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BRUHAT ORDER AND TRANSFER FOR COMPLEX REDUCTIVE GROUPS

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ABSTRACT. Let G be a complex reductive group, and $G^\wedge$ its set of irreducible admissible representations. The *Bruhat order* on $G^\wedge$ is defined in a natural way. We prove that this Bruhat order is preserved by transfer. This gives new proofs of some results by the author on L-functions.

1. **Introduction and statement of results.** Let us consider a complex reductive algebraic connected group G with Lie algebra $\mathfrak{g}$ and maximal compact subgroup K, and $G^\wedge$ the set of (equivalence classes) of its irreducible $(\mathfrak{g}_\mathbb{C}, K)$-modules. The Langlands classification establishes a one to one correspondence between $G^\wedge$ and the set of ${}^LG^0$-conjugacy classes in the set $\Phi(G)$ of admissible morphisms of the Weil group $W_\mathbb{C} = \mathbb{C}^\times$ into the *dual group* ${}^LG^0$. Elements of $\Phi(G)$ are called *sections of the L-group*. Let ϕ be an element of $\Phi(G)$: we associate to ϕ a principal series representation $\tau(\phi)$ and its distinguished irreducible subquotient $\pi(\phi)$.

We now define the *Bruhat order* on $\Phi(G)/{}^LG^0$ as follows:

$$\phi_1 \prec \phi_2 \text{ if and only if } \pi(\phi_2) \text{ is a subquotient of } \tau(\phi_1).$$

Alternatively, the Bruhat order can be defined on $G^\wedge$ by

$$\pi(\phi_1) \prec \pi(\phi_2) \text{ if and only if } \pi(\phi_2) \text{ is a subquotient of } \tau(\phi_1).$$

The equivalence relation generated by the Bruhat order is what Vogan ([V1], see also [A-V]) calls *block equivalence*. As is well known (the details will be provided below), the composition series of $\tau(\phi)$ is in one to one correspondence with the Weyl group W of G (at least in the regular integral case—things are slightly more complicated in the general case), so that our Bruhat order coincides with the usual Bruhat order on W.

Let us now consider another group G' such that there is a rational homomorphism r of ${}^LG^0$ into ${}^LG'^0$. This induces a map (*transfer*) from $\Phi(G)$ into $\Phi(G')$ which we call $\Phi(r)$. We also call $\Phi(r)$ the induced map from the set of ${}^LG^0$-conjugacy classes in $\Phi(G)$ into the set of ${}^LG'^0$-conjugacy classes in $\Phi(G')$—and also from $G^\wedge$ to $G'^\wedge$. The main result of this paper is the following.

THEOREM. *The map $\Phi(r)$ from $\Phi(G)/{}^LG^0$ to $\Phi(G')/{}^LG'^0$ is increasing with respect to the Bruhat orders in $\Phi(G)/{}^LG^0$ and $\Phi(G')/{}^LG'^0$.*

Let us spell out a different form of the theorem.

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THEOREM$'$. *The map r, considered from $G^\wedge$ to $G'^\wedge$, has the following property: Let π_1 and π_2 be in $G^\wedge$ with π_2 a subquotient in the principal series whose Langlands subquotient is π_1. Then $r(\pi_2)$ is a subquotient in the principal series whose Langlands subquotient is $r(\pi_1)$.*

Our motivation for asking this question comes from the study of the L-functions associated to representations of G (or alternatively to sections of the L-group). We specialize now the previous situation to the case $G' = \mathrm{GL}(N, \mathbb{C})$, so that ${}^LG'^0 = \mathrm{GL}(N, \mathbb{C})$ and r is a rational N-dimensional representation of the L-group.

For each π in $G^\wedge$, we can consider the L-function $L(\pi, r, s)$. In [A], we proved that $\pi_1 \prec \pi_2$ implies that the quotient $L(\pi_2, r, s)/L(\pi_1, r, s)$ is an entire function. We also proved that the quotients $L(\pi_i, r, s)/L(\pi_i^\vee, r, 1-s)$ (where $\pi_i^\vee$ denotes the contragredient representation of π_i) are equal *up to sign*. Our proof was purely combinatorial, using the Langlands parameters rather than the representations themselves.

A special case of those properties, for the group $G' = \mathrm{GL}(N, \mathbb{C})$ and r the standard N-dimensional representation of G' is a consequence of the work of Godement and Jacquet ([G-J], [J]) expressing the standard L-functions for the linear group directly in terms of the quasi-coefficients of the representation. Our theorem allows us to reduce the general case to the special case considered by Godement and Jacquet, and hence obtain a new proof.

COROLLARY 1. *Assume that π_1 and π_2 are in $G^\wedge$ and that $\pi_1 \prec \pi_2$. For any rational representation r of the L-group, the quotient*

$$\frac{L(\pi_2, r, s)}{L(\pi_1, r, s)}$$

is an entire function.

COROLLARY 2. *Assume that π_1 and π_2 are in $G^\wedge$ and that they are subquotients of the same principal series representation. Then*

$$\frac{L(\pi_1, r, s)}{L(\pi_1^\vee, r, 1-s)} = \pm \frac{L(\pi_2, r, s)}{L(\pi_2^\vee, r, 1-s)}.$$

REMARK. For $\mathrm{GL}(N, \mathbb{C})$, Godement and Jacquet only prove the equality with a constant rather than ± 1. We therefore deduce from our theorem and their work only the weaker result. However, using our result in [A] in the case of $\mathrm{GL}(N, \mathbb{C})$ and the standard representation (which is a substantially simpler calculation than the general case), we get the corollary as stated.

The proof of the theorem is geometric, using the relation of the Bruhat order with the inclusion of Schubert cells in generalized flag varieties *associated to the L-group.* Indeed, the dual group is in several ways nicer than the group itself. For instance, in the non integral case, the description of the composition series involves the Weyl group of a sub-system of the root system. It is well known that the corresponding complex reductive group is *not* in general a subgroup of the group G, whereas its L-group *is* a subgroup of ${}^LG^0$.

Our method is inspired by a new formulation by Vogan of the Langlands classification in the complex case ([V2]). We present Vogan's approach in an independent appendix, for the sake of completeness and for lack of a written reference.

There are several possible extensions of this work, which we propose to investigate in later papers. One of them is the investigation of the real case. Another is to ask whether a converse exists: are the embeddings of the L-group into $\mathrm{GL}(N,\mathbb{C})$ and the Bruhat order in $\mathrm{GL}(N,\mathbb{C})^\wedge$ enough to determine the Bruhat order on $G^\wedge$?

2. **Notations and background.** Let G be a complex connected reductive algebraic group, T a maximal torus and B a Borel subgroup containing T. Let $\mathfrak{g}$, $\mathfrak{t}$, $\mathfrak{b}$ be the Lie algebras of G, T, B; $R \subset \mathfrak{t}^*$ is the root system, R^+ the positive roots corresponding to the choice of B, and Δ the set of simple roots. For any root system, the set of integral weights is denoted by $P(R)$ and the set generated by the roots by $Q(R)$. We write $R^\vee \subset \mathfrak{t}$ for the dual root system, $X^*(T)$ and $X_*(T)$ respectively for the set of characters of T and the set of one parameter subgroups of T. We have the inclusions

$$Q(R) \subset X^*(T) \subset P(R)$$

and

$$Q(R^\vee) \subset X_*(T) \subset P(R^\vee)$$

and a duality between the two lines. The sextuplet $\left(X^*(T), R, R^+, X_*(T), R^\vee, R^{+\vee}\right)$ is called the *based root datum* associated to (G,T,B) (see [B] or [L] for all this).

The L-group ${}^LG^0$, with torus ${}^LT^0$ and Borel subgroup ${}^LB^0$ is by definition the complex connected reductive algebraic group whose based root datum is dual to the root datum associated to (G,T,B), so that for instance $X_*({}^LT^0) = X^*(T)$. The Weyl groups for G and ${}^LG^0$ are canonically isomorphic (and also the Coxeter systems corresponding to the choices of simple roots), so we use the same letter, W, for both. The length with respect to the choice of simple reflections is denoted ℓ.

Note that $\mathfrak{t} = X_*(T) \otimes_{\mathbb{Z}} \mathbb{C}$ and that $T = \mathfrak{t}/2i\pi X_*(T)$, so $\ker(\exp) \cap \mathfrak{t} = 2i\pi X_*(T)$, and similarly $\ker(\exp) \cap {}^L\mathfrak{t}^0 = 2i\pi X_*({}^LT^0)$.

Now a conjugacy class of sections of the L-group is defined by a pair (p,q) in $\left(X_*({}^LT^0) \otimes_{\mathbb{Z}} \mathbb{C}\right) \times \left(X_*({}^LT^0) \otimes_{\mathbb{Z}} \mathbb{C}\right)$ with $p - q \in X_*({}^LT^0)$, up to the diagonal action by W.

Taking the various identifications into account, $X_*({}^LT^0) \otimes_{\mathbb{Z}} \mathbb{C}$ is equal to the (complex) dual $\mathfrak{t}^*$ of $\mathfrak{t}$, so that it makes sense, for $p \in X_*({}^LT^0) \otimes_{\mathbb{Z}} \mathbb{C}$ to define

$$R_{(p)} = \{\alpha \in R \mid (p, \alpha^\vee) \in \mathbb{Z}\}.$$

It is well known that $R_{(p)}$ is a root system with Weyl group

$$W_{(p)} = \{w \in W \mid wp - p \in Q(R)\}.$$

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If (p,q) are chosen as above with $p-q \in X_*({}^LT^0)$, then $R_{(p)} = R_{(q)}$ and $W_{(p)} = W_{(q)}$. We also need the stabilizers $W^0_{(p)}$ and $W^0_{(q)}$ of p and q in W. Of course $W^0_{(p)}$ and $W^0_{(q)}$ are included in $W_{(p)}$.

Our convention for the Bruhat order $\prec$ on W (recall that a positive set of roots has been chosen) is compatible with the length ℓ, so that e is the smallest element and the *longest* element w_0 is also the *largest*. The Bruhat order can be interpreted in terms of Schubert cells—and for our purpose it is more convenient to work in the dual flag variety ${}^LG^0/{}^LB^0$:

$$w_1 \prec w_2 \text{ if and only if } \overline{{}^LB^0 w_1 {}^LB^0} \subset \overline{{}^LB^0 w_2 {}^LB^0}.$$

More generally we need to consider double cosets of the form $W_I \setminus W/W_J$ where I and J are subsets of Δ and W_I and W_J are the corresponding parabolic subgroups of W, generated by the simple reflections with respect to roots in I and J respectively. In each coset $C(w) = W_I w W_J$ there is a unique element $\tilde{w}$ of smallest length, also characterized by the property:

$$\ell(s\tilde{w}) \geq \ell(\tilde{w}) \text{ for all } s \in W_I,$$
$$\ell(\tilde{w}s) \geq \ell(\tilde{w}) \text{ for all } s \in W_J.$$

(See [Bou], Chapitre IV, Exercise 1.3.)

The Bruhat order on $W_I \setminus W/W_J$ is defined as the order induced on the set of representatives $\tilde{w}$. We also have a geometric characterization of the Bruhat order on $W_I \setminus W/W_J$. We consider the standard parabolic subgroups ${}^LP^0_I$ and ${}^LP^0_J$ associated respectively to I and J, and ${}^LP^0_I$-orbits in the generalized flag variety ${}^LG^0/{}^LP^0_J$. We have:

$$W_I w_1 W_J \prec W_I w_2 W_J \text{ if and only if } \overline{{}^LP^0_I w_1 {}^LP^0_J} \subset \overline{{}^LP^0_I\, w_2 {}^LP^0_J}.$$

The choice of a positive set of roots in R induces the choice of a positive system of roots in $R_{(p)}$. Assuming that p and q are *dominant*, that is $(p,\alpha^\vee) \neq -1,-2,-3\cdots$ and $(q,\alpha^\vee) \neq -1,-2,-3\cdots$ the stabilizers $W^0_{(p)}$ and $W^0_{(q)}$ are Weyl groups of Levi subgroups of $W_{(p)}$, so as a special case of the definition above, we have a Bruhat order on $W^0_{(p)} \setminus W_{(p)}/W^0_{(q)}$.

In order to give the corresponding geometric interpretation of this Bruhat order we need to introduce further notations. For m semi simple in ${}^L\mathfrak{g}^0$ we set $c(m) = \exp 2i\pi m \in {}^LG^0$. Set ${}^LG^0_{(m)}$ to be the connected component of the centralizer of $c(m)$ in ${}^LG^0$. It is easy to see that the Lie algebra ${}^L\mathfrak{g}^0_{(m)}$ of ${}^LG^0_{(m)}$ has root system $R^\vee_{(m)}$, and the Weyl group of ${}^LG^0_{(m)}$ is $W_{(m)}$. To m we associate the parabolic subalgebra ${}^L\mathfrak{p}^0_{(m)}$ of ${}^L\mathfrak{g}^0_{(m)}$ whose roots are

$$\{\alpha^\vee \mid (m,\alpha^\vee) \geq 0\},$$

and the corresponding parabolic subgroup ${}^LP^0_{(m)}$.

Take once again $(p,q) \in \left(X_*({}^LT^0)\otimes_{\mathbb{Z}}\mathbb{C}\right) \times \left(X_*({}^LT^0)\otimes_{\mathbb{Z}}\mathbb{C}\right)$ with $p-q \in X_*({}^LT^0)$ and p,q both dominant. The elements $c(p)$ and $c(q)$ are equal, therefore $G_{(p)} = G_{(q)}$. The two parabolic subgroups ${}^LP^0_{(p)}$ and ${}^LP^0_{(q)}$ are standard. As in the integral case, we characterize the Bruhat order on $W^0_{(p)}\backslash W_{(p)}/W^0_{(q)}$ in terms of Schubert cells $\overline{{}^LP^0_{(p)}w{}^LP^0_{(q)}}$ in ${}^LG^0_{(p)}/{}^LP^0_{(q)}$:

$$W^0_{(q)}w_1W^0_{(p)} \prec W^0_{(q)}w_2W^0_{(p)} \text{ if and only if } \overline{{}^LP^0_{(q)}w_1{}^LP^0_{(p)}} \subset \overline{{}^LP^0_{(q)}w_2{}^LP^0_{(p)}}.$$

Often, one considers ${}^LG^0_{(p)}$-orbits in $({}^LG^0_{(p)}/{}^LP^0_{(p)}) \times ({}^LG^0_{(p)}/{}^LP^0_{(q)})$, which are in one to one correspondence with ${}^LP^0_{(p)}$-orbits in ${}^LG^0_{(p)}/{}^LP^0_{(q)}$. And the Bruhat order can just as well be characterized by

$$W^0_{(q)}w_1W^0_{(p)} \prec W^0_{(q)}w_2W^0_{(p)} \text{ if and only if}$$
$$\overline{{}^LG^0_{(p)}({}^LP^0_{(p)}, w_1{}^LP^0_{(q)})} \subset \overline{{}^LG^0_{(p)}({}^LP^0_{(p)}, w_2{}^LP^0_{(q)})}.$$

Let us now recall the Langlands classification for complex groups (which is due to Zhelobenko; see [D1]) and the composition series of a principal series representation, due to Hirai, Bernstein-Gelfand and Duflo (see [H], [B-G] and [D2])—We are not concerned here with multiplicities!

As above, we have chosen a torus T and a Borel subgroup B. Let $\phi \in \Phi(G)$, and $\phi_\sharp$ an element of the conjugacy class with $\operatorname{Im}\phi_\sharp \subset {}^LT^0$. We can write $\phi_\sharp(z) = z^p\bar{z}^q$ with $(p,q) \in X_*({}^LT^0)\otimes_{\mathbb{Z}}\mathbb{C} \times X_*({}^LT^0)\otimes_{\mathbb{Z}}\mathbb{C}$ with $p-q \in X_*({}^LT^0)$. Since $X_*({}^LT^0) = X^*(T)$, it makes sense to define $\tau(\phi_\sharp) = \tau(p,q)$ as the Harish Chandra module underlying the principal series representation with parameters (p,q), *i.e.* the representation unitarily induced from the character of $B = TN$ (N the unipotent radical of B):

$$b = tn \longmapsto t^p\bar{t}^q,\ t \in T,\ n \in N = (B,B),$$

where the conjugation is relative to the split real form of $\mathfrak{g}$. (Note that in this setting, the compact parameter is $p-q$, so that the spherical case corresponds to (p,p).)

The class in the Grothendieck group of the representation $\tau(\phi_\sharp)$ is independent of the choice of $\phi_\sharp$; it has finite length, an infinitesimal character which depends only on the $W\times W$-orbit of (p,q). There is a unique subquotient $\pi(\phi)$ of $\tau(\phi)$ containing a certain K-type, and the map $\phi \in \Phi(G) \longmapsto \pi(\phi)$ is a bijection of $\Phi(G)$ onto $G^\wedge$. Often we write $\pi(p,q)$ and $\tau(p,q)$ instead of $\pi(\phi)$ and $\tau(\phi)$. If we choose p and q to be both dominant, and $w_1 \in W^0_{(p)}\backslash W_{(p)}/W^0_{(q)}$ the set of irreducible subquotients of $\tau(p,w_1q)$ is the set of $\pi(p,w_2q)$ with $w_2 \in W^0_{(p)}\backslash W_{(p)}/W^0_{(q)}$, $w_1 \prec w_2$ for the Bruhat order on $W^0_{(p)}\backslash W_{(p)}/W^0_{(q)}$.

This gives an explicit description of the Bruhat order on $\Phi(G)$ (and justifies the terminology): two elements ϕ^1 and ϕ^2 represented by $\phi^1_\sharp = (p_1,q_1)$ and $\phi^2_\sharp = (p_2,q_2)$ in $\Phi(G)$ are not comparable unless $p_2 \in Wp_1$ and $q_2 \in Wq_1$. If they are comparable, we can assume that $p_1 = p_2 = p$, and $q_2 = wq_1$ for some $w \in W$, and w must be in $W_{(q_1)}$. Finally, comparable elements of $\Phi(G)$ must be of the form (p,w_1q) and (p,w_2q), with w_1 and w_2 in $W^0_{(p)}\backslash W_{(p)}/W^0_{(q)}$, and for those the order is given as above by the usual Bruhat order.

REMARK. We can have the situation where $\phi^1_\sharp = (p,q)$, $\phi^2_\sharp = (p,wq)$ for w in W and *not* in $W_{(p)}$, so that $\pi(p,q)$ and $\pi(p,wq)$ have same infinitesimal character but are not comparable for the Bruhat order (not block equivalent): since $p - q \in X_*({}^LT^0)$ and $p - wq \in X_*({}^LT^0)$ we must have $wq - q \in X_*({}^LT^0)$. If G is not adjoint, we can find q and w so that $wq - q \in X_*({}^LT^0)$ and $wq - q \notin Q(R)$. This occurs for $\mathrm{SL}(2,\mathbb{C})$: Let α be the positive root. Take $p = q = \alpha/4$ and w the non trivial element of the Weyl group. The subgroup $W_{(\alpha/4)}$ is trivial, therefore $\pi(\alpha/4, \alpha/4)$ and $\pi(\alpha/4, -\alpha/4)$ are both irreducible.

3. **Proof of the theorem.** We now consider two groups G and G', containing maximal tori T and T', Borel subgroups B and B'; we write W and W' for the two Weyl groups. Consider a rational morphism r of ${}^LG^0$ into ${}^LG'^0$. We can assume that $r({}^LT^0) \subset {}^LT'^0$. We also use the letter r for the derived morphism from ${}^L\mathfrak{g}^0$ into ${}^L\mathfrak{g}'^0$.

We make a technical, but important choice of an element m in $X_*({}^LT^0)$ which is regular; we assume further that for β in R', $\left(r(m), \beta^\vee\right) = 0$ only if $r({}^L\mathfrak{t}^0) \subset \mathrm{Ker}(\beta^\vee)$—*i.e.* the non zero weights of the representation $ad \circ r$ of ${}^L\mathfrak{t}^0$ in ${}^L\mathfrak{g}'^0$ do not vanish on m. We then choose positive systems of roots in ${}^L\mathfrak{g}^0$ and ${}^L\mathfrak{g}'^0$ so that m and $r(m)$ are dominant. This has the consequence that the imbedding of $r({}^LG^0)$ in ${}^LG'^0$ is "nice", in the sense that the stabilizer $W'^0_{r({}^L\mathfrak{t}^0)}$ of $r({}^L\mathfrak{t}^0)$ in W' is a parabolic subgroup, namely the Weyl group of the Levi subgroup of the standard parabolic subgroup ${}^LP'^0_{r(m)}$ of ${}^LG'^0$ (whose roots are $\{\beta^\vee \in R'^\vee \mid \left(r(m), \beta^\vee\right) \geq 0\}$). See the example below to illustrate this choice.

Systematically, if K is a subgroup of H, we write $N_H(K)$ and $Z_H(K)$ for the normalizer and the centralizer of K in H.

LEMMA 1. *(1)* $N_{{}^LG'^0}\left(r({}^LT^0)\right) = \left(N_{{}^LG'^0}({}^LT'^0) \cap N_{{}^LG'^0}\left(r({}^LT^0)\right)\right) Z_{{}^LG'^0}\left(r({}^LT^0)\right)$, *and* $Z_{{}^LG'^0}\left(r({}^LT^0)\right)$ *is normal in* $N_{{}^LG'^0}\left(r({}^LT^0)\right)$.

(2) The decomposition in (1) induces a homomorphism

$$N\left({}_{{}^LG^0}({}^LT^0)\right) \longrightarrow \left(N_{{}^LG'^0}({}^LT'^0) \cap N_{{}^LG'^0}\left(r({}^LT^0)\right)\right) \Big/ \left(N_{{}^LG'^0}({}^LT'^0) \cap Z_{{}^LG'^0}\left(r({}^LT^0)\right)\right),$$

therefore a homomorphism

$$W \longrightarrow N_{W'}\left(r({}^L\mathfrak{t}^0)\right) / W'^0_{(r({}^L\mathfrak{t}^0))} \subset W' / W'^0_{(r({}^L\mathfrak{t}^0))},$$

where $N_{W'}\left(r({}^L\mathfrak{t}^0)\right)$ *is the set of* $w' \in W'$ *such that* $w'\left(r({}^L\mathfrak{t}^0)\right) \subset r({}^L\mathfrak{t}^0)$.

PROOF. Let $x' \in r\left(N_{{}^LG^0}({}^LT^0)\right)$. The maximal tori ${}^LT'^0$ and $x'^{-1}\,{}^LT'^0x'$ are included in $Z_{{}^LG'^0}\left(r({}^LT^0)\right)$. By conjugation of maximal tori in $Z_{{}^LG'^0}\left(r({}^LT^0)\right)$ there exists an element u in $Z_{{}^LG'^0}\left(r({}^LT^0)\right)$ such that $x'^{-1}\,{}^LT'^0x' = u^{-1}\,{}^LT'^0u$. Therefore ux'^{-1} belongs to $(N_{{}^LG'^0}({}^LT'^0)$. This proves (1); (2) follows easily. ∎

We write $\check{r}$ for the map from W to $W'/W'^0_{(r({}^L\mathfrak{t}^0))}$ defined in Lemma 1. It is easy to give a geometric interpretation of $\check{r}$: Recall that ${}^LB^0$ orbits in ${}^LG^0/{}^LB^0$ are parametrized by W, and that ${}^LB'^0$ orbits in ${}^LG'^0/{}^LP'^0_{(m)}$ are parametrized by $W'/W'^0_{(r({}^L\mathfrak{t}^0))}$. Consider the map

$$r_1 \colon {}^LB^0 \setminus {}^LG^0/{}^LB^0 \longrightarrow {}^LB'^0 \setminus {}^LG'^0/{}^LP'^0_{(r(m))}$$

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induced from r. From the definitions, the diagram

$$(*)\qquad \begin{array}{ccc} {}^L B^0 \backslash {}^L G^0 / {}^L B^0 & \xrightarrow{r_1} & {}^L B'^0 \backslash {}^L G'^0 / {}^L P'^0_{(r(m))} \\ \downarrow & & \downarrow \\ W & \xrightarrow{\check r} & W' / W'^0_{(r({}^L\mathfrak{t}^0))} \end{array}$$

is commutative.

We must now look at the behavior of the data attached to $p \in {}^L\mathfrak{t}^0$ under r (subroot system $R_{(p)}$, its Weyl group $W_{(p)}$, ${}^L G^0_{(p)}$ and ${}^L\mathfrak{g}^0_{(p)}$). Recall that r maps ${}^L\mathfrak{t}^0 = X_*({}^L T^0) \otimes \mathbb{C}$ into ${}^L\mathfrak{t}'^0 = X_*({}^L T'^0) \otimes \mathbb{C}$. Since $X_*({}^L T^0) = X^*(T)$, $P(R)$, $X^*(T)$, $Q(R)$ are subsets of ${}^L\mathfrak{t}^0$, so that it makes sense to consider $r\big(P(R)\big)$, $r\big(Q(R)\big)$ etc.

LEMMA 2. *Let p be in ${}^L\mathfrak{t}^0$.*

(1) $W'_{r({}^L\mathfrak{t}^0)} \subset W'_{(r(p))}$.

(2) $r(wp) = \check r(w)\big(r(p)\big)$.

(3) $r\big(Q(R)\big) \subset Q(R')$.

(4) $\check r(W_{(p)}) \subset W'_{(r(p))}/W'^0_{(r({}^L\mathfrak{t}^0))}$ *and* $\check r(W^0_{(p)}) \subset W'^0_{(r(p))}/W'^0_{(r({}^L\mathfrak{t}^0))}$.

(5) ${}^t r^{-1}(R_{(p)}{}^\vee) \subset R'_{(r(p))}{}^\vee$ *where* ${}^t r$ *is the transposed of* r.

(6) $r({}^L\mathfrak{g}^0_{(p)}) \subset {}^L\mathfrak{g}'^0_{(r(p))}$ *and* $r({}^L G^0_{(p)}) \subset {}^L G'^0_{(r(p))}$.

(7) $r({}^L\mathfrak{p}^0_{(p)}) \subset {}^L\mathfrak{p}'^0_{(r(p))}$

PROOF. (1) $W'_{(r({}^L\mathfrak{t}^0))}$ is generated by reflections with respect to roots which vanish on $r({}^L\mathfrak{t}^0)$. These roots have to vanish on $r(p)$, so the corresponding reflections belong to $W'^0_{(r(p))}$.

(2) is obvious.

(3) Consider the simply connected coverings ${}^L\widetilde{G^0}$ and ${}^L\widetilde{G'^0}$ of ${}^L G^0$ and ${}^L G'^0$, and ${}^L\widetilde{T^0}$, ${}^L\widetilde{T'^0}$ the corresponding maximal tori covering ${}^L T^0$ and ${}^L T'^0$. We have $X^*({}^L\widetilde{T^0}) = P(R^\vee)$ and $X^*({}^L\widetilde{T'^0}) = P(R'^\vee)$, so $X_*({}^L\widetilde{T^0}) = Q(R)$ and $X_*({}^L\widetilde{T'^0}) = Q(R')$. The morphism r lifts to a morphism from ${}^L\widetilde{G^0}$ to ${}^L\widetilde{G'^0}$, which maps $X_*({}^L\widetilde{T^0})$ into $X_*({}^L\widetilde{T'^0})$.

(4) $r(wp - p) = \check r(w)\big(r(p)\big) - r(p)$ belongs to $r\big(Q(R)\big) \subset Q(R')$. The second statement is obvious.

(5) Let $\alpha^\vee$ in $R'^\vee$ such that $\beta^\vee = {}^t r(\alpha^\vee)$ belongs to $R^\vee_{(p)}$. We have:

$$\begin{aligned}(p, \beta^\vee) &= \big(p, {}^t r(\alpha^\vee)\big) \\ &= \big(r(p), \alpha^\vee\big)\end{aligned}$$

so that $\big(r(p), \alpha^\vee\big)$ is an integer.

(6) If $g \in {}^L G^0_{(p)}$, $r(g)c\big(r(p)\big)\big(r(g)\big)^{-1} = r\big(gc(p)g^{-1}\big) = c\big(r(p)\big)$, so $r({}^L G^0_{(p)}) \subset {}^L G'^0_{(r(p))}$. The results follows for the Lie algebras.

(7) Let ${}^L X_{\beta^\vee}$ be a root vector in ${}^L\mathfrak{g}^0_{(p)}$ for the root $\beta^\vee$. One sees that $r(X_{\beta^\vee})$ belongs to $\sum {}^L\mathfrak{g}'^{0\,\alpha^\vee}$, the sum being over the set of $\alpha^\vee$ such that ${}^t r(\alpha^\vee) = \beta^\vee$. The statement follows from $(p, \beta^\vee) = \big(r(p), \alpha^\vee\big) \geq 0$. ∎

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Let p and q be in ${}^L\mathfrak{t}^0$ with $p - q$ in $X_*({}^LT^0)$, and w in $W_{(p)} = W_{(q)}$. Assume that p, q and $r(p)$ are dominant. Then

$$r(wq) = \check{r}(w)\big(r(q)\big)$$

by definition of $\check{r}$, and $\check{r}(w)$ belongs to $W'_{(r(p))}/W'_{(r({}^L\mathfrak{t}^0))}$.

This proves that block equivalence (see introduction) is preserved under Φ. But it is not enough for the Bruhat order. Indeed, with the assumptions of the proposition, there is no reason why $r(q)$ should be dominant (see an example below). So we have to write $r(q) = w'_1 q'_1$ with w'_1 in $W'_{(r(p))}$ and q'_1 dominant, and the that map we have to study is roughly the map $w \longrightarrow \check{r}(w)w'_1$. It is easy to see, as a consequence of the geometric interpretation of $\check{r}$ (diagram (*)), that $w \longrightarrow \check{r}(w)$ is increasing with respect to the Bruhat orders at both ends. But it does not follow that the map $w \longrightarrow \check{r}(w)w'_1$ is increasing.

EXAMPLE. Consider ${}^LG^0 = G = \mathrm{GL}(3, \mathbb{C})$ with its standard torus and Borel subgroups. Write α and β for the simple roots. Let r be the eight dimensional irreducible subrepresentation of the adjoint representation of $\mathrm{GL}(3, \mathbb{C})$. We order its weights in the following way: $\alpha + \beta, \alpha, \beta, 0, 0, -\beta, -\alpha, -\alpha - \beta$. Taking a basis of weight vectors (with an arbitrary choice for the 0-weight vectors), r becomes a rational map from $\mathrm{GL}(3, \mathbb{C})$ into $\mathrm{GL}(8, \mathbb{C})$, the image of the standard torus is included in the standard torus, and the image of the standard Borel is included in the standard Borel. Elements of $X_*({}^LT^0)$ are diagonal matrices, so it is enough to give their diagonal entries.

In terms of the technical choice of $m = (m_1, m_2, m_3)$ described at the beginning of this section, the choice here corresponds to an m such that $m_1 > m_2 > m_3$ and $m_1 - m_2 > m_2 - m_3$. The parabolic ${}^LP^0_{(m)}$ is the standard Borel subgroup of $\mathrm{GL}(3, \mathbb{C})$. There are two choices of positive roots for $\mathrm{GL}(8, \mathbb{C})$, which corresponds to the ambiguity in the choice of 0-weight vectors. The parabolic ${}^LP'^0_{(r(m))}$ is the set of matrices of the form

$$\begin{pmatrix} * & * & * & * & * & * & * & * \\ & * & * & * & * & * & * & * \\ & & * & * & * & * & * & * \\ & & & * & * & * & * & * \\ & & & * & * & * & * & * \\ & & & & & * & * & * \\ & & & & & & * & * \\ & & & & & & & * \end{pmatrix}$$

in $\mathrm{GL}(8, \mathbb{C})$.

Choose $p = (p_1, p_2, p_3), q = (q_1, q_2, q_3)$ strictly dominant elements of $X_*({}^LT^0)$: $p_i, q_i \in \mathbb{Z}$ and $p_1 - p_2 > 0, q_1 - q_2 > 0, p_2 - p_3 > 0, q_2 - q_3 > 0$. We have $r(p) = (p_1 - p_3, p_1 - p_2, p_2 - p_3, 0, 0, p_3 - p_2, p_2 - p_1, p_3 - p_1)$ and $r(q) = (q_1 - q_3, q_1 - q_2, q_2 - q_3, 0, 0, q_3 - q_2, q_2 - q_1, q_3 - q_1)$. So if we choose (p, q) so that $p_1 - p_2 > p_2 - p_3$ and $q_1 - q_2 < q_2 - q_3$, $r(p)$ is dominant whereas $r(q)$ is not.

For this example, the calculations can be carried out explicitly using a combinatorial description of the Bruhat order, and one shows that the map $w \longrightarrow \check{r}(w)w'_1$ is increasing.

For the general case, the combinatorial difficulties are quite formidable, so we need to work differently. Indeed, the difficulty lies with the fact that although the relative position of two parabolic subalgebras is given by the Weyl group, it is not enough to look at pairs of parabolic subalgebras, because pairs of subalgebras do not behave properly under r: there is a choice involved, of the type of parabolic subalgebras one needs to look at in ${}^L\mathfrak{g}'^0_{(p)}$. What is needed is some additional information. We incorporate this by considering some fiber bundles over the flag varieties (see Lemma 3 below). The idea can be put in perspective by looking at Vogan's formulation of the classification of $G^\wedge$ explained in the appendix.

Going back to the general case, we fix p and q as before:

$$p, q \in {}^L\mathfrak{t}^0, p - q \in X_*({}^LT^0), p \text{ and } q \text{ dominant}, r(p) \text{ dominant}.$$

We choose w'_1 in $W'_{(r(p))}$ such that $r(q) = w'_1 q'_1$ with q'_1 dominant. The map $w \longmapsto ww'_1$ from $W'_{(r(p))}$ into itself induces a bijection from $W'^0_{(r(p))} \setminus W'_{(r(p))} / W'^0_{(r(q))}$ to $W'^0_{(r(p))} \setminus W'_{(r(p))} / W'^0_{(q'_1)}$. This bijection is independent of the choice of w'_1. Also, $\check{r}$ induces a map, which we denote $\check{\check{r}}$ from $W^0_{(p)} \setminus W_{(p)} / W^0_{(q)}$ to $W'^0_{(r(p))} \setminus W'_{(r(p))} / W'^0_{(r(q))}$. By composition, we get a map, called $r^\vee$ from $W^0_{(p)} \setminus W_{(p)} / W^0_{(q)}$ to $W'^0_{(r(p))} \setminus W'_{(r(p))} / W^0_{(q'_1)}$ such that for any choice of a representative w in the class $\dot{w}$, we have:

$$r^\vee(\dot{w}) = W'^0_{(r(p))} \check{\check{r}}(w) w'_1 W'^0_{(q'_1)}.$$

REMARK. Note that $r^\vee$ depends on q. If we identify the block $\mathbf{B}_{(p,q)}$ containing the representation $\pi(p, q)$ with $W^0_{(p)} \setminus W_{(p)} / W^0_{(q)}$, and the block $\mathbf{B}_{(r(p),q'_1)}$ with $W'^0_{(r(p))} \setminus W'_{(r(p))} / W^0_{(q'_1)}$, then $r^\vee$ coincides with the restriction of $\Phi(r)$ to $\mathbf{B}_{(p,q)}$. Now we need to prove that the map $r^\vee$ is increasing for the Bruhat orders, and for that we nee a geometric description of $\check{\check{r}}$.

For m semi-simple in ${}^L\mathfrak{g}^0$, recall the definitions above of $c(m)$, ${}^LG^0_{(m)}$ and ${}^LP^0_{(m)}$.

LEMMA 3. *The map from ${}^LG^0_{(m)}.\, m$ to ${}^LG^0_{(m)} / {}^LP^0_{(m)}$*

$$g.m \longmapsto gP_{(m)}g^{-1}$$

gives $X(m) = {}^LG^0_{(m)}.\, m$ the structure of a vector bundle over ${}^LG^0_{(m)} / {}^LP^0_{(m)}$ with fiber over ${}^LP^0_{(m)}$ isomorphic to the unipotent radical ${}^L\mathfrak{u}^0_{(m)}$ of ${}^L\mathfrak{p}^0_{(m)}$.

PROOF. Since $P_{(m)}$ is parabolic, it is equal to its normalizer. Therefore the fiber containing gm is ${}^LP^0_{(m)}.\, m = m + {}^L\mathfrak{u}^0_{(m)}$. ∎

As we have noted before, $c(p) = c(q) = c(wq)$ for any w in $W_{(p)}$. In particular we have ${}^LG^0_{(p)} = {}^LG^0_{(q)} = {}^LG^0_{(wq)}$. We have seen (Lemma 2) that $r\big({}^LG^0(p)\big)$ is included in ${}^LG'^0_{(r(p))}$. We write $X(p)$, $X(q)$, $X'\big(r(p)\big)$ and $X'\big(r(q)\big)$ for the fiber bundles defined in Lemma 3 associated to p, q, $r(p)$ and $r(q)$.

LEMMA 4. *The map r defines an* ${}^LG^0_{(p)}$*-equivariant map of fiber bundles from* $X(p)\times X(q)$ *to* $X'\big(r(p)\big)\times X'\big(r(q)\big)$. *The induced map* r_2 *from the set of* ${}^LG^0_{(p)}$*-orbits in the base* ${}^LG^0_{(p)}/{}^LP^0_{(p)}\times{}^LG^0_{(q)}/{}^LP^0_{(q)}$ *of* $X(p)\times X(q)$ *to the set of* ${}^LG'^0_{(r(p))}$*-orbits in the base* ${}^LG'^0_{(r(p))}/{}^LP'^0_{(r(p))}\times{}^LG'^0_{(r(q))}/{}^LP'^0_{(r(q))}$ *of* $X'\big(r(p)\big)\times X'\big(r(q)\big)$ *coincides with* $\check{\check{r}}$ *after the appropriate identifications, i.e. the diagram*

(∗∗)

$$\begin{array}{ccc} W^0_{(p)}\setminus W_{(p)}/W^0_{(q)} & \xrightarrow{\check{\check{r}}} & W'^0_{(r(p))}\setminus W'_{(r(p))}/W'^0_{(r(q))}\\ \downarrow & & \downarrow\\ ({}^LG^0_{(p)}/{}^LP^0_{(p)}\times{}^LG^0_{(p)}/{}^LP^0_{(q)})/{}^LG^0_{(p)} & \xrightarrow{r_2} & ({}^LG'^0_{(r(p))}/{}^LP'^0_{(r(p))}\times{}^LG'^0_{(r(p))}/{}^LP'^0_{(r(q))})/{}^LG'^0_{(r(p))} \end{array}$$

is commutative.

PROOF. We have seen that $r\big({}^LG^0_{(p)}\big)$ is included in ${}^LG'^0_{(r(p))}$ and that $r({}^LP^0_{(p)})$ is included in ${}^LP'^0_{(r(p))}$ (Lemma 2). Since r is a homomorphism, the first part of the lemma holds.

For the second part, take w in $W^0_{(p)}\setminus W_{(p)}/W^0_{(q)}$. By Lemma 2 we have $r(wq)=\check{r}(w)\big(r(q)\big)$. Starting with the ${}^LG^0_{(p)}$-orbit of (p,wq) in $X(p)\times X(q)$, and applying r, we get the ${}^LG'^0_{(r(p))}$ orbit of $\Big(r(p),\check{r}(w)\big(r(q)\big)\Big)$. Its image in the base is the ${}^LG'^0_{(r(p))}$-orbit of $\big({}^LP'^0_{(r(p))},{}^LP'^0_{(\check{r}(w)(r(q)))}\big)$ in ${}^LG'^0_{(r(p))}/{}^LP^0_{(r(p)}\times{}^LG'^0_{(r(q))}/{}^LP^0_{r(q)}$. ∎

PROOF OF THE THEOREM. Assume that $\dot{w}_1\prec\dot{w}_2$ for the Bruhat order in $W^0_{(p)}\setminus W_{(p)}/W^0_{(q)}$. Then

$$\overline{{}^LG^0_{(p)}(p,\dot{w}_1q)}\subset\overline{{}^LG^0_{(p)}(p,\dot{w}_2q)}.$$

This must be conserved by r:

$$\overline{{}^LG'^0_{(r(p))}\big(r(p),\check{r}(\dot{w}_1)r(q)\big)}\subset\overline{{}^LG'^0_{(r(p))}\big(r(p),\check{r}(\dot{w}_2)r(q)\big)}.$$

By diagram (∗∗) we go down to the base ${}^LG'^0_{(r(p))}/{}^LP^0_{(r(p)}\times{}^LG'^0_{(r(q))}/{}^LP^0_{r(q)}$ where we have:

$$\overline{{}^LG'^0_{(r(p))}\left({}^LP'^0_{(r(p))},\check{r}(w_1){}^LP'^0_{(r(q))}\right)}\subset\overline{{}^LG'^0_{(r(p))}\left({}^LP'^0_{(r(p))},\check{r}(w_2){}^LP'^0_{(r(q))}\right)}.$$

Since $\check{r}(w)w_1'=r^\vee(w)$, for $i=1,2$ the ${}^LG'^0_{(r(p))}$-orbit of $({}^LP'^0_{(r(p))},\check{r}(w){}^LP'^0_{(r(q))})$ in ${}^LG'^0_{(r(p))}/{}^LP^0_{(r(p)}\times{}^LG'^0_{(r(q))}/{}^LP^0_{r(q)}$ is the ${}^LG'^0_{(r(p))}$-orbit of $({}^LP'^0_{(r(p))},r^\vee(w){}^LP'^0_{(q_1')})$ in ${}^LG'^0_{(r(p))}/{}^LP^0_{(r(p))}\times{}^LG'^0_{(r(q))}/{}^LP^0_{(q_1')}$. Now we can apply the geometric characterization of the Bruhat order to conclude that $r^\vee(\dot{w}_1)\prec r^\vee(\dot{w}_2)$. ∎

REMARK. The replacement of pairs of parabolics by pairs of flats amounts to adding to each parabolic subalgebra ${}^L\mathfrak{p}^0$ a p which defines it (${}^L\mathfrak{p}^0={}^L\mathfrak{p}^0_{(p)}$). This extra piece of information is superfluous to define an element of the coset of the Weyl group, but becomes useful under the homomorphism r, because $r(p)$ determines a subalgebra of ${}^L\mathfrak{g}'^0_{(r(p))}$.

4. **Application to L-functions.** Let us first recall Langlands' definition of the L-functions associated to an irreducible representation $\pi(\phi)$. Here r is a rational representation of the L-group into $\mathrm{GL}(N,\mathbb{C})$, so $r\circ\phi$ is an N-dimensional representation of the Weil group. By definition, the L-function $L(\pi,r,s)=L(\pi(\phi),r,s)$ is the Artin L-function $L(r\circ\phi,s)$ associated to $r\circ\phi$. If we decompose $r\circ\phi$ into a sum of characters $\sum_i\chi_i$ with $\chi_i(z)=z^{p_i}\bar{z}^{q_i}$ $(p_i-q_i\in\mathbb{Z})$, we have

$$L\big(\pi(\phi),r,s\big)=\prod_i L(\chi_i,s),$$

where

$$L(\chi_i,s)=2(2\pi)^{-\sup(p_i,q_i)-s}\Gamma\big(\sup(p_i,q_i)+s\big).$$

A special instance of this construction arises when one considers $G=\mathrm{GL}(N,\mathbb{C})$ and r the standard representation i of ${}^LG^0=\mathrm{GL}(N,\mathbb{C})$. Clearly, the L-factor associated with the representation $\pi(\phi)$ of G and the rational N-dimensional representation r of ${}^LG^0$ is equal to the L-factor associated to the representation $\pi(r\circ\phi)$ of $\mathrm{GL}(N,\mathbb{C})$ and i.

In the case of the linear group and the standard representation, Jacquet and Langlands (for $N=2$), and Godement-Jacquet (in the general case) (see [J-L], [G-J], [J]) have given a direct construction of the L-function $L\big(\pi(\phi),s\big)=L(\phi,s)$ using Mellin transforms of the pseudo-coefficients of π. In their construction, the L-function associated to $\pi(\phi)$ is equal to the L-function associated to the whole principal series $\tau(\phi)$, and is a greatest common denominator of the L-functions associated to the various subquotients of $\tau(\phi)$. Therefore, the divisibility property is trivial: see [J], paragraphs 4 and 5.

Since the Bruhat order is preserved by $\Phi(r)$, assuming that $\phi_1\prec\phi_2$, we have $r\circ\phi_1\prec r\circ\phi_2$. Therefore, we know by Jacquet-Godement that

$$\frac{L(r\circ\phi_2,s)}{L(r\circ\phi_1,s)}$$

is entire. This proves Corollary 1.

For the special case of $\mathrm{GL}(N,\mathbb{C})$ and the standard representation, the property of Corollary 2 is that with appropriate normalizations, the ϵ-factor is a constant, and that the γ-factors are the same for all constituents of a principal series representation. These facts are contained in [J], paragraphs 4 and 5. The same argument as for Corollary 1 reduces the proof of Corollary 2 to the special case. The more precise statement that the constant is in fact ± 1 is a consequence of the same fact in the special case. As was pointed out in the introduction, this is not contained in [J]—but is in [A].

5. **Appendix.** The idea of the proof of the theorem, considering the fiber bundle

$${}^LG^0_{(m)}.\,m\longrightarrow {}^LG^0_{(m)}/{}^LP^0_{(m)}$$

is extracted from a deeper observation of David Vogan's [V2] which we are going to outline now.

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Call *special flats* the fibers of the bundles ${}^LG^0_{(m)}.m \longrightarrow {}^LG^0_{(m)}/{}^LP^0_{(m)}$ for m semi-simple. The special flats are all of the form $f_m = m + {}^L\mathfrak{u}^0_{(m)}$ where ${}^L\mathfrak{u}^0_{(m)}$ is the Lie algebra of the unipotent radical of ${}^LP^0_{(m)}$. Consider the set $\mathcal{F}$ of all special flats. It is a subset of the grassmannian of affine subspaces of ${}^L\mathfrak{g}^0$. For a special flat f, we write P_f for the parabolic $P_{(m)}$ for any m in f. It is clear that for $m' \in f_m$, $c(m') = \exp 2i\pi m' = c(m)$ since m' and m are conjugate under $G_{(m)}$. So we can write $c(f)$ instead of $c(m)$ for m in a flat f. Conversely, we have:

LEMMA 5. *Assume that the special flats f_1 and f_2 are such that $c(f_1) = c(f_2)$. Then there exist m_1 in f_1 and m_2 in f_2 such that m_1 and m_2 commute. The pair (m_1, m_2) is unique up to conjugacy under $P_{f_1} \cap P_{f_2}$.*

PROOF. Since $c(f_1) = c(f_2)$, the parabolics ${}^LP^0_{f_1}$ and ${}^LP^0_{f_2}$ are parabolics of the same subgroup ${}^LG^0_c$. Consider a Cartan subalgebra ${}^l\mathfrak{h}^0$ included in both ${}^L\mathfrak{p}^0_{f_1}$ and ${}^L\mathfrak{p}^0_{f_2}$. We set $f_1 \cap {}^l\mathfrak{h}^0 = \{m_1\}$ and $f_2 \cap {}^l\mathfrak{h}^0 = \{m_2\}$. Conversely, if m_1 and m_2 commute, they have to belong to the same Cartan subalgebra. And two Cartan subalgebras contained in both ${}^L\mathfrak{p}^0_{f_1}$ and ${}^L\mathfrak{p}^0_{f_2}$ are conjugate under $P_{f_1} \cap P_{f_2}$. ∎

We consider now the set of ${}^LG^0$-orbits in $\Psi(G) = \{(f_1,f_2) \in \mathcal{F} \times \mathcal{F} \text{ such that } c(f_1) = c(f_2)\}$.

PROPOSITION. *The sets $\Psi(G)/{}^LG^0$ and $\Phi(G)/{}^LG^0$ are in one-to-one correspondence.*

PROOF. Let ϕ be in $\Phi(G)/{}^LG^0$, and choose a representative $\phi_\sharp$ of ϕ in the given Cartan subalgebra of ${}^L\mathfrak{g}^0$: we have $\phi_\sharp(z) = p(z)q(\bar{z})$ for $(p,q) \in X_*({}^LT^0)\otimes_{\mathbb{Z}}\mathbb{C} \times X_*({}^LT^0)\otimes_{\mathbb{Z}}\mathbb{C}$ with $p - q \in X_*({}^LT^0)$. The pair (p,q) defines a pair of special flats f_p,f_q. Since $p-q$ belongs to $X_*({}^LT^0)$, $c(p) = c(q)$, so that the ${}^LG^0$-orbit of (f_p,f_q) is an element of $\Psi(G)/{}^LG^0$. Conversely, take (f_1,f_2) a pair of special flats with $c(f_1) = c(f_2)$. By Lemma 5, we choose $(p,q) \in f_1 \times f_2$ with (p,q) in some Cartan subalgebra. Since we are free to conjugate by ${}^LG^0$, we can assume that p and q both belong to the given Cartan subalgebra of ${}^L\mathfrak{g}^0$. Since $c(p) = c(q)$, we must have $p - q \in X_*({}^LT^0)$. So (p,q) defines an element of $\Phi(G)$. Clearly, the ${}^LG^0$-orbit of (p,q) is independent of the choice. ∎

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