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A PERMUTABILITY PROBLEM IN INFINITE GROUPS AND RAMSEY'S THEOREM

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We use Ramsey's theorem to generalise a result of L. Babai and T.S. Sós on Sidon subsets and then use this to prove that for an integer $n > 1$ the class of groups in which every infinite subset contains a rewritable n-subset coincides with the class of groups in which every infinite subset contains n mutually disjoint non-empty subsets $X_1, \ldots, X_n$ such that $X_1 \cdots X_n \cap X_{\sigma(1)} \cdots X_{\sigma(n)} \neq \emptyset$ for some non-identity permutation σ on the set $\{1, \ldots, n\}$.

INTRODUCTION AND RESULTS

In [4], Babai and Sós called a subset S of a group G a Sidon subset of the first (second) kind, if for any $x, y, z, w \in S$ of which at least 3 are different, $xy \neq zw$ ($xy^{-1} \neq zw^{-1}$, respectively). Among other things, they proved [4, Proposition 8.1] that an infinite subset of a group contains an infinite subset which is a Sidon subset of both kinds simultaneously.

We generalise the above definition as follows:

Let n be a positive integer greater than 1 and $\alpha_1, \ldots, \alpha_{2n}$ be non-zero integers. We say that a subset S of a group is a Sidon subset of kind $(\alpha_1, \ldots, \alpha_{2n})$ if and only if for any $x_1, \ldots, x_{2n} \in S$ of which at least $n+1$ are different, $x_1^{\alpha_1} \cdots x_n^{\alpha_n} \neq x_{n+1}^{\alpha_{n+1}} \cdots x_{2n}^{\alpha_{2n}}$. Thus in our terminology every Sidon set of kind $(1, 1, 1, 1)$ ($(1, -1, 1, -1)$) is a Sidon set of the first (respectively, second) kind.

Using Ramsey's Theorem [14], we prove the following which generalises [4, Proposition 8.1].

THEOREM A. *Let $n > 1$ be an integer, $\alpha_1, \ldots, \alpha_{2n}$ be non-zero integers and X be an infinite subset of a group such that for all $i \in \{1, \ldots, 2n\}$ and for all distinct elements $x, y \in X$, $x^{\alpha_i} \neq y^{\alpha_i}$. Then X contains an infinite Sidon subset of kind $\left(\alpha_{f(1)}, \ldots, \alpha_{f(2k)}\right)$ for all $k \in \{2, \ldots, n\}$ and for all functions $f : \{1, \ldots, 2k\} \to \{1, \ldots, 2n\}$ simultaneously.*

B.H. Neumann proved in [13] that a group is centre-by-finite if and only if every infinite subset of the group contains a pair of commuting elements. Extensions of problems of this type are to be found in [10] and [12]. The notion of commutativity was extended

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to rewritable products in [7] with a complete description obtained in [9]. Detailed study of rewritable groups may be found in [5] and [6]. In [8], the authors introduced a class R_n of groups, where n is an integer greater than 1, as follows: a group G is called an R_n group if every infinite subset X of G contains a subset $\{x_1,\dots,x_n\}$ of n elements such that $x_1,\dots,x_n = x_{\sigma(1)},\dots,x_{\sigma(n)}$ for some non-identity permutation σ on the set $\{1,\dots,n\}$. They proved there that a group G is an R_n group for some integer n if and only if G has a normal subgroup F such that G/F is finite, F is an FC-group and the exponent of $F/Z(F)$ is finite.

In [1] and [2] we considered the following condition on a group. Let $n > 1$ be an integer. A group G is called restricted (∞, n)-permutable if and only if for all n infinite subsets $X_1,\dots,X_n$ of G there exists a non-identity permutation σ on the set $\{1,\dots,n\}$ such that $X_1\cdots X_n \cap X_{\sigma(1)}\cdots X_{\sigma(n)} \neq \emptyset$, where, as usual, for any non-empty subsets $Y_1,\dots,Y_k$ of a group, $Y_1\cdots Y_k$ equals the set of products $y_1\cdots y_k$ where $y_1 \in Y_1,\dots,y_k \in Y_k$.

In [1] we showed that every infinite restricted $(\infty,2)$-permutable group is Abelian and in [2] we extended this result by proving that every restricted (∞,n)-permutable group is n-permutable for all integers $n > 1$. Also in [3] we considered a class $\overline{Q_n}$ of groups which is defined as follows: a group G is a $\overline{Q}_n$-group if for all n infinite subsets $X_1,\dots,X_n$ of G there exists two distinct permutations σ,τ on the set $\{1,\dots,n\}$ such that $X_{\tau(1)}\cdots X_{\tau(n)} \cap X_{\sigma(1)}\cdots X_{\sigma(n)} \neq \emptyset$. We proved in [3] that every infinite $\overline{Q_n}$-group is n-rewritable.

Now, for every integer n greater than 1, we consider another class of groups called $\overline{R}_n$-groups and defined as follows: a group G is called an $\overline{R}_n$-group if every infinite subset X of G contains n mutually disjoint non-empty subsets $X_1,\dots,X_n$ such that $X_1\cdots X_n \cap X_{\sigma(1)}\cdots X_{\sigma(n)} \neq \emptyset$ for some non-identity permutation σ on the set $\{1,\dots,n\}$. Clearly, every R_n-group is an $\overline{R}_n$-group and we use Theorem A to prove that the converse also holds; namely we have:

Theorem B. *For every integer n greater than 1, the class R_n coincides with the class $\overline{R}_n$.*

Proofs

To prove Theorem A we need Ramsey's Theorem [14].

Proof of Theorem A: Let $k \in \{2,\dots,n\}$, let r be an arbitrary element of the set $\{1,\dots,k\}$ and let Y be an infinite subset of X. Suppose that f is a function from the set $\{1,\dots,2k\}$ to the set $\{1,\dots,2n\}$ and put $\varepsilon_i = \alpha_{f(i)}$ for all $i \in \{1,\dots,2k\}$. List the elements of Y as $x_1, x_2,\dots$ under some well order $\leqslant$ so that $x_i < x_j$ if $i < j$. Consider the set $Y^{(k+r)}$ of all $(k+r)$-element subsets of Y. For each $s \in Y^{(k+r)}$, list the elements $x_{i_1},\dots,x_{i_{k+r}}$ of s in the ascending order given by $\leqslant$ and write $\overline{s} = (x_{i_1},\dots,x_{i_{k+r}})$. Now let $(t_1, t_2,\dots,t_{2k})$ be a $2k$-tuple of elements of $\{1,2,\dots,k+r\}$ such that $\left|\{t_1,\dots,t_{2k}\}\right| = k+r$,

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and let T_r be the set of all such $2k$-tuples. Define $|T_r|+1$ sets, one $U_t(Y)$ for each element t of T_r and $V_r(Y)$, as follows. For each $s \in Y^{(k+r)}$, $\overline{s} = (x_{i_1}, \dots, x_{i_{k+r}})$, put $s \in U_t(Y)$ if $x_{i_{t_1}}^{\varepsilon_1} \cdots x_{i_{t_k}}^{\varepsilon_k} = x_{i_{t_{k+1}}}^{\varepsilon_{k+1}} \cdots x_{i_{t_{2k}}}^{\varepsilon_{2k}}$ and put s in $V_r(Y)$ if $s \notin U_t(Y)$ for any t. By Ramsey's Theorem, there exists an infinite subset $Z \subseteq Y$ such that $Z^{(k+r)} \subseteq U_t(Y)$ for some t or $Z^{(k+r)} \subseteq V_r(Y)$. By restricting the order $\leqslant$ to Z, we may assume that $Z = \{x_1, x_2, \dots\}$ and $x_i < x_j$ if $i < j$. Suppose, if possible, that $Z^{(k+r)} \subseteq U_t(Y)$ for some $t = (t_1, \dots, t_{2k}) \in T_r$. Hence for any $i_1 < i_2 < \dots < i_{k+r}$,

$$x_{i_{t_1}}^{\varepsilon_1} \cdots x_{i_{t_k}}^{\varepsilon_k} = x_{i_{t_{k+1}}}^{\varepsilon_{k+1}} \cdots x_{i_{t_{2k}}}^{\varepsilon_{2k}}.$$

Since $\left|\{t_1, \dots, t_{2k}\}\right| = k + r > k$, there exists $l \in \{1, \dots, 2k\}$ such that $t_l \neq t_h$ for any $h \in \{1, \dots, 2k\} \backslash \{l\}$. Thus we may write $x_{i_{t_l}}^{\varepsilon_l}$ as a product of $x_{i_{t_h}}^{\pm\varepsilon_h}$'s where $t_h \in \{t_1, \dots, t_{2k}\} \backslash \{t_l\}$. Now, since Z is infinite there exist sequences $j_1 < j_2 < \dots < j_{k+r}$ and $p_1 < p_2 < \dots < p_{k+r}$ such that $j_{t_l} \neq p_{t_l}$ and $j_{t_h} = p_{t_h}$ when $h \in \{1, \dots, 2k\} \backslash \{l\}$. Therefore $x_{j_{t_l}}^{\varepsilon_l} = x_{p_{t_l}}^{\varepsilon_l}$, contrary to the hypothesis on the set X. Hence $Z^{(k+r)} \subseteq V_r(Y)$.

Now, list the elements of X as $x_1, x_2, \dots$ under some well order $\leqslant$ so that $x_i < x_j$ if $i < j$. Replace Y by X and put $r = 1$ in the above argument, then there is an infinite subset X_1 of X such that $X_1^{(k+1)} \subseteq V_1(X)$. By restricting the order $\leqslant$ to each infinite subset of X, the above process yields a chain of infinite subsets $X_k \subseteq \dots \subseteq X_1 \subseteq X_0 = X$ such that $X_i^{(k+i)} \subseteq V_i(X_{i-1})$ for all $i = 1, 2, \dots, k$. Thus X_k is a Sidon subset of kind $(\varepsilon_1, \dots, \varepsilon_{2k})$. Since the set of such $2k$-tuples is finite, where k ranges over the set $\{2, \dots, n\}$, the proof is complete. □

As we mentioned before, Theorem A generalises [**4**, Proposition 8.1]. In fact we have:

Corollary 1. *Let G be an infinite group, $n > 1$ an integer and $\varepsilon_1, \dots, \varepsilon_{2n} \in \{-1, 1\}$. Then every infinite subset of G contains a Sidon subset of kind $\left(\varepsilon_{f(1)}, \dots, \varepsilon_{f(2k)}\right)$ for all $k \in \{2, \dots, n\}$ and for all functions $f : \{1, \dots, 2k\} \to \{1, \dots, 2n\}$ simultaneously.*

Using the fact that in a torsion-free nilpotent group no two distinct elements can have the same non-zero power (for example see [**11**, Theorem 16.2.8]), we also have the following corollary of Theorem A on nilpotent groups.

Corollary 2. *Let G be a nilpotent group and let T be the torsion subgroup of G. If $n > 1$ and $\alpha_1, \dots, \alpha_{2n}$ are non-zero integers, then every infinite subset X of G such that $xT \neq yT$ for all distinct elements $x, y \in X$, contains a Sidon set of kind $\left(\alpha_{f(1)}, \dots, \alpha_{f(2k)}\right)$ for all $k \in \{2, \dots, n\}$ and for all functions $f : \{1, \dots, 2k\} \to \{1, \dots, 2n\}$ simultaneously.*

To prove Theorem B we need only to prove that every infinite $\overline{R}_n$-group is an R_n-group for which we use Theorem A.

Proof of Theorem B: Let G be an infinite $\overline{R}_n$-group. Let X be an infinite subset of G. By Corollary 1, X contains an infinite Sidon subset X_0 of kind $(\overbrace{1, \dots, 1}^{2n})$. Now since $G \in \overline{R}_n$, X_0 contains mutually disjoint non-empty subsets $X_1, \dots, X_n$ such that

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$X_1 \cdots X_n \cap X_{\sigma(1)} \cdots X_{\sigma(n)} \neq \emptyset$ for some non-identity permutation σ on the set $\{1, \ldots, n\}$. Therefore there exist elements $x_1, \ldots, x_{2n} \in X_0$ of which at least n are distinct, such that $x_1 \cdots x_n = x_{n+1} \cdots x_{2n}$, $(x_1, \ldots, x_n) \neq (x_{n+1}, \ldots, x_{2n})$, and $x_1, \ldots, x_n$ are distinct as well as $x_{n+1}, \ldots, x_{2n}$. Thus by the property of the set X_0, we must have $\{x_1, \ldots, x_n\} = \{x_{n+1}, \ldots, x_{2n}\}$. Therefore there exists a non-identity permutation τ on the set $\{1, \ldots, n\}$ such that $x_{n+i} = x_{\tau(i)}$ for all $i \in \{1, \ldots, n\}$, and so $x_1 \cdots x_n = x_{\tau(1)} \cdots x_{\tau(n)}$. This completes the proof. ∎

By [**8**, Theorem A] and Theorem B, we obtain the following corollary which generalises the key lemmas of [**2**] and [**3**] ([**2**, Lemma 2.3] and [**3**, Lemma 4]).

Corollary 3. *A group G is an $\overline{R}_n$-group for some integer $n > 1$ if and only if G has a normal subgroup F such that G/F is finite, F is an FC-group and the exponent of $F/Z(F)$ is finite.*

Lastly in this paper we consider another class of groups. Let $n > 1$ be an integer and $m > 0$ be a cardinality of a countable set (finite or infinite). We say that a group G is an $R^*(n, m)$-group if and only if every infinite set of m-sets (a set with cardinality m is said to be an m-set) in G contains n distinct members $X_1, \ldots, X_n$ such that

$$X_1 \cdots X_n \cap X_{\sigma(1)} \cdots X_{\sigma(n)} \neq \emptyset$$

for some non-identity permutation σ on the set $\{1, \ldots, n\}$. It is natural to ask what are the relations between this class of groups and $\overline{R}_n$-groups. In fact we have:

Proposition 4. *$R^*(n, m) = R_n$ for all n and m.*

Proof: By Theorem B, it is enough to prove $R^*(n, m) = \overline{R}_n$. Suppose, for a contradiction, that G is an infinite $R^*(n, m)$-group which is not in $\overline{R}_n$. Thus there exists an infinite subset X of G such that for every n mutually disjoint non-empty subsets $X_1, \ldots, X_n$ and for all non-identity permutation σ on the set $\{1, 2, \ldots, n\}$, we have $X_1 \cdots X_n \cap X_{\sigma(1)} \cdots X_{\sigma(n)} = \emptyset$. Since X is infinite and m is the cardinality of a countable set, there exists an infinite set of mutually disjoint m-sets of X. Now the existence of the latter set contradicts the property $R^*(n, m)$. Conversely, suppose that $G \in \overline{R}_n = R_n$. Let $\mathcal{X}$ be an infinite set of m-sets of G. If two distinct members X_1, X_2 of $\mathcal{X}$ intersect in an element x then by considering $n - 2$ other arbitrary different members $X_3, \ldots, X_n$ of $\mathcal{X}$, we have $X_1 X_2 \cdots X_n \cap X_2 X_1 X_3 \cdots X_n \neq \emptyset$, so we may assume that the members of $\mathcal{X}$ are mutually disjoint. Thus by choosing one element from each member of $\mathcal{X}$, we obtain an infinite set X such that each element of X belongs to one and only one member of $\mathcal{X}$. Now since $G \in R_n$, there exist n elements $x_1, \ldots, x_n$ and a permutation σ on the set $\{1, 2, \ldots, n\}$ such that $x_1 \cdots x_n = x_{\sigma(1)} \cdots x_{\sigma(n)}$. Now by the choice of X for each $i \in \{1, 2, \ldots, n\}$, there exists an element $X_i \in \mathcal{X}$ such that $x_i \in X_i$; that is, there exist $X_1, X_2, \ldots, X_n \in \mathcal{X}$ such that

$$x_1 \cdots x_n \in X_1 X_2 \cdots X_n \cap X_{\sigma(1)} \cdots X_{\sigma(n)},$$

so that $G \in R^*(n, m)$. This completes the proof. ∎

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