Glasgow Math. J. **47** (2005) 303–307. © 2005 Glasgow Mathematical Journal Trust.
doi:10.1017/S001708950500251X. Printed in the United Kingdom

ON A DIOPHANTINE EQUATION OF CASSELS

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(Received 30 June, 2004; accepted 11 April, 2005)

Abstract. J.W.S. Cassels gave a solution to the problem of determining all instances of the sum of three consecutive cubes being a square. This amounts to finding all integer solutions to the Diophantine equation $y^2 = 3x(x^2 + 2)$. We describe an alternative approach to solving not only this equation, but any equation of the type $y^2 = nx(x^2 + 2)$, with n a natural number. Moreover, we provide an explicit upper bound for the number of solutions of such Diophantine equations. The method we present uses the ingenious work of Wilhelm Ljunggren, and a recent improvement by the authors.

2000 *Mathematics Subject Classification.* 11D25.

1. Introduction. In [**2**], J.W.S. Cassels gave a solution to the problem of determining all instances of the sum of three consecutive cubes being a square. This amounts to finding all integer solutions to the Diophantine equation $y^2 = 3x(x^2 + 2)$. He showed that the nonzero values of x that correspond to solutions are $x = 1, 2, 24$. The proof essentially uses some clever arguments involving arithmetic in certain quartic number fields. In this note we point out that this result, and a generalization thereof, can be deduced from the classical work of Wilhelm Ljunggren. Moreover, by using this approach, we can describe an effective algorithm for finding all solutions to any equation of the form

$$y^2 = nx(x^2 + 2), \tag{1.1}$$

and provide an upper bound for the number of integer solutions to such an equation. The algorithm we present is simplified somewhat by a recent improvement of Ljunggren's work by the authors in [**5**]. We remark that an analogous result was proved by Bennett on Diophantine equations of the form $y^2 = nx(x + 1)(x + 2)$ in [**1**].

In what follows, let n denote a positive integer. There is no loss in generality by assuming that n is square-free, and so we henceforth make this assumption for the statement and proof of the following theorem. Furthermore, we let $\omega(n)$ denote the number of distinct prime factors of n.

THEOREM 1. *Equation* (1.1) *has at most* $3 \cdot 2^{\omega(n)-1}$ *solutions in positive integers* (x, y)*, and there is an effective algorithm to compute all such integer solutions.*

In the case that n is prime, we have the following corollary.

Corollary 1. *For any prime p, the equation* $y^2 = px(x^2 + 2)$ *has at most three solutions in positive integers* (x, y).

Evidently, we see that for $p = 3$, the upper bound of 3 for the number of solutions is attained, and so this corollary provides a new proof of Cassels' result.

Another extension of Cassels' theorem, is to replace the factor $x^2 + 2$ by $x^2 - 2$. Unfortunately, the necessary analogues of the theorems of Ljunggren we use to treat the former case do not exist for this case.

2. Preliminary results. In order to prove Theorem 1, we require three theorems of Ljunggren. The first can be found in [**3**].

Lemma 1. *Let d denote a positive non-square integer, and let* ϵ_d *denote the fundamental unit in the quadratic field* $\mathbf{Q}(\sqrt{d})$*. If the Diophantine equation*

$$X^2 - dY^4 = 1 \tag{1.2}$$

has two positive integer solutions $(X_1, Y_1), (X_2, Y_2), (Y_1 < Y_2)$*, then either*

$$\epsilon_d = X_1 + Y_1^2\sqrt{d}, \epsilon_d^2 = X_2 + Y_2^2\sqrt{d},$$

or

$$\epsilon_d = X_1 + Y_1^2\sqrt{d}, \epsilon_d^4 = X_2 + Y_2^2\sqrt{d},$$

with the latter case occurring for only finitely many d.

For computational purposes, the following improvement to this result was proved in [**7**].

Lemma 2. *Let* $\epsilon_d = T + U\sqrt{d} > 1$ *denote the minimal solution to* $X^2 - dY^2 = 1$*, and for* $k \geq 1$*, let* $T_k + U_k\sqrt{d} = \epsilon_d^k$*. Assume that* $U = lv^2$ *with l square-free. If* $U_k = Y^2$ *for some integer* Y*, then either* $k = 1$*,* $k = 2$*,* $(k, d) = (4, 1785)$*, or* $k = l$ *and l is a prime of the form* $4t + 3$.

This lemma shows that the problem of determining all integer solutions to an equation of the form $X^2 - dY^4 = 1$ essentially amounts to determining the fundamental solution of $X^2 - dY^2 = 1$.

The following was proved by Ljunggren in [**4**], and deals with the more general family of quartic curves

$$aX^2 - bY^4 = 1, \tag{1.3}$$

with a a nonsquare integer.

Lemma 3. *Let* a, b *be positive integers, a non-square, for which the equation* $aX^2 - bY^2 = 1$ *is solvable in positive integers* X, Y*. Let* $\tau_{a,b} = x_1\sqrt{a} + y_1\sqrt{b}$ *denote the biquadratic unit corresponding to the minimal solution of* $aX^2 - bY^2 = 1$*, and put* $\tau_{a,b}^{2k+1} = x_{2k+1}\sqrt{a} + y_{2k+1}\sqrt{b}$*, which represents all positive integer solutions to* $aX^2 - bY^2 = 1$*. Let* $y_1 = lv^2$ *with l a square-free integer. If l is even, then equation* (1.3) *has* *no solutions in positive integers. If l is odd, then the only possible solution to* (1.3) *is* $(X, Y) = (x_l, \sqrt{y_l})$.

We remark to the reader that for each of $l = 1, 3, 5$, there are infinitely many pairs a, b for which equation (1.3) has a solution at the lth power of $\tau_{a,b}$. Furthermore, it was conjectured in [**6**] that (1.3) is not solvable whenever $l > 5$. This remains an open problem.

We state one more result of Ljunggren, proved in [**4**], which essentially solves the Diophantine equation

$$aX^2 - bY^4 = 2. \tag{1.4}$$

LEMMA 4. *Let a, b be odd positive integers for which the equation $aX^2 - bY^2 = 2$ is solvable in odd positive integers X, Y. Let $\tau_{a,b} = (x_1\sqrt{a} + y_1\sqrt{b})/\sqrt{2}$ denote the minimal solution of $aX^2 - bY^2 = 2$, and put $\tau_{a,b}^{2k+1} = (x_{2k+1}\sqrt{a} + y_{2k+1}\sqrt{b})/\sqrt{2}$, which represents all positive integer solutions to $aX^2 - bY^2 = 2$. Let $y_1 = lv^2$ with l a square-free integer. Then the only possible solutions to* (1.4) *are* $(X, Y) = (x_l, \sqrt{y_l})$ *and* $(X, Y) = (x_{3l}, \sqrt{y_{3l}})$.

We remark that in [**5**], we improved upon this result by showing that equation (1.4) is not solvable if $l > 1$, and the only possible solutions arise from $(X, Y) = (x_1, \sqrt{y_1})$and $(X, Y) = (x_3, \sqrt{y_3})$. This is evidently quite useful for computational purposes.

LEMMA 5. *An odd square-free positive integer $n > 1$ has at most $2^{\omega(n)-1}$ divisors d satisfying $d \equiv 3 \pmod 8$.*

Proof. Assume first that $n \not\equiv 1 \pmod 8$. In this case, for each divisor d of n, we have that $d \not\equiv n/d \pmod 8$, and so n has at most $2^{\omega(n)-1}$ divisors which are $3 \pmod 8$. We remark that this argument also shows that for each $i \in \{1, 3, 5, 7\}$, n has at most $2^{\omega(n)-1}$ divisors which are $i \pmod 8$.

Now assume that $n \equiv 1 \pmod 8$, and assume that n has at least one prime divisor p for which $p \not\equiv 1 \pmod 8$, for otherwise the result follows immediately. We partition the set of divisors of n as $S_1 \bigcup S_2$, where $S_1 = \{d \mid d \mid n/p\}$ and $S_2 = \{pd \mid d \in S_1\}$. By the first part of this proof, we see that S_1 has at most $2^{\omega(n/p)-1} = 2^{\omega(n)-2}$ elements which are $3 \pmod 8$. The number of elements in S_2 which are $3 \pmod 8$ is the same as the number of elements in S_1 which are $3p^{-1} \pmod 8$, and by the remark at the end of the previous paragraph, we see that this number is at most $2^{\omega(n/p)-1} = 2^{\omega(n)-2}$. The lemma now follows. □

3. Proof of Theorem 1. Let n be a positive square-free integer, and let x, y be positive integers satisfying

$$y^2 = nx(x^2 + 2).$$

As n is square-free, replacing y/n by w, we see that

$$nw^2 = x(x^2 + 2).$$

Consider first the case in which x is odd. Then there are integers n_1, n_2, u, v for which $n = n_1n_2$, $x = n_1u^2$, and $x^2 + 2 = n_2v^2$. This yields the equation

$$n_2v^2 - n_1^2u^4 = 2.$$

https://doi.org/10.1017/S001708950500251X Published online by Cambridge University Press

Assume that the equation $n_1X^2 - n_2^2Y^4 = 2$ has two solutions in positive integers $(x_1, y_1), (x_2, y_2)$, $(y_1 < y_2)$, which is the maximum possible number of solutions by Lemma 4. Then, by Lemma 4, we have that

$$\left(x_2\sqrt{n_1} + y_2^2\sqrt{n_2^2}\right)/\sqrt{2} = \left(\left(x_1\sqrt{n_1} + y_1^2\sqrt{n_2^2}\right)/\sqrt{2}\right)^3,$$

from which it follows that $y_2^2 = y_1^2(2y_1^4n_2^2 + 3)$, which in turn implies that $2y_1^4n_2^2 + 3$ is the square of an integer. This equation is not possible (mod 9). We therefore conclude that for a given pair of integers n_1, n_2 satisfying $n = n_1n_2$, the equation $n_1X^2 - n_2^2Y^4 = 2$ has at most one solution in odd positive integers X, Y. If such an equation is solvable, then necessarily $n_1 \equiv 3 \pmod 8$. Therefore, by Lemma 5, the total number of solutions to $y^2 = nx(x^2 + 2)$, with x odd, is at most $2^{\omega(n)-1}$.

Now consider the case in which a solution to $y^2 = nx(x^2 + 2)$ has x is even, and put $x = 2z$. As before, since n is square-free, we shall replace $y/2n$ by w to get

$$nw^2 = z(2z^2 + 1).$$

In this case there are integers n_1, n_2, u, v for which $n = n_1n_2$, $z = n_1u^2$, and $2z^2 + 1 = n_2v^2$, which yields the equation

$$n_2v^2 - 2n_1^2u^4 = 1.$$

By Lemma 3, if $n_2 > 1$, there can be at most one solution. On the other hand, if $n_2 = 1$, Lemma 1 states that there are at most two solutions, but we shall show that in fact there can be at most one solution. If two solutions exist then, by Lemma 1, they must correspond to the fundamental unit in $\mathbf{Q}(\sqrt{2n_1^2}) = \mathbf{Q}(\sqrt{2})$, and either its square or fourth power. But the fundamental unit in the ring of integers of this field is $1 + \sqrt{2}$, which has norm -1, and hence cannot give rise to a solution of $n_2v^2 - 2n_1^2u^4 = 1$. Therefore, in total, there are at most $2^{\omega(n)}$ solutions to $y^2 = nx(x^2 + 2)$ with x even. Combining this bound with the bound for solutions with x odd yields a total bound of $3 \cdot 2^{\omega(n)-1}$ solutions to (1.1) in positive integers x, y.

The effectiveness of this result stems from the fact that all integer solutions to the equation $y^2 = nx(x^2 + 2)$ arise from specific powers of certain units in either quadratic, or biquadratic, number fields whose discriminant depends entirely on the integer n. This completes the proof of Theorem 1.

ACKNOWLEDGEMENTS. The first author was supported in part by the grant SEP-CONACyT 37259E. The second author gratefully acknowledges support from the Natural Sciences and Engineering Research Council of Canada.

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