ON SANDS' QUESTIONS CONCERNING STRONG AND HEREDITARY RADICALS

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(Received 18 April, 1984)

In [**7**] Sands raised the following questions:

(1) Must a hereditary radical which is right strong be left strong?

(2) Must a right hereditary radical be left hereditary?

(3) (Example 6) Does there exist a right strong radical containing the prime radical β which is not left strong or hereditary?

Negative answers to questions (1) and (2) were given by Beidar [**1**].

In this paper we present different examples to answer (1) and (2), and we answer (3). We prove that the strongly prime radical defined in [**4, 5**] is right but not left strong. In the proof we use an example given by Parmenter, Passman and Stewart [**6**]. The same example and the strongly prime radical are used to answer (2) and (3).

All rings considered are associative, but do not necessarily have a unity. As usual, $I \lhd A$ ($I <_l A, I <_r A$) means that I is an ideal (left ideal, right ideal) of the ring A. The right (left) annihilator of a subset F of a ring R will be denoted by r_RF (l_RF).

The fundamental definitions and properties of radicals may be found in [**2**].

Let us recall that a radical S is said to be *left* (*right*) *strong* [**3**] if every S-semisimple ring contains no non-zero left (right) S-ideals. A radical S is said to be *hereditary* (*left hereditary*, *right hereditary*) if the class S is closed under taking ideals (left ideals, right ideals).

A ring A is said to be (*right*) *strongly prime* if every non-zero ideal I of A contains a finite subset F such that $r_RF = 0$.

The (*right*) *strongly prime radical* U is defined as the upper radical determined by the class of all strongly prime rings, i.e. for any ring R,

$$U(R) = \bigcap \{I \lhd R \mid R/I \text{ is strongly prime}\}.$$

It is known that the radical U is special; so, in particular, U is hereditary and contains the prime radical β.

PROPOSITION 1. *If* $0 \neq J <_r R$ *and the ring* R *is strongly prime then so is the ring* $J/\beta(J)$.

Proof. We prove first that $\beta(J) = l_JJ$. Clearly, $l_JJ \subseteq \beta(J)$. Conversely, $\beta(J)J \subseteq \beta(J)$; so $\beta(J)J \in \beta$. Since $\beta(J)J <_r R$ and R is prime, $\beta(J)J = 0$.

Now suppose that $0 \neq I/\beta(J) \lhd J/\beta(J)$. Since R is prime, $RIJ \neq 0$. Thus $0 \neq RIJ \lhd R$; so there exists a finite subset $F = \{x_1, \ldots, x_n\} \subseteq RIJ$ such that $r_RF = 0$. Let $x_k = \sum r_{kl}i_{kl}j_{kl}$, where $r_{kl} \in R$, $i_{kl} \in I$, $j_{kl} \in J$, and $F_1 = \{i_{kl}j_{kl} \mid k, l$ run over all relevant subscripts$\}$. Obviously $F_1 \subseteq I$. Now if, for some $j \in J\backslash\beta(J)$, $F_1j \subseteq \beta(J)$ then $F_1jJ = 0$. In consequence $FjJ = 0$. Now $jJ \neq 0$ as $j \notin \beta(J)$ and $\beta(J) = l_JJ$. Hence $r_RF \neq 0$, a contradiction.

Theorem 9 of [**3**] and Proposition 1 imply the following result.

COROLLARY 1. *The radical U is right strong.*

Now we shall show that the radical U is not left strong.

EXAMPLE 1. Let G be a free group on countably many generators and K a field on which G acts faithfully. In [6] Parmenter, Passman and Stewart proved that the skew group ring $D = K * G$ is simple and the ring R of all infinite matrices over D containing only finitely many non-zero rows is U-semisimple. Let L be the set of all infinite matrices over D which have only finitely many non-zero entries. It is clear that $L <_l R$. Moreover $L \in U$ as L is simple and, for every finite subset $F \subseteq L$, $r_L F \neq 0$. This shows that the radical U is not left strong.

To construct an example of a right but not left hereditary radical we need the following proposition.

PROPOSITION 2. *If $0 \neq J <_r R$, $0 \neq I/\beta(J) \lhd J/\beta(J)$ and the ring R is simple then $J^2 \subseteq I$.*

Proof. The result is clear when $R^2 = 0$. Thus let R be prime. Then $0 \neq RIJ \lhd R$; so $RIJ = R$ and $JRIJ = JR$. On the other hand $JRIJ \subseteq JIJ \subseteq I$. Hence $JR \subseteq I$ and, in particular, $J^2 \subseteq I$.

EXAMPLE 2. Let R and L be those of Example 1 and let

$$M = \{J \mid \text{there exists a chain } J = J_1 <_r J_2 <_r \dots <_r J_n <_r R\}.$$

It is clear that the class M is right hereditary; so the lower radical l_M determined by M is right hereditary. We shall show that l_M is not left hereditary. Certainly $L <_l R \in l_M$. We claim that $L \notin l_M$. If not, then since the ring L is simple, there exists $J \in M$ and a homomorphism $f: J \to L$ mapping J onto L. If $J = J_1 <_r \dots <_r J_n <_r R$ then $J_1 J_2 \cdots J_n <_r R$, $J^n \subseteq J_1 J_2 \dots J_n \subseteq J$. Also $f(J_1 J_2 \dots J_n) = L$ as $L = L^n = f(J)^n = f(J^n) \subseteq f(J_1 J_2 \dots J_n) \subseteq f(J) = L$. Thus we can assume that $J <_r R$. Now since $\beta(L) = 0$, $\beta(J) \subseteq \operatorname{Ker} f$. Moreover Proposition 2 implies that if $\beta(J) \neq \operatorname{Ker} f$ then $J^2 \subseteq \operatorname{Ker} f$. This is impossible as $J/\operatorname{Ker} f \approx L$ and $L^2 = L$. Thus $\beta(J) = \operatorname{Ker} f$. Now Proposition 1 implies that the ring $J/\beta(J)$ is strongly prime. This is impossible as $J/\beta(J) = J/\operatorname{Ker} f \approx L$ and $L \in U$.

Now we answer question (3).

EXAMPLE 3. Let K, L and R be those of Example 1 and let F be a field with $\operatorname{card} F > \operatorname{card} R$. We claim that the lower right strong radical S determined by U and the polynomial ring $F[x]$ is not left strong or hereditary. To prove that S is not hereditary it suffices to check that $I = xF[x] \notin S$. But if $I \in S$ then [3, Lemma 3] there exists a chain $0 \neq I_0 <_r \dots <_r I_n = I$ such that $I_0 \in U$ or I_0 is a homomorphic image of $F[x]$. The former is impossible because I_0, being a domain, is strongly prime and the latter because I_0 contains no non-zero idempotents.

Now we shall prove that $R \notin S$ which together with the fact that $L \in S$ implies that S is not left strong. For this it is enough to show that if A is a non-zero strongly prime S-ring then $\operatorname{card} A > \operatorname{card} R$. It is known [3, Theorem 2] that $S = \bigcup M_\alpha$, where

$$M_1 = U \cup \{T \mid T \text{ is a homomorphic image of } F[x]\}$$

and, for $\alpha \geqslant 2$,

$M_\alpha = \{T \mid$ every non-zero homomorphic image of T contains a nonzero right ideal in $M_{\alpha'}$ for some $\alpha' < \alpha\}$.

If $A \in M_1$ then, since A is strongly prime, A is a homomorphic image of $F[x]$. Thus in this case card $A =$ card $F >$ card R. If $A \in M_\alpha$ for some $\alpha \geqslant 2$ then A contains a non-zero right ideal $I \in M_{\alpha'}$ for some $\alpha' < \alpha$. By Proposition 1, $I/\beta(I)$ is strongly prime. Since $M_{\alpha'}$ is homomorphically closed, $I/\beta(I) \in M_{\alpha'}$. Thus card $I \geqslant$ card $I/\beta(I) >$ card R. This ends the proof.

The following remark is motivated by the comments to Example 7 of [**7**].

Remark. Let S be the lower radical determined by $M = \{Z, Z^0\}$, where Z is the ring of integers and Z^0 is the zero-ring on the additive group of Z. It is clear that $\beta \subseteq S$. Since the ring $2Z$ of even integers is S-semisimple and $Z \in S$, the radical S is not hereditary. Now let $M_2(Z)$ be the ring of 2×2-matrices over Z. Then $I_1 = \begin{bmatrix} Z & 0 \\ Z & 0 \end{bmatrix} <_l M_2(Z)$ and $I_2 = \begin{bmatrix} Z & Z \\ 0 & 0 \end{bmatrix} <_r M_2(Z)$. It is easy to check that $I_1, I_2 \in S$ and $S(M_2(Z)) = 0$. Hence S is neither left nor right strong.

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