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# PRIME POWERS AS CONJUGACY CLASS LENGTHS OF $\pi$-ELEMENTS

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Let $G$ be a finite group and $\pi$ an arbitrary set of primes. We investigate the structure of $G$ when the lengths of the conjugacy classes of its $\pi$-elements are prime powers. Under this condition, we show that such lengths are either powers of just one prime or exactly $\{1, q^a, r^b\}$, with $q$ and $r$ two distinct primes lying in $\pi$ and $a, b > 0$. In the first case, we obtain certain properties of the normal structure of $G$, and in the second one, we provide a characterisation of the structure of $G$.

## 1. INTRODUCTION

If $G$ is a finite group, there are many theorems showing that the conjugacy class lengths of $G$ strongly control the structure of $G$. Moreover, several results show that imposing arithmetical conditions on the conjugacy class lengths of certain elements of $G$, for instance when such lengths are prime powers, also reflect on the structure of $G$. In [**1**], Baer proved that if every element of prime power order of $G$ has a conjugacy class of prime power size, then $G$ is a direct product of factors of coprime orders, each of which is either a $p$-group or a group with Abelian Sylow subgroups whose order is divisible by just two primes. Recently, Camina and Camina ([**7**]) restricted the above hypotheses, studying the structure of those groups whose $q$-elements, for some fixed prime $q$, have conjugacy classes of prime power size ($q$-Baer groups). On the other hand, several authors obtained a complete characterisation of those finite groups whose conjugacy class lengths are prime powers ([**8**, Theorem 2 and Corollary 2.2], or [**6**, Theorem 3]). The authors have studied the structure of $p$-solvable groups whose $p'$-elements have conjugacy classes of prime power size ([**4**, Theorem D]).

Let $\pi$ be an arbitrary set of primes and denote by $G_\pi$ the set of $\pi$-elements of a group $G$ and by $\mathrm{Con}(G_\pi)$ the set of conjugacy classes in $G_\pi$. In this paper, we extend some of the above mentioned results, describing the structure of $G$ when every class in $\mathrm{Con}(G_\pi)$ has prime power size. We recall that a group $G$ is said to be quasi-Frobenius (following [**5**]) when $G/\mathbf{Z}(G)$ is a Frobenius group. Then we shall refer to the kernel and complement of $G$ as the inverse image in $G$ of the kernel and a complement of $G/\mathbf{Z}(G)$. We state our main result.

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**Theorem A.** *Let $G$ be a finite group. Suppose that $|C|$ is a prime power for any $C \in \mathrm{Con}(G_\pi)$. Then one of the two following possibilities occurs.*

(a) *$|C|$ is a $q$-power for some fixed prime $q$. Moreover,*

(1) *$q \notin \pi$ if and only if $G$ has an Abelian Hall $\pi$-subgroup $H$. In this case $H\mathbf{O}_q(G) \trianglelefteq G$.*

(2) *$q \in \pi$ if and only if $G = H \times K$ where $K$ is a $\pi$-complement and $H$ is a nilpotent Hall $\pi$-subgroup having all Abelian Sylow subgroups except for the prime $q$.*

(b) *The lengths of the conjugacy classes in $\mathrm{Con}(G_\pi)$ are powers of exactly two different primes, say $q$ and $r$. This happens if and only if $q, r \in \pi$ and $G = H \times K$, where $K$ is a $\pi$-complement of $G$ and $H$ is a Hall $\pi$-subgroup which is a quasi-Frobenius group with Abelian kernel and complement of orders $q^a$ and $r^b$ for some $a, b > 0$. In particular, the set of conjugacy class lengths in $\mathrm{Con}(G_\pi)$ is exactly $\{1, q^a, r^b\}$.*

*Furthermore, in all cases $G$ has $\pi$-length 1 and $G/\mathbf{O}_{\pi'}(G)$ is solvable.*

As a result of Theorem A, when $\pi = \{p\}$ we provide another proof of [**7**, Theorem A(b)], which is the following.

**Corollary B.** *Let $G$ be a group and $p$ a fixed prime such that all the conjugacy classes of $p$-elements have prime power size. Then there exists only one prime $q$, equal to or distinct from $p$, such that all classes of $p$-elements have $q$-power size.*

We remark that the proof given there depends, among others, on a result of Kazarin ([**10**]), requiring Modular Representation Theory. However, our proof is simpler since we only make use of Burnside's $p^a$-Lemma to prove the fact that groups satisfying the hypotheses of Theorem A are $\pi$-separable.

One of the main tools to show Theorem A will be to define and study a graph $\Gamma_\pi(G)$ associated to the set of non-central classes in $\mathrm{Con}(G_\pi)$ of a $\pi$-separable group $G$. This graph generalises the graphs $\Gamma(G)$ and $\Gamma_p(G)$, which were defined and studied earlier in [**5, 11, 2, 3**]. We shall obtain some properties of the maximal size classes in $\mathrm{Con}(G_\pi)$, as we did for $\pi = \{p'\}$ in [**3**], so as to show Theorem A(b).

In order to prove Theorem A(a) we have also developed certain properties relating to conjugacy class lengths in $\pi$-separable groups. The most remarkable one is the following extension of a Theorem of Wielandt.

**Theorem C.** *Let $G$ be a $\pi$-separable group. If $x \in G$ with $|x^G|$ a $\pi$-number, then $[x^G, x^G] \subseteq \mathbf{O}_\pi(G)$. Consequently, $x \in \mathbf{O}_{\pi,\pi'}(G)$.*

## 2. The graph $\Gamma_\pi(G)$

**Lemma 1.** *Let $G$ be a $\pi$-separable group and let $B = b^G$, $C = c^G \in \mathrm{Con}(G_\pi)$ such that $(|B|, |C|) = 1$. Then*

(a) *$C_G(b)C_G(c) = G$.*

(b) *$BC = CB$ is a conjugacy class in* $\mathrm{Con}(G_\pi)$ *and* $|BC|$ *divides* $|B|\,|C|$.

(c) *Let* $B_0 \in \mathrm{Con}(G_\pi)$ *of maximal length and assume that* $\big(|B_0|, |C|\big) = 1$. *Then* $C^{-1}CB_0 = B_0$ *and* $\big|\langle C^{-1}C\rangle\big|$ *divides* $|B_0|$.

PROOF: (a) and (b) Mimic the proof of [**11**, Lemma 1], replacing the set $p'$ by $\pi$.

(c) We know by (b) that $CB_0 \in \mathrm{Con}(G_\pi)$ and by maximality $|CB_0| = |B_0|$. Again by (b), we have that $C^{-1}CB_0$ is a class containing $B_0$, so $C^{-1}CB_0 = B_0$. Thus, $\langle C^{-1}C\rangle B_0 = B_0$ and consequently, $B_0$ is union of some cosets of the normal subgroup $\langle C^{-1}C\rangle$. Then $\big|\langle C^{-1}C\rangle\big|$ divides $|B_0|$. ∎

We are going to use some properties of the following graph, which is a generalisation of the graph $\Gamma(G)$, defined for ordinary conjugacy classes (see [**5**]) and the graph $\Gamma_p(G)$, defined for $p$-regular classes (see [**2**]). We define $\Gamma_\pi(G)$ to be the graph having as vertices the non-central classes in $\mathrm{Con}(G_\pi)$ and two classes, $C$ and $D$, are connected by an edge when there exists a common prime dividing $|C|$ and $|D|$.

**THEOREM 2.** *Suppose that $G$ is $\pi$-separable. Then $\Gamma_\pi(G)$ has at most two connected components.*

PROOF: It is sufficient to rewrite the proof of [**2**, Theorem 1], replacing $p'$ by $\pi$ and using Lemma 1(b). ∎

In [**3**, Theorem 2], the authors obtained some properties related to $p$-regular conjugacy classes of maximal size in $p$-solvable groups. The following theorem extends them for conjugacy classes of $\pi$-elements of maximal size in $\pi$-separable groups.

From now on, $\pi(m)$ will denote the set of prime divisors of a positive integer $m$. We shall also write $\pi(X)$ to denote the set of primes dividing $|X|$ for any group $X$ or any $X \in \mathrm{Con}(G_\pi)$.

**THEOREM 3.** *Suppose that $G$ is $\pi$-separable. Let $B_0 \in \mathrm{Con}(G_\pi)$ of maximal length and write*

$$M = \Big\langle D \in \mathrm{Con}(G_\pi) \mid \big(|D|, |B_0|\big) = 1 \Big\rangle.$$

*Then $M$ is an Abelian $\pi$-subgroup of $G$. Furthermore, $Z_\pi := \mathbf{Z}(G)_\pi \subseteq M$ and $\pi(M/Z_\pi) \subseteq \pi(B_0)$.*

PROOF: We define $N = \langle D^{-1}D \mid D \in \mathrm{Con}(G_{p'}), \big(|D|, |B_0|\big) = 1\rangle$. From the definition of $M$ and $N$, it is clear that $N = [M, G]$. On the other hand, if $C \in \mathrm{Con}(G_\pi)$ with $\big(|C|, |B_0|\big)$=1, then $C^{-1}CB_0 = B_0$ by Lemma 1(c), and hence, $NB_0 = B_0$. This means that $B_0$ is union of cosets of $N$, so $|N|$ divides $|B_0|$ and $\pi(N) \subseteq \pi(B_0)$. Now, for such a class $c^G = C$, we have $\big(|N|, |C|\big) = 1$. Since $\big|N : C_N(c)\big|$ divides $\big(|N|, |C|\big)$, it follows that $N = C_N(c)$, so $N \leqslant \mathbf{Z}(M)$. But $M/N$ is contained in the centre of $G/N$, and this forces $M$ to be nilpotent. As all the generators of $M$ lie in the Hall $\pi$-subgroup of $M$, we conclude that $M$ is a $\pi$-group.

Now, it is obvious that $Z_\pi \subseteq M$. Let $r \in \pi(M/Z_\pi)$ and choose $R \in \mathrm{Syl}_r(M)$. Notice that $R \trianglelefteq G$ and that $1 \neq [R, G] \leqslant [M, G] = N$. Therefore, $r \in \pi(N) \subseteq \pi(B_0)$ and thus

$\pi(M/Z_\pi) \subseteq \pi(B_0)$ as wanted.

Finally, if $d^G = D$ is a generating class of $M$ and $R$ is a Sylow $r$-subgroup as above, then $|R : C_R(d)|$ divides $(|R|, |D|) = 1$, so $R = C_R(d)$ and we get $R \leqslant \mathbf{Z}(M)$. Hence $M$ is Abelian. ∎

**LEMMA 4.** *Let $G$ be a $\pi$-separable group such that $\Gamma_\pi(G)$ has two connected components, $X_1$ and $X_2$, and assume that $X_2$ contains the maximal length classes. Then $|A| < |B|$, $A^{-1}AB = B$ and $|\langle A^{-1}A\rangle|$ divides $|B|$ for all $A \in X_1$ and all $B \in X_2$.*

PROOF: Let $B \in X_2$, $A \in X_1$ and choose $B_0$ any maximal length class. Since $(|A|, |B|) = 1$ then $AB$ is again a class by Lemma 1(b), satisfying that $|AB|$ divides $|A|\,|B|$. As $A$ and $B$ belong to different components, we have two possibilities: either $|AB| = |A|$ or $|AB| = |B|$. If $|AB| = |A|$ then $B^{-1}BA = A$, and thus we deduce that $|\langle B^{-1}B\rangle|$ divides $|A|$ and $\langle B^{-1}B\rangle \subseteq \langle AA^{-1}\rangle$. But $|\langle AA^{-1}\rangle|$ also divides $|B_0|$ by Lemma 1(c), so we conclude that $|\langle B^{-1}B\rangle|$ divides $(|A|, |B_0|) = 1$, a contradiction. Consequently, $|AB| = |B|$, so in particular $|A| < |B|$. Moreover, $A^{-1}AB = B$ and then $|\langle A^{-1}A\rangle|$ divides $|B|$. ∎

**THEOREM 5.** *Suppose that $G$ is $\pi$-separable and that $\Gamma_\pi(G)$ has two connected components $X_1$ and $X_2$. Assume that $X_2$ contains the maximal length classes in* $\mathrm{Con}(G_\pi)$ *and let*

$$M = \langle D \in \mathrm{Con}(G_\pi) \mid D \in X_1 \rangle.$$

*Then*

(a) *$M$ coincides with the subgroup defined in Theorem 3. Accordingly, $M$ is an Abelian $\pi$-subgroup and $\pi(M/\mathbf{Z}(G)_\pi) \subseteq \pi(X_2)$.*

(b) *there is no class in $X_2$ whose size is a $\pi'$-number.*

PROOF: (a) Let $A \in X_1$ and let $B_0$ be a class of $\pi$-elements of maximal length. In Lemma 4, we proved that $|\langle AA^{-1}\rangle|$ divides $|B|$ for every class $B \in X_2$. Thus, if $C \in \mathrm{Con}(G_\pi)$ with $(|C|, |B_0|) = 1$, then either $C \in X_1$ or $|C| = 1$. If $C = \{c\}$, then we need to show that $c \in M = \langle D \in \mathrm{Con}(G_\pi) \mid D \in X_1\rangle$. Observe that $cA = \{ca \mid a \in A\}$ is a conjugacy class of $\pi$-elements of $G$ with $|cA| = |A|$. Therefore, $cA \in X_1$, and $ca \in M$ for every $a \in A$. Since $a^{-1} \in M$, we conclude that $c \in M$.

(b) Fix a class $A \in X_1$. We know that $\langle A^{-1}A\rangle \subseteq M$, which is a $\pi$-group. For any class $B \in X_2$, we proved in Lemma 4 that $|\langle A^{-1}A\rangle|$ divides $|B|$, and thus, $|B|$ is not a $\pi'$-number. ∎

## 3. Class lengths in $\pi$-separable groups

We state and prove now Theorem C of the Introduction.

**THEOREM 6.** *Suppose that $G$ is $\pi$-separable and let $x \in G$ such that $|x^G|$ is a $\pi$-number. Then $[x^G, x^G] \subseteq \mathbf{O}_\pi(G)$. Consequently, $x \in \mathbf{O}_{\pi,\pi'}(G)$.*

PROOF: We argue by induction on $|G|$. As the hypotheses of the theorem are inherited by quotient groups we can assume that $\mathbf{O}_\pi(G) = 1$ and we shall show that $[x^G, x^G] = 1$.

Let $N = \langle x^G \rangle$ and suppose that $N < G$. Since $\mathbf{O}_\pi(N) = 1$, it follows by induction that $[x^N, x^N] = 1$. In particular, $\langle x^N \rangle$ is Abelian and subnormal in $G$. Thus $x \in \langle x^N \rangle \subseteq F(G)$, which is a $\pi'$-group. But since $|x^G|$ is a $\pi$-number, then $F(G) \subseteq C_G(x)$, and so $x$ is central in $F(G)$. Therefore, $\langle x^G \rangle$ is also central in $F(G)$, so we have $[x^G, x^G] = 1$, as required.

Accordingly, we can assume that $N = G$. Now take $K$ a minimal normal subgroup of $G$, which must be a $\pi'$-group. By assumption, we have $K \subseteq C_G(x)$ and, as $G = \langle x^G \rangle$, then $K \subseteq \mathbf{Z}(G)$. It follows that $\mathbf{O}_\pi(G/K) = \mathbf{O}_\pi(G)K/K = 1$ and, by applying induction, we obtain $\left[(xK)^{G/K}, (xK)^{G/K}\right] = 1$, that is to say, $[x^G, x^G] \subseteq K \subseteq \mathbf{Z}(G)$. Then $G$ is nilpotent. Consequently, $G$ is a $\pi'$-group, so $x$ is central in $G$ and the theorem is trivially true.

Now, $\langle x^G \rangle \mathbf{O}_\pi(G)/\mathbf{O}_\pi(G)$ is an Abelian normal subgroup of $G/\mathbf{O}_\pi(G)$. Hence

$$\langle x^G \rangle \mathbf{O}_\pi(G)/\mathbf{O}_\pi(G) \subseteq F\big(G/\mathbf{O}_\pi(G)\big),$$

which is necessarily a $\pi'$-group. In particular, $x \in \mathbf{O}_{\pi,\pi'}(G)$, as required. ∎

REMARKS. Theorem 6 is a generalisation of the fact that if $G$ is a $\pi$-separable group, then every $x \in G_\pi$ with $|x^G|$ a $\pi$-number must lie in $\mathbf{O}_\pi(G)$ (see for instance, [**9**, Lemma 33.3]). On the other hand, Theorem 6 is simply not true when the $\pi$-separability hypothesis is eliminated. For instance, let $G$ be a nonabelian simple group and choose $x \neq 1$ to belong to the centre of a Sylow $p$-subgroup of $G$, for some prime $p$. Then $|x^G|$ is a $p'$-number and $\mathbf{O}_{p'}(G) = 1$, while the property $[x^G, x^G] = 1$ is clearly not satisfied.

We also need two lemmas. We shall denote by $l_\pi(G)$ the $\pi$-length of a group $G$.

**LEMMA 7.** *Let $G$ be a $\pi$-separable group. Then the conjugacy class length of any $\pi$-element in $G$ is a $\pi'$-number if and only if $G$ has Abelian Hall $\pi$-subgroups. In this case, $l_\pi(G) \leqslant 1$.*

PROOF: The first assertion can easily be proved by using induction on $|G|$ (it is exactly [**3**, Lemma 5]). The second assertion follows immediately by applying Theorem 6. ∎

**LEMMA 8:** *Let $G$ be a $\pi$-separable group. Then the conjugacy class length of every $\pi$-element of $G$ is a $\pi$-number if and only if $G = H \times K$, where $H$ and $K$ are a Hall $\pi$-subgroup and a $\pi$-complement of $G$, respectively.*

PROOF: Suppose first that $|C|$ is a $\pi$-number for any $C \in \mathrm{Con}(G_\pi)$ and take $H$ and $K$ a Hall $\pi$-subgroup and a $\pi$-complement of $G$ respectively. Then for each $x \in H$, there exists some $g \in G$ such that $K^g \subseteq C_G(x)$. Thus, $x \in C_G(K^g)$ and

$$H \subseteq \bigcup_{g \in G} C_G(K^g).$$

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Since $G = HK$, we have

$$H \subseteq \bigcup_{h \in H} C_G(K^h) \cap H \subseteq \bigcup_{h \in H} C_H(K)^h \subseteq H.$$

This yields $H = C_H(K)$ and then $G = H \times K$, as required. The converse direction of the lemma is trivial. ∎

## 4. Proofs of Theorem A and Corollary B

**Lemma 9.** *Let $G$ be a group. Suppose that $|C|$ is a prime power for any $C \in \mathrm{Con}(G_\pi)$. Then $G$ is $\pi$-separable.*

Proof: If any $\pi$-element of $G$ is central, then $G$ trivially has a central Hall $\pi$-subgroup and the lemma is proved. Thus we can assume that there exists some $C \in \mathrm{Con}(G_\pi)$ such that $1 < |C|$ is a prime power. By Burnside's Lemma, $G$ is not simple, and then the result follows by induction on $|G|$, since the hypothesis is inherited by normal subgroups and quotients. ∎

Suppose that all the conjugacy classes of elements in $G_\pi$ have prime power length. Then Theorem 2 implies that the primes appearing in this set of lengths can be at most two. Therefore, to show Theorem A of the Introduction, we only have to study two cases, depending on whether just one or two primes appear. Each one of these cases corresponds to Theorems 12 and 10, respectively.

**Theorem 10.** *Let $G$ be a group and $\pi$ an arbitrary set of primes. Then the lengths of the conjugacy classes in $\mathrm{Con}(G_\pi)$ are powers of exactly two different primes $q$ and $r$ if and only if $q, r \in \pi$ and $G = H \times K$, where $K$ is a $\pi$-complement of $G$ and $H$ is a Hall $\pi$-subgroup which is quasi-Frobenius with Abelian kernel and complement of orders $q^a$ and $r^b$, with $a, b > 0$. In particular, the set of conjugacy class lengths in $\mathrm{Con}(G_\pi)$ is exactly $\{1, q^a, r^b\}$.*

Proof: First, we note that $G$ is $\pi$-separable by Lemma 9. Suppose that the lengths of the classes in $\mathrm{Con}(G_\pi)$ are powers of exactly two primes, $q$ and $r$, and suppose without loss that $q^a$, with $a > 0$, is the maximal size. Then $\Gamma_\pi(G)$ has two connected components and Theorem 5(b) asserts that $q \in \pi$. Now, we use the subgroup $M$ defined in the statement of Theorem 5. By definition, we have $M = \langle D \in \mathrm{Con}(G_\pi) \mid |D| \text{ is an } r\text{-power}\rangle$, which further is an Abelian $\pi$-group by Theorem 5(a). We claim that $G$ has a normal Hall $\pi$-subgroup. Let us consider $G/M$, which satisfies that all conjugacy classes of $\pi$-elements have $q$-power length. Then, by Lemma 8, we can factor $G/M = H/M \times KM/M$, where $H$ and $K$ are a Hall $\pi$-subgroup and a $\pi$-complement of $G$, respectively. Accordingly, $G$ has a normal Hall $\pi$-subgroup, as claimed. Also, we shall use later the fact that $KM \trianglelefteq G$, for any $\pi$-complement $K$ of $G$.

Notice that the conjugacy class lengths of $H$ are also $r$-powers or $q$-powers, since they divide the conjugacy class sizes of $G$. If we prove that $H$ has conjugacy classes

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whose size are $q$-numbers and $r$-numbers, we shall obtain, via [**5**, Theorem A], that $H$ is quasi-Frobenius with Abelian kernel and complement. Suppose first that every class in $H$ has $r$-power size and assume that $M < H$. Take $x \in H - M$ and observe that $|x^G|$ must be a $q$-number by definition of $M$, so $|x^H| = 1$. This forces $H = M\mathbf{Z}(H)$, and thus $H$ is Abelian (observe that if $M = H$ then $H$ is also Abelian). It follows that the length of every $C \in \mathrm{Con}(G_\pi)$ is a $\pi'$-number, contradicting the fact that $q \in \pi$.

Suppose now that all the conjugacy classes in $H$ have $q$-power size and take $x \in H$ such that $|x^G|$ is an $r$-power. Then $|x^H| = 1$, that is, $x \in \mathbf{Z}(H)$, and therefore, $M \subseteq \mathbf{Z}(H)$. If $H = M$, we get a contradiction as above. Thus we can take $x \in H - M$. By definition of $M$, $1 \neq |x^G|$ is a $q$-number, so in particular, it is a $\pi$-number. Then $K^g \subseteq C_G(x)$ for some $g \in G$. But on the other hand, we have proved that $M \subseteq C_G(x)$, and then $KM = K^gM \subseteq C_G(x)$ for every $x \in H - M$. We conclude that $K \subseteq C_G(H)$ and this means that all conjugacy classes in $G_\pi$ have $q$-power length, a contradiction.

Thus, we have shown that $H$ has conjugacy classes of $r$-power size and classes of $q$-power size. Therefore $q, r \in \pi$, and by Lemma 8, we obtain $G = H \times K$. Moreover, from the fact that $H$ is quasi-Frobenius with Abelian kernel and complement one can easily check that the conjugacy class sizes of $H$ are exactly $\{1, q^a, r^b\}$ for some positive integers $a, b$, where $q^a$ and $r^b$ are exactly the orders of the kernel and a complement of $H$, respectively.

The converse direction of the theorem is trivial. ∎

The following is Corollary B of the Introduction.

**COROLLARY 11.** *Let $G$ be a group and $p$ a fixed prime such that all the conjugacy classes of $p$-elements have prime power size. Then there exists only one prime $q$, equal to or distinct from $p$, such that all classes of $p$-elements have $q$-power size.*

PROOF: Take $\pi = \{p\}$ in Theorem 10. By Theorem 2 there exist at most two primes dividing the conjugacy class sizes in $G_\pi$. If there are exactly two, then they must belong to $\pi$ by Theorem 10, and this is not possible. ∎

Now we study the remaining case, that is, when all classes in $G_\pi$ have $q$-power size for a fixed prime $q$.

**THEOREM 12.** *Suppose that for any $C \in \mathrm{Con}(G_\pi)$, $|C|$ is a power of a fixed prime $q$. Then*

(a) *$q \notin \pi$ if and only if $G$ has an Abelian Hall $\pi$-subgroup $H$. In this case $H\mathbf{O}_q(G) \trianglelefteq G$.*

(b) *$q \in \pi$ if and only if $G = H \times K$ where $K$ is a $\pi$-complement and $H$ is a nilpotent Hall $\pi$-subgroup having all Abelian Sylow subgroups except for the prime $q$.*

*Furthermore, in both cases $l_\pi(G) \leqslant 1$ and $G/\mathbf{O}_{\pi'}(G)$ is solvable.*

PROOF: (a) The direct sense is immediate by Lemmas 7 and 9, and the converse direction is trivial. In this case, it remains to be proved that if $H$ is a Hall $\pi$-subgroup of $G$ then $H\mathbf{O}_q(G) \trianglelefteq G$ and that $G/\mathbf{O}_{\pi'}(G)$ is solvable.

We first notice that $l_\pi(G) \leqslant 1$ by applying Lemma 7. To prove that $H\mathbf{O}_q(G)$ is normal in $G$, we first see, arguing by induction on $|G|$, that we may assume that $\mathbf{O}_\pi(G) = 1$. If $1 \neq M := \mathbf{O}_\pi(G)$ then, as the hypotheses are inherited by factor groups, by induction we have $\mathbf{O}_q(G/M)H/M \trianglelefteq G/M$. On the other hand, we can write $\mathbf{O}_q(G/M) = Q_0M/M$ for some $q$-subgroup $Q_0$ of $G$, and thus $HQ_0 \trianglelefteq G$. Moreover, $\mathbf{O}_q(HQ_0) = \mathbf{O}_q(G)$ since $\mathbf{O}_q(G) \subseteq Q_0M \subseteq HQ_0$. Now we distinguish two possibilities. If $HQ_0 < G$, by induction it follows that $H\mathbf{O}_q(G) \trianglelefteq HQ_0$. This implies that $H\mathbf{O}_q(G) \trianglelefteq G$, because we observe that $H\mathbf{O}_q(G)/\mathbf{O}_q(G) = \mathbf{O}_\pi\bigl(HQ_0/\mathbf{O}_q(G)\bigr)$. If $HQ_0 = G$ then $\mathbf{O}_{\pi'}(G) = \mathbf{O}_q(G)$ and since $l_\pi(G) \leqslant 1$, we also conclude that $H\mathbf{O}_q(G) \trianglelefteq G$, as required.

We prove now that we can also assume that $\mathbf{O}_{\pi',\pi}(G) = G$, that is, $\mathbf{O}_{\pi'}(G)H = G$. Suppose that $N =: \mathbf{O}_{\pi'}(G)H < G$. As $l_\pi(G) \leqslant 1$, then $N \triangleleft G$. Since the hypotheses are inherited by normal subgroups, by applying induction we get $H\mathbf{O}_q(N) \trianglelefteq N$, and thus $H\mathbf{O}_q(N) \trianglelefteq G$. But notice that $\mathbf{O}_q(N) = \mathbf{O}_q(G)$, so we conclude that $H\mathbf{O}_q(G)$ is normal in $G$ and the proof finishes.

Now, for any $x \in H$ and any prime $r \in \pi'$, $r \neq q$, by assumption there exists some $R \in \mathrm{Syl}_r(G)$ such that $R \subseteq C_G(x)$, that is, $x \in C_G(R)$. Hence

$$H \subseteq \bigcup_{g \in G} C_G(R)^g.$$

Then

$$G = H\mathbf{O}_{\pi'}(G) \subseteq \bigcup_{g \in G} C_G(R^g)\mathbf{O}_{\pi'}(G) \subseteq G,$$

so we have $G = C_G(R)\mathbf{O}_{\pi'}(G)$ for some $R \in \mathrm{Syl}_r(G)$. In particular, $\bigl|G : C_G(R)\bigr|$ is a $\pi'$-number and, replacing by some $G$-conjugated of $R$, we may assume that $H \subseteq C_G(R)$, or equivalently, $R \subseteq C_G(H)$. This has been proved for every prime $r \in \pi' - \{q\}$ and since $H$ is Abelian, it follows that $\bigl|G : C_G(H)\bigr|$ is a $q$-number. Then $G = C_G(H)Q$ for some $Q \in \mathrm{Syl}_q(G)$. Furthermore, we can choose $Q$ to be normalised by $H$, because, by the Frattini argument, $G = N_G(Q)\mathbf{O}_{\pi'}(G)$, so in particular, $N_G(Q)$ has $\pi'$-index in $G$, and consequently, there exists some Hall $\pi$-subgroup of $G$ normalising $Q$.

Let us consider $[H, G] = \bigl[H, C_G(H)Q\bigr] = [H, Q] \leqslant Q$. Then $[H, G]$ is a normal $q$-subgroup of $G$, which cannot be trivial because this contradicts the assumption $\mathbf{O}_\pi(G) = 1$. This forces $\mathbf{O}_q(G) \neq 1$ and then we can apply the inductive hypothesis to $G/\mathbf{O}_q(G)$ to conclude that $H\mathbf{O}_q(G) \trianglelefteq G$ and the proof is finished.

Finally, the solvability of $G/\mathbf{O}_{\pi'}(G)$ is a consequence of [**7**, Theorem A(d)], which asserts that, under our hypotheses, $G/\mathbf{O}_{r'}(G)$ is solvable for any prime $r \in \pi$. In fact, it is enough to notice that $\mathbf{O}_{\pi'}(G) = \cap_{r \in \pi}\mathbf{O}_{r'}(G)$ and that $G/\mathbf{O}_{\pi'}(G)$ can be immerged into $\prod_{r \in \pi} G/\mathbf{O}_{r'}(G)$, which is solvable too.

(b) Suppose that $q \in \pi$. Then $G = H \times K$ by Lemma 8. Notice that the conjugacy class sizes of $H$ are also $q$-powers, and this implies (see for instance [**8**, Proposition 4]) that $H = Q \times L$ with $Q$ a Sylow $q$-subgroup of $H$ and $L$ Abelian, as wanted. Conversely, if $G$ is as described in the statement, it clearly follows that the conjugacy classes of $G_\pi$ have $q$-power size and that $q \in \pi$. Finally, the fact that $l_\pi(G) \leqslant 1$ and the solvability of $G/\mathbf{O}_{\pi'}(G)$ are trivial in this case. $\square$

REMARK. Let us assume the hypotheses of Theorem 12 with $\pi = \{p\}$. Then we have provided a more elementary proof of the fact that $\mathbf{O}_q(G)P \trianglelefteq G$, with $P \in \mathrm{Syl}_p(G)$, which is part of [**7**, Theorem A(d)].

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