The lost boarding pass problem: converse results

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1. *Introduction and results*

This Article is a follow-up to a recent *Gazette* Article about the lost boarding pass problem by Grimmett and Stirzaker [1]. According to their book [2, 1.8.39, p. 10], it seems that they recognised this lovely problem in 2000 or earlier. We quote it with suitable minor changes.

> (*The lost boarding pass problem*) The n passengers for a Bell-Air flight in an airplane with n seats have been told their seat numbers. They get on the plane one by one. The first person loses his or her boarding pass, and sits in a randomly chosen seat. Subsequent passengers sit in their assigned seats whenever they find them available, or otherwise in a randomly chosen empty seat.

(I) Suppose that the first person sits in a seat chosen uniformly at random from n available. What is the probability that the last passenger finds his or her assigned seat to be free?

(II) Suppose that the first person sits in a seat chosen uniformly at random *except* his or her assigned seat. What is the probability in the previous question?

For the time being we assume $n \geqslant 2$. The solutions of (I) and (II) given in [2, 1.8.39, p.197] are

$$\frac{1}{2} \qquad \text{and} \qquad \frac{n-2}{2(n-1)}, \tag{1}$$

respectively. To discuss the problem we use some notation. For $l \in \{1, \dots, n\}$ let N_l be the random seat number of the passenger l, so that $(N_1, \dots, N_n)$ is a permutation of $(1, \dots, n)$. Let p_i be the probability that the seat number of the first passenger 1 is i, namely,

$$p_i = \mathrm{P}(N_1 = i) \text{ for } i \in \{1, \dots, n\}. \tag{2}$$

Then

$$\sum_{i=1}^{n} p_i = 1, \tag{3}$$

and the assumptions of (I) and (II) are expressed as

$$\begin{cases} \text{(I)} \ p_1 = \dots = p_n = \frac{1}{n}, \\ \text{(II)} \ p_1 = 0 \text{ and } p_2 = \dots = p_n = \frac{1}{n-1}, \end{cases} \tag{4}$$

respectively. Let A_l be the event that the passenger l sits in his or her assigned seat, i.e.

$$A_l = \{N_l = l\} \text{ for } l \in \{1, \dots, n\}. \tag{5}$$

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Since many authors investigate (I) (see [3], [4], [5]), we briefly explain some results for (I). Both [4, (1)] and [5] state that

$$\text{if (I) of (4) holds then } \mathrm{P}(A_l) = \frac{n - l + 1}{n - l + 2} \text{ for } l \in \{2, \dots, n\}, \quad (6)$$

in particular $\mathrm{P}(A_n) = \frac{1}{2}$. Bollobás [3, p. 177] proves it without using mathematical expressions. Moreover Henze and Last [4, Theorem 1] show that $A_2, \dots, A_n$ are independent, but a simpler proof is given by [1, Theorem 1].

In this Article, we study this problem when the first passenger randomly chooses a seat in the sense of (2). Throughout this Article, we assume

$$p_k > 0 \text{ for } k \in \{2, \dots, n - 1\}, \quad (7)$$

which includes (4), since $p_1 = 0$ or $p_n = 0$ is allowed. Under (7), we establish a necessary [as well as sufficient] condition on $p_1, \dots, p_n$ for the independence of $A_2, \dots, A_n$ as follows.

Theorem 1: Suppose that $n \geqslant 3$ and the first passenger chooses his or her seat with probability $p_1, \dots, p_n$ satisfying (7). Then we have

$$p_1 = p_3 = \dots = p_n \text{ if, and only if, } A_2, \dots, A_n \text{ are independent.} \quad (8)$$

Note that the following example shows that the natural condition (I) of (4) above, i.e. $p_1 = p_2 = \dots = p_n$, is *not* necessary.

Example 1: Let $n = 3$, and if $p_1 = p_3$ then simple calculations show that we have $\mathrm{P}(A_2) = 2p_1$, $\mathrm{P}(A_3) = \frac{1}{2}$ and $\mathrm{P}(A_2 \cap A_3) = p_1$, which gives $\mathrm{P}(A_2 \cap A_3) = \mathrm{P}(A_2)\,\mathrm{P}(A_3)$. Hence A_2 and A_3 may be independent even if $p_1 = p_2 = p_3 = \frac{1}{3}$ fails.

This Article is organised as follows. Section 2 provides preliminary results for Theorem 1. We prove Theorem 1 in Section 3, and make concluding remarks in Section 4.

2. *Preliminary results*

Let us introduce notation for the conditional probabilities

$$\alpha_k(l) = \mathrm{P}(A_l \mid N_1 = k) \text{ for } k \in \{2, \dots, n-1\} \text{ and } l \in \{1, 2, \dots, n\},$$

which are well-defined because of (7). When the first passenger sits in a seat k for $k \in \{2, \dots, n-1\}$, the following lemma holds.

Lemma: For $k \in \{2, \dots, n-1\}$ we obtain

$$\begin{cases} \alpha_k(1) = \alpha_k(k) = 0, \\ \alpha_k(l) = 1 \text{ for } l \in \{2, \dots, k-1\} \text{ with } k \geqslant 3, \end{cases} \quad (9)$$

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and

$$\alpha_k(l) = \frac{n - l + 1}{n - l + 2} \text{ for } l \in \{k + 1, \dots, n\}. \qquad (10)$$

Proof: Let us fix $k \in \{2, \dots, n - 1\}$. From the statement of the problem, (9) follows. For simplicity we set $\mathrm{P}_k(\cdot) = \mathrm{P}(\cdot \mid N = k)$. Since the passenger k randomly chooses a seat in $\{1\} \cup \{k + 1, \dots, n\}$, it turns out that

$$\mathrm{P}_k(N_k = i) = \frac{1}{n - k + 1} \text{ for } i \in \{1\} \cup \{k + 1, \dots, n\}. \qquad (11)$$

When $k \in \{2, \dots, n - 2\}$ we have

$$\mathrm{P}_k(A_l \mid N_k = i) = \alpha_i(l) \text{ for } i \in \{k + 1, \dots, n - 1\} \text{ and } l \in \{i + 1, \dots, n\}, \quad (12)$$

and when $k = n - 1$ we have

$$\mathrm{P}_{n-1}(A_n \mid N_{n-1} = n) = 0, \qquad \mathrm{P}_{n-1}(A_n \mid N_{n-1} = 1) = 1. \quad (13)$$

Moreover

$$\begin{cases} \mathrm{P}_k(A_l \mid N_k = 1) = 1 & \text{for } l \in \{k + 1, k + 2, \dots, n\}, \\ \mathrm{P}_k(A_l \mid N_k = i) = 1 & \text{for } k \in \{2, \dots, n - 2\}, i \in \{k + 2, \dots, n\}, \\ & \quad l \in \{k + 1, k + 2, \dots, i - 1\}, \\ \mathrm{P}_k(A_l \mid N_k = l) = 0 & \text{for } l \in \{k + 1, k + 2, \dots, n\}. \end{cases} \qquad (14)$$

Then it follows that for $k \in \{2, \dots, n - 2\}$ and $l \in \{k + 1, k + 2, \dots, n\}$

$$\begin{aligned} \alpha_k(l) &= \mathrm{P}_k(A_l) = \sum_{i \in \{1\} \cup \{k+1,\dots,n\}} \mathrm{P}_k(A_l \mid N_k = i)\, \mathrm{P}_k(N_k = i) \\ &= \frac{1}{n - k + 1}\left\{\mathrm{P}_k(A_l \mid N_k = 1) + \sum_{i=k+1}^{l-1} \mathrm{P}_k(A_l \mid N_k = i) \right. \\ &\qquad \left. + \sum_{i=l+1}^{n} \mathrm{P}_k(A_l \mid N_k = i)\right\} \\ &\overset{(12),(14)}{=} \begin{cases} \dfrac{n - k}{n - k + 1} & \text{if } l = k + 1, \\ \dfrac{n - l + 1 + \sum_{i=k+1}^{l-1} \alpha_i(l)}{n - k + 1} & \text{if } l \in \{k + 2, \dots, n\}. \end{cases} \end{aligned} \qquad (15)$$

Although solving this equation under (9) yields (10), we prove it by induction with k as in [2, 1.8.39, p.197]. If $k = n - 1$ then

$$\begin{aligned} \alpha_{n-1}(n) &= \mathrm{P}_{n-1}(A_n) = \mathrm{P}_{n-1}(A_n \mid N_{n-1})\, \mathrm{P}_{n-1}(N_{n-1} = 1) \\ &\quad + \mathrm{P}_{n-1}(A_n \mid N_{n-1} = n)\, \mathrm{P}_{n-1}(N_{n-1} = n) \overset{(11),(13)}{=} \frac{1}{2}. \end{aligned}$$

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Next we suppose that (10) is true for $k \in \{n - j, \dots, n - 1\}$. Then we check (10) with $k = n - j - 1 \geqslant 2$. If $l = n - j$ then $\alpha_{n-j-1}(n-j) = \dfrac{j+1}{j+2}$ from (15). If $l \in \{n - j + 1, \dots, n\}$ then we have

$$\alpha_{n-j-1}(l) = \frac{n - l + 1 + \sum_{i=n-j}^{l-1} \alpha_i(l)}{n - (n - j - 1) + 1} = \frac{n - l + 1}{n - l + 2}.$$

Hence we obtain (10), which completes the proof of the Lemma.

Remark 1:

(i) Equation (10) with $l = n$ implies $\alpha_k(n) = \frac{1}{2}$ for $k \in \{2, \dots, n-1\}$. This suggests that if the first passenger sits in a seat $k \in \{2, \dots, n-1\}$ then the seats 1 and n are chosen with the same probability.

(ii) Equation (12) means that whether the first passenger or the passenger k sits in the seat i, the conditional probability for the passenger l does not change. We use this *memoryless property* when proving the independence of $A_2, \dots, A_n$ in Theorem 1.

Theorem 2: Make the same assumption of Theorem 1. Then the probability that the passenger l sits in his or her assigned seat is

$$\mathrm{P}(A_l) = \begin{cases} p_1 & \text{for } l = 1, \\ 1 - p_2 & \text{for } l = 2, \\ 1 - \frac{1}{n-l+2} \sum_{k=2}^{l-1} p_k - p_l & \text{for } l \in \{3, \dots, n\}. \end{cases} \tag{16}$$

Proof: If $l = 1$ then $\mathrm{P}(A_1) = \mathrm{P}(N_1 = 1) = p_1$. Let us assume $l \in \{2, \dots, n - 1\}$. Conditioned by N_1, we have

$$\begin{aligned}\mathrm{P}(A_l) &= \mathrm{P}(A_l \cap \{N_1 = 1\}) + \left(\sum_{k=2}^{n-1} \mathrm{P}(A_l \mid N_1 = k)\mathrm{P}(N_1 = k)\right) + \mathrm{P}(A_l \cap \{N_1 = n\}) \\ &= p_1 + \sum_{k=2}^{n-1} \alpha_k(l) p_k + p_n.\end{aligned}$$

The Lemma implies the following.

- If $l = 2$ then $\mathrm{P}(A_2) = p_1 + \sum_{k=3}^{n} p_k = 1 - p_2$.
- If $l \in \{3, \dots, n - 1\}$ then

$$\begin{aligned}\mathrm{P}(A_l) &= p_1 + \sum_{k=2}^{i-1} \alpha_k(l) p_k + \sum_{k=l+1}^{n-1} \alpha_k(l) p_k + p_n \\ &\overset{(10)}{=} p_1 + \frac{n-l+1}{n-l+2} \sum_{k=2}^{l-1} p_k + \sum_{k=l+1}^{n} p_k \overset{(3)}{=} 1 - \frac{1}{n-l+2} \sum_{k=2}^{l-1} p_k - p_l.\end{aligned}$$

Finally, if $l = n$ then $\mathrm{P}(A_n) = p_1 + \sum_{k=2}^{n-1} \alpha_k(n) p_k \overset{(10),(3)}{=} 1 - \frac{1}{2} \sum_{k=2}^{n-1} p_k - p_n$. Hence (16) holds, which completes the proof.

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Remark 2: Theorem 2 tells us that for $l \in \{3, \dots, n\}$ the probability $\mathrm{P}(A_l)$ depends only on $p_2, \dots, p_l$, and is smaller than $1 - p_l$, which is the probability that the first passenger sits anywhere except the seat l. In addition it implies that

$$p_1 = p_n \text{ if, and only if, } \mathrm{P}(A_n) = \frac{1}{2}. \tag{17}$$

In fact, combining (16) with $l = n$ and (3) yields $\mathrm{P}(A_n) = \frac{1}{2}(1 + p_1 - p_n)$, which gives (17). Note that (17) corresponds to Remark 1 (i).

Example 2:

- Case (I): Equation (16) with $p_1 = p_2 = \dots = p_n = 1/n$ implies (6).
- Case (II): Equation (16) with $p_1 = 0$ and $p_2 = \dots = p_n = 1/(n-1)$ implies

  $$\mathrm{P}(A_l) = \frac{n - l + 1}{n - l + 2} - \frac{1}{(n-1)(n-l+2)} \text{ for } l \in \{2, \dots, n\},$$

  whose form suggests the difference from (6).

We remark that (1) follows from Cases (I) and (II) with $l = n$, respectively.

3. *Proof of Theorem* 1

Suppose $p_1 = p_3 = \dots = p_n$. Then we show

$$\mathrm{P}(A_j \mid A_i^c) = \mathrm{P}(A_j) \text{ for } 2 \leqslant i < j \leqslant n, \tag{18}$$

noting that $\mathrm{P}(A_j \mid A_i^c)$ is well-defined since $\mathrm{P}(A_i^c) \geqslant p_i > 0$ for $i \in \{2, \dots, n-1\}$. It follows that

$$\mathrm{P}(A_j \mid A_i^c) = \mathrm{P}(A_j \mid N_1 = i) = \alpha_i(j) \overset{(10)}{=} \frac{n - j + 1}{n - j + 2}, \tag{19}$$

where the first equality holds for the same reason as (12). Using (16) and

$$p_2 = 1 - (n-1)p_1, \tag{20}$$

we have

$$\mathrm{P}(A_j) = \frac{n - j + 1}{n - j + 2} \text{ for } j \in \{3, \dots, n\}, \tag{21}$$

because

- if $j \in \{3, \dots, n-1\}$ then $\mathrm{P}(A_j) \overset{(16)}{=} 1 - \dfrac{p_2 + (j-3)p_1}{n-j+2} - p_1 \overset{(20)}{=} \dfrac{n-j+1}{n-j+2}$,
- if $j = n$ then $\mathrm{P}(A_n) \overset{(17)}{=} \frac{1}{2}$.

Therefore (18) holds, which implies that A_i and A_j are independent by using [2, 1.5.1, p.3]. Similarly, to show that $A_2, \dots, A_n$ are independent, it is sufficient to prove for any $m \in \{2, 3, \dots, n-2\}$ and $2 \leqslant j_0 < \dots < j_m \leqslant n$,

$$\mathrm{P}\left(\bigcap_{s=1}^{m} A_{j_s}^c \,\middle|\, A_{j_0}^c\right) = \prod_{s=1}^{m} \mathrm{P}(A_{j_s}^c), \tag{22}$$

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which follows from

$$\text{LHS of (22)} \overset{(12)}{=} \mathrm{P}_{j_0}\left(\bigcap_{s=1}^{m} A_{j_s}^c\right) = \mathrm{P}_{j_0}\left(A_{j_m}^c \,\middle|\, \bigcap_{s=1}^{m-1} A_{j_s}^c\right) \mathrm{P}_{j_0}\left(\bigcap_{s=1}^{m-1} A_{j_s}^c\right)$$

$$\overset{(12)}{=} \mathrm{P}_{j_{m-1}}\left(A_{j_m}^c\right) \mathrm{P}_{j_0}\left(\bigcap_{s=1}^{m-1} A_{j_s}^c\right) = \left\{1 - \alpha_{j_{m-1}}\left(j_m\right)\right\} \mathrm{P}_{j_0}\left(\bigcap_{s=1}^{m-1} A_{j_s}^c\right)$$

$$= \prod_{s=1}^{m}\left\{1 - \alpha_{j_{s-1}}\left(j_s\right)\right\} \overset{(10)}{=} \prod_{s=1}^{m} \frac{1}{n - j_s + 2} \overset{(21)}{=} \text{RHS of (22)}.$$

Note that $\mathrm{P}_{j_0}\left(A_{j_m}^c \,\middle|\, \cap_{s=1}^{m-1} A_{j_s}^c\right)$ is also well-defined because it turns out that $\mathrm{P}_{j_0}\left(\cap_{s=1}^{m-1} A_{j_s}^c\right) > 0$ from (7). Hence $A_2, \dots, A_n$ are independent.

Next, we suppose that $A_2, \dots, A_n$ are independent. Then (21) is obtained by (18) and (19). Hence (16) yields

$$1 - \frac{1}{n - l + 2} \sum_{k=2}^{l-1} p_k - p_l = \frac{n - l + 1}{n - l + 2},$$

so that

$$p_l = \frac{p_1 + p_l + \dots + p_n}{n - l + 2} \text{ for } l \in \{3, \dots, n\}.$$

If $l = n$ then $p_1 = p_n$. If $l = n - 1$ then $p_{n-1} = \frac{1}{3}\left(p_1 + p_{n-1} + p_n\right)$, which implies $p_{n-1} = p_1 = p_n$. Repeating this procedure leads to $p_1 = p_3 = \dots = p_n$, which completes the proof.

4. *Conclusion*

Let us remark that the condition (7) is required for Theorem 1. Indeed, if (7) is violated then $A_2, \dots, A_n$ are independent for $p_1 = 1$ or $p_n = 1$ which does not satisfy $p_1 = p_3 = \dots = p_n$. Finally, it would be interesting to have an intuitively clear reason why the value of p_2 is independent of the result of Theorem 1.

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Quotations for *Nemo* (continued from page 224)

2. What odds and ends of knowledge we picked up on those long days in the saddle! That if lightning strikes a pine even lightly, it kills, but that a fir will ordinarily survive; that mountain miles are measured air-line, so that twenty-five miles may really be forty, and that, even then, they are calculated on the level, so that one is credited with only the base of the triangle while he is laboriously climbing up its hypotenuse. I am personally acquainted with the hypotenuses of a good many mountains, and there is no use trying to pretend that they are bases. They are not.

3. Englishmen will not even believe that the square on the hypotenuse is equal to the squares on the containing sides until they have measured and weighed as well as they are able by rude experimental devices a few selected pieces of rudely shaped rectangular paper..

4. It usually takes a hypotenuse a long time to discover that it is the longest side of a triangle. But it's a long line that has no turning.

5 ‘Sh, Jemima! Daddy’s talking.’
‘…that the square on the hypotenuse is equal to…’
‘But *Mummy*…’
‘*Sh,* Jemima! You mustn’t interrupt when someone’s speaking! How many times have I had to tell you?’
‘…equal to the sum of the squares on the other two sides!’

6. He proposed to take some leading proposition of Euclid's, and show by construction that its truth was known to us, to demonstrate, for example, that the angles at the base of an isosceles triangle are equal, and that if the equal sides be produced the angles on the other side of the base are equal also, or that the square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the two other sides. By demonstrating our knowledge of these things we should demonstrate our possession of a reasonable intelligence.