}. \tag{3.24}$$

If $r = 1$, then by (2.6),

$$E_{r,k}(x) = O\left(\sum_{d \leqq x^{1/k}} \frac{1}{d^k} \log \frac{x}{d^k}\right) = O\left(\log x \sum_{d \leqq x^{1/k}} \frac{1}{d^k}\right) = O(\log^2 x) \text{ or } O(\log x),$$

according as $k = 1$ or $k > 1$.

If $r > 1$, by (3.24) and (2.6),

$$E_{r,k}(x) = O\left(\sum_{d \leqq x^{1/k}} \frac{1}{d^k}\right) = O(1).$$

Hence the lemma follows.

LEMMA 3.7.

$$(3.25)\qquad G_{r,k}(x) \equiv \sum_{n \leqq x} \frac{I_k(n)}{n^r} = \begin{cases} O(\log^3 x) & \text{if } r = 1, k = 1 \\ O(\log x) & \text{if } r = 1, k > 1 \\ O(1) & \text{if } r > 1, \end{cases}$$

where $I_k(n)$ is given by (3.18).

PROOF.

$$G_{r,k}(x) = \sum_{n \leqq x} \frac{1}{n^r} \sum_{d^k | n} \vartheta(d) = \sum_{d^k \delta \leqq x} \frac{\vartheta(d)}{d^{kr} \delta^r}$$

$$(3.26)\qquad = \sum_{d \leqq x^{1/k}} \frac{\vartheta(d)}{d^k} \sum_{\delta \leqq x/d^k} \frac{1}{\delta}.$$

It is clear that $\vartheta(n) \leqq \tau(n)$, where $\tau(n) = \sum_{d|n} 1$. If $r = 1$, then by (2.6),

$$G_{r,k}(x) = O\left(\sum_{d \leqq x^{1/k}} \frac{\tau(d)}{d^k} \log \frac{x}{d^k}\right) = O\left(\log x \sum_{d \leqq x^{1/k}} \frac{\tau(d)}{d^k}\right)$$
$$= O(\log^3 x) \text{ or } O(\log x),$$

according as $k = 1$ or $k > 1$, since

$$\sum_{d \leqq x^{1/k}} \frac{\tau(d)}{d^k} = O(\log^2 x) \quad \text{or } O(1),$$

according as $k = 1$ or $k > 1$.

If $r > 1$, then by (3.26) and (2.6),

$$G_{r,k}(x) = O\left(\sum_{d \leqq x^{1/k}} \frac{\tau(d)}{d^{kr}}\right) = O(1).$$

Hence the lemma follows.

4. Main Theorems

In this section we prove some theorems and deduce the average orders of magnitude of the functions considered in (1.3) and (1.6).

THEOREM 4.1. *If in* (1.1) *and* (1.2) $g(n)$ *is bounded and* $f(n) = n^r$ *then for* $r \geqq 1$, $x \geqq 2$, $k \geqq 1$,

$$(4.1)\qquad \sum_{n \leqq x} F_k^*(n) = \frac{x^{r+1}}{r+1} \sum_{n=1}^{\infty} \frac{g(n)\Phi_k(n)}{n^{r+2}} + x^r E_{r,k}(x)$$

$$(4.2)\qquad \sum_{n \geqq x} F_k^{s*}(n) = \frac{x^{r+1}}{r+1} \sum_{n=1}^{\infty} \frac{g(n)\phi_k^*(n)}{n^{r+2}} + x^r E_{r,k}(x),$$

https://doi.org/10.1017/S1446788700012908 Published online by Cambridge University Press

where $\Phi_k(n)$ *and* $\phi_k^*(n)$ *are respectively given by* (2.3) *and* (2.4) *and* $E_{r,k}(x)$ *is given by* (3.23).

PROOF. We have by (3.1) and Remark 3.1.

$$\begin{aligned}
\sum_{n\leq x} F_k^*(n) &= \sum_{n\leq x} \sum_{\substack{d\delta=n\\(d,\delta)_k=1}} d^r g(\delta) = \sum_{\substack{d\delta\leq x\\(d,\delta)_k=1}} d^r g(\delta) = \sum_{\delta\leq x} g(\delta) \sum_{\substack{\delta\leq x/\delta\\(d,\delta)_k=1}} d^r\\
&= \sum_{\delta\leq x} g(\delta)\phi_{r,k}\left(\frac{x}{\delta},\delta\right)\\
&= \sum_{\delta\leq x} g(\delta)\left\{\frac{1}{r+1}\left(\frac{x}{\delta}\right)^{r+1}\frac{\Phi_k(\delta)}{\delta} + O\left(x^r\frac{\tau_k(\delta)}{\delta^r}\right)\right\}\\
&= \frac{x^{r+1}}{r+1}\sum_{n\leq x}\frac{g(n)\Phi_k(n)}{n^{r+2}} + O\left(x^r\sum_{n\leq x}\frac{\tau_k(n)}{n^r}\right)\\
&= \frac{x^{r+1}}{r+1}\sum_{n=1}^{\infty}\frac{g(n)\Phi_k(n)}{n^{r+2}} + O\left(x^{r+1}\sum_{n>x}\frac{\Phi_k(n)}{n^{r+2}}\right)\\
&\qquad + O(x^r E_{r,k}(x)),
\end{aligned}$$

by (3.23) and the boundedness of $g(n)$.

Since $\Phi_k(n) \leqq n$, the first O-term is $O(x)$ by (2.7) and the second O-term is $x^r E_{r,k}(x)$ by (3.23). Hence (4.1) follows.

The proof of (4.2) follows similarly by using (3.2) instead of (3.1).

COROLLARY 4.1.1. $(g(n) = 1)$. *For* $r \geqq 1,\ x \geqq 2,\ k \geqq 1$,

(4.3) $$\sum_{n\leq x}\sigma_{r,k}^*(n) = \frac{x^{r+1}}{r+1}\sum_{n=1}^{\infty}\frac{\Phi_k(n)}{n^{r+2}} + x^r E_{r,k}(x)$$

(4.4) $$\sum_{n\leq x}\sigma_{r,k}^{s*}(n) = \frac{x^{r+1}}{r+1}\sum_{n=1}^{\infty}\frac{\phi_k^*(n)}{n^{r+2}} + x^r E_{r,k}(x).$$

REMARK 4.1. The case $k = 1$, $r = 1$ in (4.1) and (4.3) has been discussed by Cohen [3] and the case $k = 1$, $r > 1$ in Theorem 4.1 and Corollary 4.1.1 has been discussed by Chidambaraswamy [1]. The case $k > 1$, $r = 1$ in (4.1) and (4.3) has been discussed in [12].

REMARK 4.2. The coefficients of $x^{r+1}/(r+1)$ in (4.3) and (4.4) can be obtained from (2.9) and (2.11) respectively by taking $s = r + 2$.

COROLLARY 4.1.2. $(g(n) = \mathscr{V}_{k+1}(n))$. *For* $r \geqq 1,\ x \geqq 2,\ k \geqq 1$,

(4.5) $$\sum_{n\leq x}\sigma_{r,k}^{*c}(n) = \frac{x^{r+1}}{r+1}\sum_{n=1}^{\infty}\frac{\mathscr{V}_{k+1}(n)\Phi_k(n)}{n^{r+2}} + x^r E_{r,k}(x).$$

REMARK 4.2. The coefficient of $x^{r+1}/(r+1)$ in (4.5) can be obtained from (2.10) by taking $s = r + 2$.

THEOREM 4.2. *If in* (1.1) *and* (1.2) $g(n)$ *is bounded and* $f(n) = v_{k+1}(n)n^r$, *then for* $r \geqq 1$, $x \geqq 2$,

$$\sum_{n \leqq x} F_k^*(n) = \frac{x^{r+1}}{(r+1)\zeta(k+1)} \sum_{n=1}^{\infty} \frac{g(n)\beta_k(n)}{n^{r+1}} + O(x^{r+1/(k+1)}) \tag{4.6}$$

$$\sum_{n \leqq x} F_k^{s*}(n) = \frac{x^{r+1}}{(r+1)\zeta(k+1)} \sum_{n=1}^{\infty} \frac{g(n)\beta_k^*(n)}{n^{r+1}} + O(x^{r+1/(k+1)}), \tag{4.7}$$

where $\beta_k(n)$ *and* $\beta_k^*(n)$ *are given by* (3.5) *and* (3.7) *respectively.*

The proof of this theorem is similar to that of Theorem 4.1.

COROLLARY 4.2.1. $(g(n) = 1)$. *For* $r \geqq 1$, $x \geqq 2$, $k \geqq 1$,

$$\sum_{n \leqq x} \sigma_{r,k}^{*\prime}(n) = \frac{x^{r+1}}{(r+1)\zeta(k+1)} \sum_{n=1}^{\infty} \frac{\beta_k(n)}{n^{r+1}} + O(x^{r+1/(k+1)}) \tag{4.8}$$

$$\sum_{n \leqq x} \sigma_{r,k}^{s*\prime}(n) = \frac{x^{r+1}}{(r+1)\zeta(k+1)} \sum_{n=1}^{\infty} \frac{\beta_k^*(n)}{n^{r+1}} + O(x^{r+1/(k+1)}). \tag{4.9}$$

REMARK 4.4. The case $k = 1$, $r = 1$ in (4.6) and (4.8) has been discussed by Cohen [3] and the case $k = 1$, $r > 1$ in Theorem 4.2 and Corollary 4.2.1 has been discussed by Chidambaraswamy [1].

REMARK 4.5. The coefficients of $x^{r+1}/(r+1)\zeta(k+1)$ in (4.8) and (4.9) can be obtained from (3.13) and (3.14) by taking $s = r + 1$.

COROLLARY 4.2.2. $(g(n) = \mathscr{V}_{k+1}(n))$. *For* $r \geqq 1$, $x \geqq 2$, $k \geqq 1$.

$$\sum_{n \leqq x} \sigma_{r,k}^{*\prime c}(n) = \frac{x^{r+1}}{(r+1)\zeta(k+1)} \sum_{n=1}^{\infty} \frac{\mathscr{V}_{k+1}(n)\beta_k(n)}{n^{r+1}} + O(x^{r+1/(k+1)}). \tag{4.10}$$

REMARK 4.6. The coefficient of $x^{r+1}/(r+1)\zeta(k+1)$ in the above can be obtained from (3.15) by taking $s = r + 1$.

THEOREM 4.3. *If in* (1.2) $f(n)$ *is bounded and* $g(n) = n^r$, *then for* $r \geqq 1$, $x \geqq 2$, $k \geqq 1$,

$$\sum_{n \leqq x} F_k^{s*}(n) = \frac{x^{r+1}}{r+1} \sum_{n=1}^{\infty} \frac{f(n)\psi_k^*(n)}{n^{r+2}} + x^r G_{r,k}(x), \tag{4.11}$$

where $\psi_k^*(n)$ *is given by* (3.17) *and* $G_{r,k}(x)$ *is given by* (3.25).

PROOF. By (3.16),

$$\sum_{n \leqq x} F_k^{s*}(n) = \sum_{n \leqq x} \sum_{\substack{d\delta = n \\ (d,\delta)_k^* = 1}} f(d)\delta^r = \sum_{\substack{d\delta \leqq x \\ (d,\delta)_k^* = 1}} f(d)\delta^r$$

https://doi.org/10.1017/S1446788700012908 Published online by Cambridge University Press

$$= \sum_{d \leq x} f(d) \sum_{\substack{\delta \leq x/d \\ (d,\delta)_k^* = 1}} \delta^r = \sum_{d \leq x} f(d) \psi_{r,k}^* \left(\frac{x}{d}, d\right)$$

$$= \sum_{d \leq x} f(d) \left\{ \frac{1}{r+1} \left(\frac{x}{d}\right)^{r+1} \frac{\psi_k^*(d)}{d} + O\left(x^r \frac{I_k(d)}{d^r}\right) \right\}$$

$$= \frac{x^{r+1}}{r+1} \sum_{n=1}^{\infty} \frac{f(n)\psi_k^*(n)}{n^{r+2}} + O\left(x^{r+1} \sum_{n > x} \frac{\psi_k^*(n)}{n^{r+2}}\right)$$

$$+ O\left(x^r \sum_{n \leq x} \frac{I_k(n)}{n^r}\right),$$

by the boundedness of $f(n)$. Since $\psi_k^*(n) \leqq n$, the first O-term is $O(x)$ by (2.7) and the second O-term is $x^r G_{r,k}(x)$ by (3.25). Hence (4.11) follows.

Corollary 4.3.1. ($f(n) = 1$). *For* $r \geqq 1$, $x \geqq 2$, $k \geqq 1$,

$$\sum_{n \leq x} \sigma_{r,k}^{cs*}(n) = \frac{x^{r+1}}{r+1} \sum_{n=1}^{\infty} \frac{\psi_k^*(n)}{n^{r+2}} + x^r G_{r,k}(x), \tag{4.12}$$

Corollary 4.3.2. ($f(n) = \mathscr{V}_{k+1}(n)$). *For* $r \geqq 1$, $x \geqq 2$, $k \geqq 1$,

$$\sum_{n \leq x} \sigma_{r,k}^{cs*\prime}(n) = \frac{x^{r+1}}{r+1} \sum_{n=1}^{\infty} \frac{\mathscr{V}_{k+1}(n)\psi_k^*(n)}{n^{r+2}} + x^r G_{r,k}(x). \tag{4.13}$$

Remark 4.7. The coefficients of $x^{r+1}/r+1$ in (4.12) and (4.13) can be obtained from (3.21) and (3.22) by taking $s = r+2$.

Theorem 4.4. *If in* (1.1) *and* (1.2) $g(n) = 1$ *and* $f(n) = h(n)/n^r$ *where* $h(n)$ *is bounded, then for* $r \geqq 1$, $x \geqq 2$, $k \geqq 1$,

$$\sum_{n \leq x} F_k^*(n) = x \sum_{n=1}^{\infty} \frac{h(n)\Phi_k(n)}{n^{r+2}} + E_{r,k}(x) \tag{4.14}$$

$$\sum_{n \leq x} F_k^{s*}(n) = x \sum_{n=1}^{\infty} \frac{h(n)\psi_k^*(n)}{n^{r+2}} + G_{r,k}(x). \tag{4.15}$$

The proof of this theorem is similar to the proofs of Theorems 4.1 and 4.3.

Corollary 4.4.1. ($h(n) = 1$). *For* $r \geqq 1$, $x \geqq 2$, $k \geqq 1$,

$$\sum_{n \leq x} \sigma_{-r,k}^*(n) = x \sum_{n=1}^{\infty} \frac{\Phi_k(n)}{n^{r+2}} + E_{r,k}(x) \tag{4.16}$$

$$\sum_{n \leq x} \sigma_{-r,k}^{s*}(n) = x \sum_{n=1}^{\infty} \frac{\psi_k^*(n)}{n^{r+2}} + G_{r,k}(x). \tag{4.17}$$

Remark 4.8. The coefficients of x in the above can be obtained from (2.9) and (3.21) by taking $s = r+2$.

COROLLARY 4.4.2. $(h(n) = \mathscr{V}_{k+1}(n))$. *For* $r \geqq 1$, $x \geqq 2$, $k \geqq 1$,

$$\sum_{n \leqq x} \sigma^{*\prime}_{-r,k}(n) = x \sum_{n=1}^{\infty} \frac{\mathscr{V}_{k+1}(n)\Phi_k(n)}{n^{r+2}} + E_{r,k}(x) \tag{4.18}$$

$$\sum_{n \leqq x} \sigma^{s*\prime}_{-r,k}(n) = x \sum_{n=1}^{\infty} \frac{\mathscr{V}_{k+1}(n)\psi_k^*(n)}{n^{r+2}} + G_{r,k}(x). \tag{4.19}$$

REMARK 4.9. The coefficients of x in the above can be obtained from (2.10) and (3.22) by taking $s = r + 2$.

THEOREM 4.5. *If in* (1.1) $g(n) = \mathscr{V}_{k+1}(n)$ *and* $f(n) = h(n)/n^r$ *where* $h(n)$ *is bounded, then for* $r \geqq 1$, $x \geqq 2$, $k \geqq 1$,

$$\sum_{n \leqq x} F_k^*(n) = \frac{x}{\zeta(k+1)} \sum_{n=1}^{\infty} \frac{h(n)\beta_k(n)}{n^{r+1}} + O\left(x^{1/(k+1)}\right). \tag{4.20}$$

The proof of this theorem is similar to that of Theorem 4.1.

COROLLARY 4.5.1. $(h(n) = 1)$. *For* $r \geqq 1$, $x \geqq 2$, $k \geqq 1$,

$$\sum_{n \leqq x} \sigma^{*c}_{-r,k}(n) = \frac{x}{\zeta(k+1)} \sum_{n=1}^{\infty} \frac{\beta_k(n)}{n^{r+1}} + O\left(x^{1/(k+1)}\right). \tag{4.21}$$

REMARK 4.10. The coefficients of $x/\zeta(k+1)$ in the above can be obtained by taking $s = r + 1$.

COROLLARY 4.5.2. $(h(n) = \mathscr{V}_{k+1}(n))$. *For* $r \geqq 1$, $x \geqq 2$, $k \geqq 1$,

$$\sum_{n \leqq x} \sigma^{*\prime c}_{-r,k}(n) = \frac{x}{\zeta(k+1)} \sum_{n=1}^{\infty} \frac{\mathscr{V}_{k+1}(n)\beta_k(n)}{n^{r+1}} + O\left(x^{1/(k+1)}\right). \tag{4.22}$$

REMARK 4.11. The coefficient of $x/\zeta(k+1)$ in the above can be obtained by taking $s = r + 1$ in (3.15).

THEOREM 4.6. *If in* (1.2) $f(n) = 1$ *and* $g(n) = h(n)/n^r$, *where* $h(n)$ *is bounded, then for* $r \geqq 1$, $x \geqq 2$, $k \geqq 1$,

$$\sum_{n \leqq x} F_k^{s*}(n) = x \sum_{n=1}^{\infty} \frac{h(n)\phi_k^*(n)}{n^{r+2}} + E_{r,k}(x). \tag{4.23}$$

The proof of this theorem is similar to that of Theorem 4.3.

COROLLARY 4.6.1. $(h(n) = 1)$. *For* $r \geqq 1$, $x \geqq 2$, $k \geqq 1$,

$$\sum_{n \leqq x} \sigma^{cs*}_{-r,k}(n) = x \sum_{n=1}^{\infty} \frac{\phi_k^*(n)}{n^{r+2}} + E_{r,k}(x). \tag{4.24}$$

REMARK 4.12. The coefficient of x in the above can be obtained from (2.11) by taking $s = r + 2$.

https://doi.org/10.1017/S1446788700012908 Published online by Cambridge University Press

THEOREM 4.7. *If in* (1.2) $f(n) = \mathscr{V}_{k+1}(n)$ *and* $g(n) = h(n)/n^r$, *where* $h(n)$ *is bounded, then* $r \geqq 1$, $x \geqq 2$, $k \geqq 1$,

$$\sum_{n \leq x} F_k^{s*}(n) = \frac{x}{\zeta(k+1)} \sum_{n=1}^{\infty} \frac{h(n)\beta_k^*(n)}{n^{r+1}} + O(x^{1/(k+1)}). \tag{4.25}$$

The proof of this theorem is similar to that of Theorem 4.3.

COROLLARY 4.7.1. $(h(n) = 1)$. *For* $r \geqq 1$, $x \geqq 2$, $k \geqq 1$,

$$\sum_{n \leq x} \sigma_{-r,k}^{cr*\prime}(n) = \frac{x}{\zeta(k+1)} \sum_{n=1}^{\infty} \frac{\beta_k^*(n)}{n^{r+1}} + O(x^{1/(k+1)}). \tag{4.26}$$

REMARK 4.13. The coefficients of $x/\zeta(k+1)$ in the above can be obtained from (3.14) by putting $s = r+1$.

References

[1] J. Chidambaraswamy, 'Sum functions of unitary and semi-unitary divisors,' *The Journal of the Indian Mathematical Society* 31 (1967), 117–126.

[2] E. Cohen, 'Some totient functions' *Duke Mathematical Journal* 23 (1956), 515–522.

[3] E. Cohen, 'Arithmetical functions associated with the unitary divisiors of an integer,' *Mathematische Zeitzchrift* 74 (1960), 66–80.

[4] L. E. Dickson, *History of the theory of numbers, vol. I*, (Chelsea publishing company, New York (reprinted) 1952).

[5] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, 4*th edn.* (Oxford (1965).

[6] V. L. Klee, 'A generalization of Euler's function', *American Mathematical Monthly* 55 (1948), 358–359.

[7] P. J. McCarthy, 'On a certain family of Arithmetical functions,' *American Mathematical Monthly* 65 (1958), 586–590.

[8] U. V. Satyanarayana and K. Pattabhirama Sastry, 'A note on the generalized ϕ-functions', *Mathematics student* 33 (1965), 81–83.

[9] D. Suryanarayana, 'The number of k-ary divisors of an integer,' *Monatschefte für Mathematik* 72 (1968), 445–450.

[10] D. Suryanarayana, 'Two arithmetic functions and asymptotic densities of related sets', *Portugaliae Mathematica* 31 (1972), 1–11.

[11] D. Suryanarayana, 'The greatest divisor of n which is prime to k', *Mathematics student* 37 (1969), 147–157.

[12] D. Suryanarayana, 'Some theorems concerning the k-ary divisors of an integer' *Mathematics student* 39 (1971), 384–394.

Department of Mathematics
Andhra University
Waltair, India

Department of Mathematics
A. M. Jain College
Meenambakkam
Madras–61, India

https://doi.org/10.1017/S1446788700012908 Published online by Cambridge University Press