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# *A priori* bounds for periodic solutions of a class of Hamiltonian systems

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*Abstract.* This paper concerns estimates for periodic solutions of a very general class of Hamiltonian systems of prescribed energy. The estimtes are *a priori* upper and lower bounds for the action integral in terms of the period.

During the past few years, several papers have studied the existence and multiplicity of periodic solutions of a Hamiltonian system on a prescribed energy surface (see [**1**]-[**10**] and [**12**]-[**17**]). One of the difficulties in treating this question is that the period of such a solution is not known *a priori.* This note contains simple upper and lower *a priori* bounds for the period for a class of such problems which contains in particular most of the cases considered in [**1**]-[**10**] and [**12**]-[**17**]. (The remaining cases such as in [**12**] can be treated even more easily.)

To state these estimates more precisely, let $H \in C^1(R^{2n}, R)$ and $p, q \in R^n$. The corresponding Hamiltonian system is

$$\text{(i)}\ \ \dot{p} = -H_q(p, q), \qquad \text{(ii)}\ \ \dot{q} = H_p(p, q). \tag{1}$$

To normalize matters, suppose we are interested in periodic solutions of (1) of energy 1, i.e. the solutions lie on $\mathscr{D} \equiv H^{-1}(1)$. Assume there exists such a solution whose period is $T(>0)$. It is convenient to make the dependence of the equation on the period explicit. Therefore rescaling the time variable by $t \to 2\pi T^{-1} t \equiv \lambda^{-1} t$, the period becomes $2\pi$ and (1) transforms to

$$\text{(i)}\ \ \dot{p} = -\lambda H_q(p, q), \qquad \text{(ii)}\ \ \dot{q} = \lambda H_p(p, q). \tag{2}$$

Let $z = (p, q)$. The action integral associated with (2) is

$$A(z) \equiv \int_0^{2\pi} p \cdot \dot{q}\, dt.$$

Our main result is the following:

THEOREM. *Suppose* $H \in C^1(R^{2n}, R)$ *and satisfies* $(\mathrm{H}_1)$ $\mathscr{D}$ *is the boundary of a compact neighbourhood of* 0 *in* $R^{2n}$ *and* $H_z(\equiv (H_p, H_q)) \neq 0$ *on* $\mathscr{D}$, $(\mathrm{H}_2)$ $p \cdot H_p(z) > 0$ *for all* $(p, q) \in R^n$, $p \neq 0$. *If* $z = (p, q)$ *is a* $2\pi$ *periodic solution of* (2) *on* $\mathscr{D}$, *then there exist constants* $\underline{a}, \bar{a} > 0$ *and independent of* $z$ *such that*

$$0 < \underline{a} A(z) \leq \lambda \leq \bar{a} A(z). \tag{3}$$

*Remarks.* (i) Note that if $H(p, q) = K(p, q) + V(q)$, where the potential energy $V$ satisfies $\mathscr{R} \equiv \{q \in R^n \mid V(q) \leq 1\}$ is compact and $V_q(q) \neq 0$ on $\partial\mathscr{R}$, and the kinetic

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energy $K$ satisfies $K(0, q)=0$, $p \cdot K_p(p, q)>0$ if $(p, q) \in R^n$ and $p \neq 0$, and e.g. $K(\alpha p, q) \to \infty$ as $|\alpha| \to \infty$ uniformly for $q \in \mathcal{R}$ and $p \in S^{n-1}$, then $H$ satisfies ($H_1$) and ($H_2$). This special case contains the situations studied e.g. in [**2**], [**7**], [**9**], [**13**] and [**15**]-[**17**].

(ii) The upper and lower bounds for $\lambda$ involve the action integral. Experience has shown that such estimates are intimately related to corresponding existence results for (1). We conjecture that if ($H_1$) and ($H_2$) hold, there exists a periodic solution of (1) on $\mathcal{D}$.

*Proof of the theorem.* We will first prove (3) for the technically simpler case of $H \in C^2(R^{2n}, R)$ and then show how the argument can be modified if $H \in C^1$. Note first that if $z(t)$ is a solution of (2), $p(t)$ cannot vanish identically on any open interval $\mathcal{I}$, for then by ($H_2$) $H_p(0, q(t))=0$ for $t \in \mathcal{I}$ and $(0, q(t)) \in \mathcal{D}$. Consequently $H_q(0, q(t)) \neq 0$ for $t \in \mathcal{I}$ by ($H_1$) contrary to (2)(i). It follows that

$$\int_0^{2\pi} p(t) \cdot H_p(z(t))\, dt > 0.$$

Taking the scalar product of (2)(ii) with $p$ and integrating yields

$$A(z)=\lambda \int_0^{2\pi} p \cdot H_p(z)\, dt. \tag{4}$$

Hence $A(z)>0$ and

$$A(z) \leq 2\pi\lambda \max_{(\xi,\eta) \in \mathcal{D}} \xi \cdot H_p(\xi, \eta). \tag{5}$$

The lower bound for $\lambda$ now follows from (5) and ($H_2$).

Next, taking the scalar product of (2)(i) with $H_q(z)$ shows

$$\begin{aligned}
-\lambda \int_0^{2\pi} |H_q|^2\, dt &= \int_0^{2\pi} \dot{p} \cdot H_q\, dt \\
&= -\int_0^{2\pi} p \cdot (H_{qp}\dot{p} + H_{qq}\dot{q})\, dt \\
&= -\lambda \int_0^{2\pi} p \cdot (H_{qq}H_p - H_{qp}H_q)\, dt
\end{aligned}$$

or

$$0=\lambda \int_0^{2\pi} [|H_q|^2 + p \cdot (H_{qp}H_q - H_{qq}H_p)]\, dt. \tag{6}$$

Multiplying (6) by a parameter $b$ to be chosen later and adding to (4) gives

$$A(z)=\lambda \int_0^{2\pi} [p \cdot H_p + b|H_q|^2 + bp \cdot (H_{qp}H_q - H_{qq}H_p)\, dt. \tag{7}$$

By ($H_1$) and ($H_2$) again, there is a constant $\gamma>0$ such that

$$|H_q(0, \eta)| \geq \gamma \qquad \text{if } (0, \eta) \in \mathcal{D}.$$

Therefore, by the continuity of $H_q$, there is a constant $\sigma>0$ such that

$$|H_q(\xi, \eta)| \geq \gamma/2 \qquad \text{if } (\xi, \eta) \in \mathcal{D} \text{ and } |\xi| \leq \sigma. \tag{8}$$

Decreasing $\sigma$ if necessary, it can also be assumed that

$$|\xi\cdot(H_{qp}(\zeta)H_q(\zeta)-H_{qq}(\zeta)H_p(\zeta))|\le\gamma^2/8 \tag{9}$$

if $\zeta=(\xi,\eta)\in\mathscr{D}$ and $|\xi|\le\sigma$. Let $T_1\equiv\{t\in[0,2\pi]\,|\,|p(t)|\le\sigma\}$, $T_2\equiv[0,2\pi]\backslash T_1$ and $l\equiv|T_1|$, where $|B|$ denotes the Lebesgue measure of the set $B$. By ($H_2$), (8) and (9),

$$\begin{aligned} I_1&\equiv\int_{T_1}[p\cdot H_p+b|H_q|^2+bp\cdot(H_{qp}H_q-H_{qq}H_p)]\,dt\equiv\int_{T_1}\mathscr{H}\,dt\\ &\ge b\left(\frac{\gamma^2}{4}-\frac{\gamma^2}{8}\right)l=\frac{b\gamma^2 l}{8}. \end{aligned} \tag{10}$$

Letting

$$M_1=\max_{\zeta=(\xi,\eta)\in\mathscr{D}}|\xi\cdot(H_{qp}(\zeta)H_q(\zeta)-H_{qq}(\zeta)H_p(\zeta))|$$

and

$$\omega\equiv(2M_1)^{-1}\min_{\zeta\in\mathscr{D},|\xi|\ge\sigma}\xi\cdot H_p(\zeta),$$

it follows that

$$I_2\equiv\int_{T_2}\mathscr{H}\,dt\ge(2\pi-l)M_1(2\omega-b). \tag{11}$$

Choosing $b=\omega$ yields

$$I_2\ge(2\pi-l)M_1\omega. \tag{12}$$

Since

$$\lambda^{-1}A(z)=I_1+I_2, \tag{13}$$

(10) and (12) imply

$$\lambda^{-1}A(z)\ge\omega\left(\frac{\gamma^2}{8}l+(2\pi-l)M_1\right)\ge2\pi\omega\min\left(\frac{\gamma^2}{8},M_1\right) \tag{14}$$

and the upper bound for $\lambda$ in (3) follows from (14). Thus (3) is proved for $H\in C^2$.

An examination of the above argument shows $\underline{a}$ depends on $C^1$ bounds for $H$ on $\mathscr{D}$ while $\bar{a}$ depends on $M_1$ and therefore on $C^2$ bounds for $H$ on $\mathscr{D}$. Consequently a better upper bound for $\lambda$ is needed to establish the $C^1$ version of (3). Let $W(z)$ be a $C^1$ function in a neighbourhood of $\mathscr{D}$ with values in $R^n$. Taking the scalar product of (1)(i) with $W$ and arguing as in (6) yields

$$0=\lambda\int_0^{2\pi}[H_q(z)\cdot W(z)+p\cdot(W_p(z)H_q(z)-W_q(z)H_p(z))]\,dt. \tag{15}$$

Suppose $W$ satisfies

$$W(\zeta)\cdot H_q(\zeta)\ge\gamma^2 \tag{16}$$

if $\zeta=(0,\eta)\in\mathscr{D}$. Then choosing $\sigma$ so that

$$W(\zeta)\cdot H_p(\zeta)\ge\gamma^2/4$$

if $\zeta=(\xi,\eta)\in\mathscr{D}$ and $|\xi|\le\sigma$ and arguing as in (9)–(14) gives

$$\lambda^{-1}A(z)\ge2\pi\bar{\omega}\min(\gamma^2/8,\bar{M}_1), \tag{17}$$

where

$$\bar{M}_1 \equiv \max_{\zeta=(\xi,\eta)\in\mathscr{D}} |\xi \cdot (W_p(\zeta)H_q(\zeta) - W_q(\zeta)H_p(\zeta))|$$

and

$$\bar{\omega} = (2\bar{M}_1)^{-1} \min_{\zeta\in\mathscr{D}, |\xi|\geq\sigma} \xi \cdot H_p(\zeta).$$

Thus (3) is established for this case provided that there exists a $C^1$ function $W$ satisfying (16).

A simple way to obtain $W$ is to use a notion due to Palais [11]. If $E$ is a real Banach space, $\mathcal{O} \subset E$ is open and $\Phi \in C^1(\mathcal{O}, R)$, then $w \in E$ is said to be a pseudogradient vector for $\Phi$ at $e \in \mathcal{O}$ if

$$\text{(i)}\ \|w\|_E \leq 2\|\Phi'(e)\|_{E^*}, \qquad \text{(ii)}\ \langle \Phi'(e), w\rangle_{E^*,E} \geq \|\Phi'(e)\|^2_{E^*}, \tag{18}$$

where $\langle\cdot,\cdot\rangle$ denotes the pairing between $E^*$ and $E$. If $\Phi \in C^1(E, R)$, $\tilde{E} = \{e \in E \mid \Phi'(e) = 0\}$, $W(e)$ is a pseudogradient vector for $\Phi$ for all $e \in \tilde{E}$ and $W$ is locally Lipschitz continuous on $\tilde{E}$, then $W(\cdot)$ is called a pseudogradient vector field on $\tilde{E}$. Palais proved [11]:

LEMMA. *If $\Phi \in C^1(E, R)$, there exists a pseudogradient vector field $W$ for $\Phi$ on $\tilde{E}$.*

Choosing $E = R^n$ and $\Phi = H(0, q)$, this lemma implies there is a pseudogradient vector field $W(q)$ for $H(0, q)$ on $\tilde{E}$. By $(H_1)$, $|H_q(\zeta)| \geq \gamma > 0$ for $\zeta = (0, \eta) \in \mathscr{D}$. Hence $\tilde{E} \cap \{q \in R^n \mid (0, q) \in \mathscr{D}\} = \varnothing$ and by (18)(ii)

$$H_q(0, \eta) \cdot W(\eta) \geq |H_q(0, \eta)|^2 \geq \gamma^2$$

for $(0, \eta) \in \mathscr{D}$ so (16) holds. Finally the proof of the lemma shows that if one uses a smooth partition of unity in the construction given there, $W$ is smooth and in particular can be assumed to be $C^1$.

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