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EXISTENCE OF POSITIVE RADIAL SOLUTIONS FOR A CLASS OF NONLINEAR SINGULAR ELLIPTIC PROBLEMS IN ANNULAR DOMAINS

by ZONGMING GUO

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We establish the existence of positive radially symmetric solutions of $\Delta u+f(r,u,u')=0$ in the domain $R_1<r<R_0$ with a variety of Dirichlet and Neumann boundary conditions. The function f is allowed to be singular when either $u=0$ or $u'=0$. Our analysis is based on Leray–Schauder degree theory.

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1.

In this paper we consider the existence of positive radially symmetric solutions of the singular equation

$$\Delta u+f(r,u,u'(r))=0 \quad \text{in } R_1<r<R_0 \tag{1.1}$$

subject to one of the following sets of boundary conditions

$$u=0 \quad \text{on } r=R_1 \quad \text{and} \quad u=0 \quad \text{on } r=R_0 \tag{1.2a}$$

$$u=0 \quad \text{on } r=R_1 \quad \text{and} \quad \frac{\partial u}{\partial r}=0 \quad \text{on } r=R_0 \tag{1.2b}$$

$$\frac{\partial u}{\partial r}=0 \quad \text{on } r=R_1 \quad \text{and} \quad u=0 \quad \text{on } r=R_0 \tag{1.2c}$$

Here $r=|x|$, $x\in\mathbb{R}^N$, $N\geqq 3$; $\partial/\partial r$ denotes differentiation in the radial direction; $0<R_1<R_0\leqq\infty$; f is continuous on $(R_1,R_0)\times(0,\infty)\times(-\infty,\infty)$. The equation is singular because f is allowed to be singular at $u=0$, $u'=0$ and $r=R_0$, R_1.

Equation (1.1) arises in many branches of mathematics and applied mathematics. It has been studied by many authors, see for example, **[5]**, **[8]**, **[14]**.

Recently, the problem of the existence of positive radially symmetric solutions of the problem

$$\left.\begin{aligned}\Delta u+f(u)=0 \quad \text{in} \quad \Omega \\ u=0 \quad \text{on} \quad \partial\Omega\end{aligned}\right\} \tag{1.3}$$

where $\Omega\subset\mathbb{R}^N$, $N\geqq 3$ is a symmetric domain and f is nondecreasing, has attracted much interest and has been treated by many authors (see [**5, 9, 15–16, 10**]). An important contribution was made by Gidas, Ni and Nirenberg [**10**] who showed that all positive solutions in $C^2(\bar{\Omega})$ of the equation (1.3) are radially symmetric provided that Ω is a N-ball. They also provided that no such result automatically applies to the annulus (see also [**13**]). On the other hand, when $f(u)$ is superlinear (i.e. $\lim_{t\to\infty} f(t)/t=\infty$) the existence of positive solutions of problem (1.3) with a general Ω has been provided under various sets of assumptions, always including a restriction on the growth of f at infinity (see [**1, 4, 14**]). It is known that such a growth condition is, in general, necessary for starlike domains [**18**]. In the case of the annulus, such a growth condition is not necessary [**2**]. Therefore, the problem of the existence of solutions of equation (1.1) in an annulus is of much interest. It is natural to look for radially symmetric solutions because an annulus is a symmetric domain. We easily give many references ([**2, 6, 11–13**]).

We shall study the problems when f depends on u and u' and f is nonincreasing for u and u'. When $f(r,u,u'(r))=f(u)$ and $f(u)$ is nonincreasing, some results about the existence and uniqueness of a positive radially symmetric solution of the problem (1.1) with the boundary condition (1.2a) has been given in [**11**].

Putting

$$\begin{aligned} t&=[(N-2)r^{N-2}]^{-1} \\ \phi(t)&=([N-2)t]^{-k} \quad k=(2N-2)/(N-2) \\ t_i&=[(N-2)R_i^{N-2}]^{-1} \quad i=0,1 \end{aligned} \tag{1.4}$$

radial solutions of (1.1) are solutions of

$$u''(t)+\phi(t)g(t,u,u'(t))=0 \quad t_0<t<t_1 \tag{1.1$'$}$$

(see [**2**]). Now the boundary conditions become

$$u(t_0)=0 \quad \text{and} \quad u(t_1)=0 \tag{1.2a$'$}$$

$$u'(t_0)=0 \quad \text{and} \quad u(t_1)=0 \tag{1.2b$'$}$$

$$u(t_0)=0 \quad \text{and} \quad u'(t_1)=0 \tag{1.2c$'$}$$

(when $R_0=\infty$, $t_0=0$). In this, or other equivalent forms, these problems have been investigated by many authors (see [**3, 7, 11, 17**]). Our results improve on the results of [**3**], [**17**] and cover many new examples not treated by [**3**], [**11**], [**17**].

By a solution u of (1.1)′ we mean $u \in C^2(t_0, t_1) \cap C^1[t_0, t_1]$.

In Sections 2–3, we use the Leray–Schauder degree to seek positive solutions of (1.1)′ subject to one of (1.2a)′, (1.2b)′ and (1.2c)′. All the results obtained in Sections 2–3 can be applied to obtain the existence of a positive radial solution of (1.1).

2.

In this section we establish the existence of positive solutions on $[t_0, t_1]$ of

$$\begin{cases} u'' + \phi(t) g(t, u, u'(t)) = 0 \\ u(t_0) = 0, u'(t_1) = b \geqq 0 \end{cases} \tag{2.1}$$

where $t_0 > 0$, $\phi(t)$ is as in (1.4).

Theorem 2.1. *Suppose that*

(i) *g is continuous on $[t_0, t_1] \times (0, \infty) \times (-\infty, \infty)$;*

(ii) *$0 < g(t, u, z) \leqq \psi(t) h(u)$ on $(t_0, t_1) \times (0, \infty) \times (-\infty, \infty)$, where*

- (a) *$h(u) > 0$ is continuous and nonincreasing on $(0, \infty)$,*
- (b) *$\psi > 0$ is continuous on $[t_0, t_1]$,*
- (c) *$1/h(k(t - t_0))$ is continuous on $[t_0, t_1]$ for each constant k, $0 < k < 1$.*
- (d) *$\int_{t_0}^{t_1} h(k(t - t_0)) \psi(t)\, dt < \infty$ for any constant k, $0 < k < 1$,*
- (e) *$\lim_{t \to \infty} h(t) \int_1^t [h(s)]^{-1} ds = \infty$.*

(iii) *for each constant $M_0 > 0$ there exists $\xi(t)$ continuous on $[t_0, t_1]$ and positive on (t_0, t_1) such that $g(t, u, z) \geqq \xi(t)$ on $[t_0, t_1] \times (0, M_0] \times (-\infty, \infty)$. Then Problem (2.1) has a positive solution.*

Example 1. Let $g(t, u, z) = t^{-2} u^{-1/2}(1 + 3u^{1/2})(2 + z^2)(1 + z^2)^{-1}$ and $\xi(t) = 3t^{-2}$. We let $h(u) = u^{-1/2}(1 + 3u^{1/2})$, $\psi(t) = 2t^{-2}$, an easy calculation shows that $g(t, u, z)$, $h(u)$ and $\xi(t)$ satisfy all the conditions of Theorem 1. Therefore, the equation

$$\Delta u + [(N-2) r^{N-2}]^2 u^{-1/2}(1 + 3u^{1/2})[2 + r^{2(N-1)}(u'(r))^2][1 + r^{2(N-1)}(u'(r))^2]^{-1} = 0$$

for $r \in [R_1, R_0]$ with the boundary condition (1.2c) has a positive radially symmetric solution.

Example 2. Let $g(t, u, z) = (t - t_0)^2 u^{-5/2}(3 + z^2)(1 + z^2)^{-1}$, $\xi(t) = M_0^{-5/2}(t - t_0)^2$, where M_0 is as in (iii). Let $h(u) = u^{-5/2}$, $\psi(t) = 3(t - t_0)^2$, we obtain from Theorem 2.1 that the equation

$$\Delta u + (N-2)^2 [r^{-(N-2)} - R_0^{-(N-2)}]^2 u^{-5/2}[3 + r^{2(N-1)}(u'(r))^2][1 + r^{2(N-1)}(u'(r))^2]^{-1} = 0$$

for $r \in [R_1, R_0]$ with the boundary condition (1.2c) has a positive radially symmetric solution.

Proof of Theorem 2.1.

We consider the problem:

$$\begin{cases} u'' + \phi(t)g(t,u,u') = 0 \\ u(t_0) = 1/n, u'(t_1) = b \geqq 0 \end{cases} \tag{2.2}$$

where $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ to avoid the possible singularity of g at $u = 0$. If u is any solution to (2.2), then $u'' < 0$ on (t_0, t_1). So, $u' > b \geqq 0$ on (t_0, t_1) which implies that u is strictly increasing on (t_0, t_1). Accordingly, we may remove the singularity at $u = 0$ by defining

$$g_n(t,u,z) = \begin{cases} g(t, |u|, z), & t_0 < t < t_1, |u| \geqq 1/n \\ g(t, 1/n, z), & t_0 < t < t_1, |u| < 1/n \end{cases}.$$

So, every solution v of

$$\begin{cases} v'' + \phi(t)g_n(t,v,v') = 0 \\ v(t_0) = 1/n, v'(t_1) = b \end{cases} \tag{2.3}$$

is a solution to (2.2). We now consider the family of problems

$$\begin{cases} u'' + (1-\lambda)\delta + \lambda\phi(t)g_n(t,u,u') = 0 \\ u(t_0) = 1/n, u'(t_1) = b \end{cases} \tag*{(2.4)$^n_\lambda$}$$

where $0 < \delta < 1$ is a positive real number which is determined below and $\lambda \in [0, 1]$. Let $u(t)$ be a solution of $(2.4)^n_\lambda$, then $u(t) \geqq 1/n$, $u'(t) \geqq b$ for $t \in [t_0, t_1]$. We also have $u'' + (1-\lambda)\delta + \lambda\phi(t)\psi(t)h(u) \geqq u'' + (1-\lambda)\delta + \lambda\phi(t)g_n(t,u,u') = 0$ and this implies $-u'' \leqq (1-\lambda)\delta + \lambda\phi(t)\psi(t)h(u)$. Integrating from t to t_1 we obtain

$$u'(t) - b \leqq (1-\lambda)\delta(t_1 - t_0) + \lambda \int_t^{t_1} \phi(s)\psi(s)h(u(s))\, ds \leqq \delta(t_1 - t_0) + h(u(t)) \int_{t_0}^{t_1} \phi(s)\psi(s)\, ds,$$

since h is nonincreasing. Thus,

$$u'(t) \leqq \delta(t_1 - t_0) + h(u(t)) \int_{t_0}^{t_1} \phi(s)\psi(s)\, ds + b. \tag{2.5}$$

Dividing by $h(u(t))$ and integrating from t_0 to t we obtain

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$$\int_{1/n}^{u(t)} \frac{dv}{h(v)} \leqq [\delta(t_1-t_0)+b]\frac{(t_1-t_0)}{h(u(t))}+(t_1-t_0)\int_{t_0}^{t_1}\phi(t)\psi(t)\,dt. \tag{2.6}$$

It follows from (*e*) that there exists a constant $C_0>0$ which is independent of λ, δ, n, such that

$$u(t)\leqq C_0 \quad t\in[t_0,t_1] \tag{2.7}$$

therefore $1/n\leqq u(t)\leqq C_0$. On the other hand, from $u''(t)\leqq -(1-\lambda)\delta-\lambda\phi(t)\xi(t)$, we obtain $u(t)\geqq\theta(t)+1/n$, where

$$\theta(t)=b(t-t_0)+2^{-1}(1-\lambda)\delta(2t_1-t_0-t)(t-t_0)+\lambda\int_{t_0}^{t}\int_{s}^{t_1}\phi(v)\xi(v)\,dv\,ds. \tag{2.8}$$

Let

$$\zeta(t)=\int_{t_0}^{t}\int_{s}^{t_1}\phi(v)\xi(v)\,dv\,ds$$

and

$$k_0=\int_{t_0}^{t_1}\phi(s)\xi(s)\,ds,$$

then $\zeta'(t_0)=k_0>0$. Hence there is an $\varepsilon>0$ such that $\zeta(t)\geqq 2^{-1}k_0(t-t_0)$ on $[t_0,t_0+\varepsilon]$. Since $\zeta(t)/(t-t_0)$ is bounded below on $[t_0+\varepsilon,t_1]$, there is a constant $k_1>0$ such that $\zeta(t)\geqq k_1(t-t_0)$ on $[t_0+\varepsilon,t_1]$. Let $\bar{k}=\min\{k_0/2,k_1\}$, therefore,

$$1/n+[b+2^{-1}(1-\lambda)\delta(t_1-t_0)+\lambda\bar{k}](t-t_0)\leqq u(t)\leqq C_0. \tag{2.9}$$

Let $0<\delta<\min\{(t_1-t_0)^{-1},\bar{k}(t_1-t_0)^{-1},1\}$. We obtain

$$1/n+[2^{-1}\delta(t_1-t_0)](t-t_0)\leqq u(t)\leqq C_0. \tag{2.10}$$

Using $|u''(t)|\leqq 1+\phi(t)\psi(t)h(u(t))$, we know that

$$|u''|\leqq 1+\phi(t)\psi(t)h([2^{-1}\delta(t_1-t_0)](t-t_0)), \tag{2.11}$$

and then

$$|u'(t)|\leqq b+(t_1-t_0)+\int_{t_0}^{t_1}\phi(t)\psi(t)h([2^{-1}\delta(t_1-t_0)](t-t_0))\,dt=C_1, \tag{2.12}$$

where $C_1>0$ is independent of λ, n. Let $x(t)=1/h([2^{-1}\delta(t_1-t_0)](t-t_0))$, then

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$$|x(t)u''(t)| \leqq x(t) + \phi(t)\psi(t) \leqq C_2. \tag{2.13}$$

For $u \in C^2(t_0, t_1) \cap C^1[t_0, t_1]$ define

$$\|u\|_0 = \sup_{[t_0, t_1]} |u(t)|,$$

$$\|u\|_1 = \max(\|u\|_0, \|u'\|_0),$$

$$\|u\|_2 = \max(\|u\|_0, \|u'\|_0, \sup_{[t_0, t_1]} |x(t)u''(t)|).$$

Set

$$\mathbb{K}_{a,b} = \{u \in C^2(t_0, t_1) \cap C^1[t_0, t_1] u(t_0) = a \geqq 0, u'(t_1) = b \geqq 0 \text{ and } \|u\|_2 < \infty\}$$

with norm $\|\cdot\|_2$ and

$$\mathbb{C} = \{u \in C(t_0, t_1) \colon \sup_{[t_0, t_1]} |u(t)| < \infty\}$$

with the obvious norm. We know that $\mathbb{K}_{a,b}$ and $\mathbb{C}$ are Banach spaces (see [3]). Define mappings $G_{\lambda,n}\colon C^1[t_0, t_1] \to \mathbb{C}$, $j\colon \mathbb{K}_{1/n,b} \to C^1[t_0, t_1]$, $L\colon \mathbb{K}_{1/n,b} \to \mathbb{C}$ by $G_{\lambda,n}(u)(t) = x(t)[(1-\lambda)\delta + \lambda\phi(t)g_n(t, u, u')]$, $ju = u$, and $Lu = x(t)u''(t)$. Clearly $G_{\lambda,n}$ is continuous by the continuity of $x(t)g_n$. By the same idea as in [3], [17] we know that j is completely continuous and L^{-1} exists and is continuous. Now, $(2.4)_\lambda^n$ is equivalent to

$$(I + L^{-1}G_{\lambda,n}j)(u) = 0. \tag{2.14}$$

Let

$$C = \max\{C_0, C_1, C_2, 1 + b(t_1 - t_0) + \delta(t_1 - t_0)^2, \delta \max_{[t_0, t_1]}(x(t)), b + \delta(t_1 - t_0)\} \tag{2.15}$$

and define

$$U = \{u \in \mathbb{K}_{1/n,b}, \|u\|_2 < C + 1\}, \tag{2.16}$$

then $(I + L^{-1}G_{\lambda,n}j)(u) \neq 0$ on ∂U. Finally, by $1/n + b(t - t_0) + 2^{-1}\delta(2t_1 - t_0 - t)(t - t_0) \in U$ and properties of the Leray–Schauder degree, we have

$$\deg(I + G_{1,n}j, U, 0) = 1. \tag{2.17}$$

Then (2.3) has a solution in U. The remainder of the proof is similar to the proof of Theorem 1 of [3].

Remark 1. The results of Theorem 1 extend the results of [3], [17]. In [17], one of the conditions on $h(u)$ is $\int_0^c h(s)\,ds<\infty$, $c\in[0,\infty)$. So, for example, no result of [17] applies to Example 2 above.

Theorem 2.2. *Suppose that*

(i) g *is continuous on* $(t_0,t_1)\times(0,\infty)\times(0,\infty)$,

(ii) $\xi(t)\leqq g(t,u,z)\leqq\psi(t)h(u)p(z)$ *on* $(t_0,t_1)\times(0,\infty)\times(0,\infty)$,

where

(a) $h(u)>0$, $p(z)>0$ *are continuous and nonincreasing on* $(0,\infty)$,

(b) $\psi(t)>0$ *and* $\xi(t)>0$ *are continuous on* (t_0,t_1); $\xi(t_1)>0$ *and* $\int_{t_0}^{t_1}\xi(t)\,dt<\infty$,

(c) $\int_{t_0}^{t_1}\psi(t)h(\alpha(t-t_0))p(\beta(t_1-t))\,dt<\infty$, *for each pair* (α,β), $0<\alpha,\beta<1$,

(d) $g(t,u,z)/[\psi(t)h(\alpha(t-t_0))p(\beta(t_1-t))]$, $[\psi(t)h(\alpha(t-t_0))p(\beta(t_1-t))]^{-1}$ *are continuous on* $[t_0,t_1]\times(0,\infty)\times(0,\infty)$ *and* $[t_0,t_1]$ *respectively for* α, β, $0<\alpha,\beta<1$. *Then Problem (2.1) has a positive solution.*

Example 3. Let $g(t,u,z)=\psi(t)h(u)p(z)$; here $\psi(t)=t$, $h(u)=u^{-3/4}(1+u^{3/4})$ and

$$p(z)=\begin{cases}z^{-1/2}, & 0<z\leqq 1\\ \frac{1}{2}(1+z^{-4/5}), & z>1\end{cases}.$$

Let $\xi(t)=t/2$. A calculation shows that the functions satisfy the conditions of Theorem 2.2.

Proof of Theorem 2.2.

We only discuss the case when $u'(t_1)=b=0$. For $b\neq 0$ this theorem follows easily from the proof of Theorem 2.1. We consider the family of problems

$$\begin{cases}u''+(1-\lambda)\delta+\lambda\phi(t)g_n(t,u,u')=0\\ u(t_0)=1/n,\, u'(t_1)=1/n\end{cases}\tag{2.18$^n_\lambda$}$$

where

$$g_n(t,u,z)=\begin{cases}g(t,|u|,|z|), & t_0<t<t_1, |u|\geqq 1/n, |z|\geqq 1/n\\ g(t,1/n,1/n), & t_0<t<t_1, |u|<1/n, |z|<1/n\end{cases}$$

$0<\delta<1$ is determined below and $\lambda\in[0,1]$. Let u be a solution of $(2.18)^n_\lambda$, then $u(t)\geqq 1/n$, $u'(t)\geqq 1/n$, for $t\in[t_0,t_1]$. It follows from (ii) that $u(t)\geqq\theta(t)+1/n$ and $u'(t)\geqq\theta'(t)+1/n$, where

$$\theta(t)=2^{-1}(1-\lambda)\delta(2t_1-t_0-t)(t-t_0)+\lambda\int_{t_0}^{t}\int_{s}^{t_1}\phi(v)\xi(v)\,dv\,ds, \tag{2.19}$$

$$\theta'(t)=(1-\lambda)\delta(t_1-t)+\lambda\int_{t}^{t_1}\phi(s)\xi(s)\,ds, \tag{2.20}$$

$$\theta''(t_1)=-(1-\lambda)\delta-\lambda\phi(t_1)\xi(t_1). \tag{2.21}$$

Then, there exists a $C_3>0$ such that $\int_{t_0}^{t_1}\phi(s)\xi(s)\,ds>C_3$ and

$$\theta'(t_0)\geqq(1-\lambda)\delta(t_1-t_0)+\lambda C_3. \tag{2.22}$$

Using the same methods as in the proof of Theorem 2.1, we have that there exists a constant k_2, $0<k_2<1$ such that

$$u(t)\geqq k_2(t-t_0) \quad \text{for} \quad t\in(t_0,t_1). \tag{2.23}$$

Whether $\xi(t_1)=\infty$ or not, there exists a constant $C_4>0$ such that $\phi(t_1)\xi(t_1)>C_4$, and

$$\theta''(t_1)\leqq-(1-\lambda)\delta-\lambda C_4. \tag{2.24}$$

Using the same idea as above, we have that there exists a constant k_3, $0<k_3<1$ such that

$$\theta'(t)\geqq k_3(t_1-t), \quad \text{for} \quad t\in(t_0,t_1). \tag{2.25}$$

Therefore, $u'(t)\geqq k_3(t_1-t)+1/n$ and $|u''(t)|\leqq 1+\phi(t)\psi(t)h(k_2(t-t_0))p(k_3(t_1-t))$. Hence, by conditions (c),

$$|u'(t)|\leqq(t_1-t_0)+\int_{t_0}^{t_1}\phi(t)\psi(t)h(k_2(t-t_0))p(k_3(t_1-t))\,dt=C_5,$$

and

$$|u(t)|\leqq C_6 \quad \text{for} \quad t\in[t_0,t_1]. \tag{2.26}$$

$C_6=C_5(t_1-t_0)$. Let $x(t)=[\psi(t)h(k_2(t-t_0))p(k_3(t_1-t))]^{-1}$, then, $|x(t)u''(t)|\leqq x(t)+\phi(t)\leqq C_7$. Here C_5, C_6 and C_7 are positive constants which are independent of λ, n. The remainder of the proof is similar to the proofs of Theorem 2.1.

If $g(t,u,z)$ has singularities at $t=t_0$, $t=t_1$ and $\lim_{u\to\infty}g(t,u,z)=0$ for $(t,z)\in(t_0,t_1)\times(-\infty,\infty)$, then the condition (ii) of Theorem 2.2 does not hold. In this case the following theorem applies:

Theorem 2.3. *Suppose that*

(i) *g is continuous on* $(t_0, t_1)\times(0,\infty)\times(-\infty,\infty)$;

(ii) $0<g(t,u,z)\leqq\psi(t)h(u)$ *on* $(t_0,t_1)\times(0,\infty)\times(-\infty,\infty)$, *where*

(a) $h>0$ *is continuous and nonincreasing on* $(0,\infty)$,

(b) $\psi>0$ *is continuous on* (t_0,t_1) *and* $\int_{t_0}^{t_1}(t-t_0)\psi(t)\,dt<\infty$,

(c) $g(t,u,z)/[\psi(t)h(k(t-t_0))]$ *and* $1/[\psi(t)h(k(t-t_0))]$ *are continuous on* $[t_0,t_1]\times(0,\infty)\times(-\infty,\infty)$ *and* $[t_0,t_1]$ *respectively, for each constant* k, $0<k<1$.

(d) $\int_{t_0}^{t_1}h(k(t-t_0))\psi(t)\phi(t)\,dt<\infty$ *for any constant* k, $0<k<1$,

(e) $\lim_{t\to\infty}h(t)\int_1^t[h(s)]^{-1}\,ds=\infty$.

(iii) *For each constant* $M_0>0$ *there exists* $\xi(t)>0$ *continuous on* (t_0,t_1) *and* $\int_{t_0}^{t_1}(t-t_0)\xi(t)\,dt<\infty$, *such that* $g(t,u,z)\geqq\xi(t)$ *on* $(t_0,t_1)\times(0,M_0]\times(-\infty,\infty)$. *Then Problem (2.1) has a positive solution.*

Proof. Let $x(t)=\{\psi(t)h([2^{-1}\delta(t_1-t_0)](t-t_0))\}^{-1}$ be as in Theorem 2.1, then the result of this theorem follows from a slight modification of the proof of Theorem 2.1 by changing the order of integration (see [**3**], [**11**]).

If $g(t,u,z)$ has singularities at $t=t_0$, $t=t_1$ and $\lim_{z\to\infty}g(t,u,z)=0$ for $(t,u)\in(t_0,t_1)\times(0,\infty)$, then the following theorem applies:

Theorem 2.4. *Suppose that*

(i) *g is continuous on* $(t_0,t_1)\times[0,\infty)\times(0,\infty)$,

(ii) $0<g(t,u,z)\leqq\psi(t)p(z)$ *on* $(t_0,t_1)\times(0,\infty)\times(0,\infty)$, *where*

(a) $p(z)>0$ *is continuous and nonincreasing on* $(0,\infty)$ *and* $zp(z)$ *is nondecreasing on* $(0,\infty)$,

(b) $\psi(t)>0$ *is continuous on* (t_0,t_1) *and* $\int_{t_0}^{t_1}\psi(t)\,dt<\infty$,

(c) $g(t,u,z)/[\psi(t)p(k(t_1-t))]$ *is continuous on* $[t_0,t_1]\times[0,\infty)\times(0,\infty)$ *for each constant* $k>0$,

(iii) *for constants* M_1, $M_2>0$ *there exists a continuous, positive function* $\xi(t)$ *on* (t_0,t_1) *and* $\int_{t_0}^{t_1}\xi(t)\,dt<\infty$ *such that* $g(t,u,z)\geqq\xi(t)$ *on* $(t_0,t_1)\times(0,M_1]\times(0,M_2]$. *Then Problem (2.1) has a positive solution.*

Proof. We only discuss the case when $u'(t_1)=b=0$. We consider the family of problems

$$\begin{cases}u''+\lambda\phi(t)g_n(t,u,u')=0\\ u(t_0)=1/n,\,u'(t_1)=1/n\end{cases}. \tag{2.27$_\lambda^n$}$$

Here $g_n(t,u,u')$, λ are as in the proof of Theorem 2.2. Let u be a solution of $(2.27)_\lambda^n$, then $u(t)\geqq 1/n$ and $u'(t)\geqq 1/n$, for $t\in[t_0,t_1]$. From (ii) we know that $(1/p(u'(t)))u''(t)+\lambda\phi(t)\psi(t)\geqq 0$. Let

$$f(z)=\int_0^z \frac{dz}{p(z)},$$

$f(z)$ is increasing since $p(z)$ is decreasing and $(f(u'(t)))'+\lambda\phi(t)\psi(t)\geqq 0$. Therefore,

$$f(u'(t))\leqq f(1/n)+\int_{t_0}^{t_1}\phi(t)\psi(t)\,dt.$$

The fact that $f(z)$ is increasing and condition (b) together imply that $|u'(t)|\leqq C_8$ and $|u(t)|\leqq C_8(t_1-t_0)$. Let $C_9=C_8(t_1-t_0)$. Then C_8 and C_9 are independent of λ, n. By condition (iii) and the same idea as in the proof of Theorem 2.2, we have that there exists $k_4>0$ such that $u'(t)\geqq\lambda k_4(t_1-t)$ and

$$|u''(t)|\leqq\lambda\phi(t)\psi(t)p(\lambda k_4(t_1-t))\leqq\phi(t)\psi(t)p(k_4(t_1-t)). \tag{2.28}$$

Let $x(t)=1/[\psi(t)p(k_4(t_1-t))]$. Define $\mathbb{K}_{1/n,1/n}$, L, j, $G_{\lambda,n}$ *as* in Theorem 2.1 for $\delta=0$, the proof is then a consequence of Leray–Schauder degree theory as in the proof of Theorem 2.1.

Remark 2. By the same methods we can discuss the following problem

$$\begin{cases} u''+\phi(t)g(t,u,u')=0\\ u'(t_0)=0, u(t_1)=0\end{cases} \tag{2.29}$$

and obtain results similar to the above theorems.

3.

In this section we examine the existence of positive solutions on $(t_0,t_1)(t_0>0)$ to

$$\begin{cases} u''+\phi(t)g(t,u,u')=0\\ u(t_0)=u(t_1)=0\end{cases} \tag{3.1}$$

Theorem 3.1. *Suppose that*

(i) *g is continuous on $(t_0,t_1)\times(0,\infty)\times(-\infty,\infty)$;*

(ii) *$0<g(t,u,z)\leqq\psi(t)h(u)$ on $(t_0,t_1)\times(0,\infty)\times(-\infty,\infty)$, where*

(a) *$h(u)$ is continuous and nonincreasing on $(0,\infty)$,*

(b) *$\psi>0$ is continuous on (t_0,t_1) and $\int_{t_0}^{t_1}(t_1-t)(t-t_0)\psi(t)\,dt<\infty$,*

(c) *$g(t,u,z)/[\psi(t)h(k(t-t_0)(t_1-t))]\in C^0([t_0,t_1]\times(0,\infty)\times(-\infty,\infty))$, for each constant k, $0<k<1$,*

(d) $\int_{t_0}^{t_1} \psi(t)h(k(t-t_0)(t_1-t))\,dt<\infty$ *for any constant* k, $0<k<1$,

(e) $\lim_{t\to\infty} h(t)\int_1^t [h(s)]^{-1}\,ds=\infty$.

(iii) *for each constant* $M_0>0$ *there exists a continuous and positive function* $\xi(t)$ *on* (t_0,t_1), $\int_{t_0}^{t_1}(t-t_0)(t_1-t)\xi(t)\,dt<\infty$ *such that*

$$g(t,u,u')\geqq \xi(t) \text{ on } (t_0,t_1)\times(0,M_0]\times(-\infty,\infty).$$

Then Problem (3.1) has a positive solution.

Proof. Use the same idea as in the proof of Theorem 1 of [**11**].

Theorem 3.2. *Suppose that*

(i) $g(t,u,z)$ *is continuous on* $[t_0,t_1]\times(0,\infty)\times(0,\infty)$;

(ii) $\xi(t)\leqq g(t,u,z)\leqq \psi(t)h(u)p(z)$ *on* $[t_0,t_1]\times(0,\infty)\times(0,\infty)$, *where*

(a) $h>0$, $p>0$ *are continuous and nonincreasing on* $(0,\infty)$, $\lim_{z\to 0}p(z)=\infty$,

(b) $\xi(t)>0$, $\psi(t)>0$ *are continuous on* $[t_0,t_1]$, $\xi>0$ *at* $t=t_0$, t_1,

(c) $\int_{t_0}^{t_0+\varepsilon}\psi(t)h(k(t-t_0))\,dt<\infty$; $\int_{t_1-\varepsilon}^{t_1}\psi(t)h(k(t_1-t))\,dt<\infty$; $\int_0^{\varepsilon}p(t)\,dt<\infty$, *for each constant* ε, k, $0<\varepsilon,k<1$,

(d) $1/h(k(t_1-t)$ *and* $1/h(k(t-t_0))$ *are continuous on* $[t_0,t_1]$. *Then problem*

$$\begin{cases} u''+\phi(t)g(t,u,|u'|)=0\\ u(t_0)=u(t_1)=0\end{cases} \tag{3.2}$$

has a positive solution in $C^2(t_0,t_1)\cap C^1[t_0,t_1]$.

Proof. We consider the family of problems

$$\begin{cases} u''+(1-\lambda)\delta+\lambda\phi(t)g_n(t,u+1/n,u')=0\\ u(t_0)=u(t_1)=0\end{cases} \tag*{(3.3)$_\lambda^n$}$$

Here $g_n(t,u,u')$ is as in the proof of Theorem 2.2. Let u be a solution of $(3.3)_\lambda^n$, then $u\geqq 0$ for $t\in[t_0,t_1]$. Let $t_2\in(t_0,t_1)$, $u(t_2)=\max_{t_0<t<t_1}u(t)$. By the proof of Theorem 2.2, we know that there exist k_5, k_6 satisfying $0<k_5,k_6<1$ such that $u(t)\geqq k_5(t-t_0)$, $u'(t)\geqq k_6(t_2-t)$ for $t\in[t_0,t_2]$. A similar argument on $[t_2,t_1]$ yields $u(t)\geqq k_7(t_1-t)$, $|u'(t)|\geqq k_8(t-t_2)$ for $t\in[t_2,t_1]$. Here, $0<k_7,k_8<1$. Then,

$$|u''(t)|\leqq(1-\lambda)\delta+\lambda\phi(t)\psi(t)h(u)p(|u'|)\leqq\begin{cases}1+\phi(t)\psi(t)h(k_5(t-t_0))p(k_6(t_2-t)), & t\in(t_0,t_2)\\ 1+\phi(t)\psi(t)h(k_7(t_1-t))p(k_8(t-t_2)), & t\in(t_2,t_1)\end{cases}. \tag{3.4}$$

Let

$$x(t)=\begin{cases}1/h(k_5(t-t_0))p(k_6(t_2-t)), & t\in(t_0,t_2)\\ 1/h(k_7(t_1-t))p(k_8(t-t_2)), & t\in(t_2,t_1)\end{cases}.$$

By condition (d), $x\in C^0[t_0,t_1]$. Therefore,

$$|x(t)u''(t)|\leqq C_{10}. \tag{3.5}$$

Condition (c) and (3.4) imply

$$|u'(t)|\leqq C_{11}, |u(t)|\leqq C_{12}. \tag{3.6}$$

Here C_{10}, C_{11} and C_{12} are positive constants which are independent of λ, n and δ. The remainder of the proofs follows from a slight modification of the proof of Theorem 1 of [**11**].

Remark 3. It follows easily that the results of Theorems 2.3 and 3.2 still hold for $t_0=0$ if the function $m(t)=\phi(t)\psi(t)$ satisfies the conditions imposed on $\psi(t)$, $n(t)=\phi(t)\xi(t)$ satisfies the conditions imposed on $\xi(t)$. The result of Theorem 2.2 holds for $t_0=0$, if $m(t)$ satisfies the conditions imposed on $\psi(t)$ and $\int_0^{t_1}\phi(t)\xi(t)\,dt<\infty$. These results prove the existence of a positive radially symmetric solution u of equation (1.1) in $\Omega=\{|x|,|x|>\mathrm{R}_1\}$ and u satisfies $u|_{r=R1}=0$ and $\lim_{r\to\infty}u(r)=0$.

Remark 4. Consider the following problem:

$$\begin{cases}u''+\lambda\phi(t)g_n(t,u+1/n,u')=0\\ u(t_0)=u(t_1)=0.\end{cases} \tag{3.7}$$

Here $g_n(t,u,u')$ is as in Theorem 3.2. Using the same idea as in the proof of Theorem 3.2, we can obtain an existence result for Problem (3.2) if $g(t,u,z)$, $\psi(t)$ and $p(z)$ satisfy all the conditions of Theorem 2.4 but (c) replaced by

(c)′ $g(t,u,z)/\psi(t)$ *is continuous on* $[t_0,t_1]\times[0,\infty)\times(0,\infty)$.

We also need $\lim_{z\to 0}p(z)=\infty$.

Remark 5. By the results obtained in [**17**], we prove the existence of positive radial solutions of (1.1) subject to one of the following sets of boundary conditions

$$u=a\geqq 0 \quad \text{on } r=R_1 \quad \text{and} \quad u=b\geqq 0 \quad \text{on } r=R_0$$

$$u=\alpha\geqq 0 \quad \text{on } r=R_1 \quad \text{and} \quad \frac{\partial}{\partial r}=b>0 \quad \text{on } t=R_0$$

$$\frac{\partial u}{\partial r}=a>0 \quad \text{on } r=R_1 \quad \text{and} \quad u=b\geqq 0 \quad \text{on } r=R_0$$

where $0<R_1<R_0\leqq\infty$.

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Department of Mathematics
University of Glasgow
Glasgow G12 8QW
Scotland

Henan Normal University
Xinxiang
P.R. China

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