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ALGEBRAS OF OPERATORS AS TOPOLOGICAL ALGEBRAS

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Introduction. An example of a non-topologizable algebra was given in [**2**]. In [**4**] Żelazko gave a simple proof of the fact that, if X is an infinite-dimensional vector space, then the algebra of all finite-rank linear operators on X is not topologizable as a topological algebra. In the following we use a similar idea to prove that, if E is a Fréchet space which is not normable, then each subalgebra A of the algebra of all bounded linear operators on E such that A contains the ideal of continuous, finite-rank operators, is non-topologizable as a topological algebra. This is a shorter proof and more general version of the result of [**1**].

Preliminaries. Let E be a locally convex space, with dual space E', and let (p_α) be the family of separating continuous seminorms which defines the topology τ of E. The spaces of all bounded linear operators and all continuous, finite-rank operators on E will be denoted by $\mathcal{B}(E)$ and $\mathcal{F}(E)$, respectively. So $\mathcal{F}(E) \subseteq \mathcal{B}(E)$.

Let $E \otimes E'$ be the tensor product of E and E', so that $E \otimes E'$ is a linear space generated by $\{x_0 \otimes \lambda_0 : x_0 \in E, \lambda_0 \in E'\}$. Here $x_0 \otimes \lambda_0$ is defined by:

$$(x_0 \otimes \lambda_0)(x) = \langle x, \lambda_0 \rangle x_0 \quad (x \in E),$$

where the notation $\langle x, \lambda_0 \rangle$ is used for $\lambda_0(x)$. We identify $E \otimes E'$ with $\mathcal{F}(E)$.

We recall that a *topological algebra A* is an associative algebra with a topology on it such that it is a (Hausdorff) topological linear space and the multiplication is jointly continuous.

Now take A to be a subalgebra of $\mathcal{B}(E)$ containing $\mathcal{F}(E)$, and let A be a topological algebra with respect to some topology. Take $\mathcal{V}$ to be a balanced, absorbing, local base for the topology of A.

Fix $x_0 \in E$. Since E is locally convex, there exists $\lambda_0 \in E'$ with $\langle x_0, \lambda_0 \rangle = 1$. Since the topology of A is Hausdorff, there exists $V \in \mathcal{V}$ for which $x_0 \otimes \lambda_0 \notin V$. Now choose W in $\mathcal{V}$ with $W^2 \subseteq V$, and define

$$K = \operatorname{conv}\{x \in E : x \otimes \lambda_0 \in W\}.$$

Clearly K is convex, absorbing, balanced subset of E. So ρ_K, its Minkowski functional, is a seminorm on E. We shall show that ρ_K is actually a norm.

For each $\lambda \in E'$, there is $m_\lambda > 0$ such that

$$x_0 \otimes \lambda \in m_\lambda W.$$

Now, if $x \in K$, $x \otimes \lambda_0 \in W$, and $\lambda \in E'$, we have

$$(x_0 \otimes \lambda) \circ (x \otimes \lambda_0) \in m_\lambda W^2 \subseteq m_\lambda V.$$

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It is easy to see that

$$(x_0 \otimes \lambda) \circ (x \otimes \lambda_0) = \langle x, \lambda \rangle (x_0 \otimes \lambda_0).$$

So $\langle x, \lambda \rangle (x_0 \otimes \lambda_0) \in m_\lambda V$, and, since $x_0 \otimes \lambda_0 \notin V$, it follows that $|\langle x, \lambda \rangle| \leq m_\lambda$. Therefore $|\langle x, \lambda \rangle| \leq m_\lambda$ for each $x \in K$, and consequently

$$|\langle x, \lambda \rangle| \leq m_\lambda \rho_K(x) \quad (x \in E). \tag{1}$$

This shows that ρ_K is a norm because, for each $x \neq 0$ in E, there exists $\lambda \in E'$ with $\langle x, \lambda \rangle \neq 0$,

Let us write $\|x\|$ for $\rho_K(x)$. Rewriting (1) we obtain:

$$|\langle x, \lambda \rangle| \leq m_\lambda \|x\| \quad (x \in E, \lambda \in E').$$

This relation also shows that $B = \{x \in E : \|x\| \leq 1\}$ is a weakly bounded set in E. Since E is locally convex, B is bounded. So, for each α, there exists $k_\alpha > 0$ with

$$p_\alpha(x) \leq k_\alpha \|x\| \quad (x \in E).$$

By replacing p_α with p_α / k_α, we can suppose that

$$p_\alpha(x) \leq \|x\| \quad (x \in E).$$

We can now state our result.

PROPOSITION *Let (E, τ) be a Fréchet space, and let A be a subalgebra of $\mathcal{B}(E)$ containing $\mathcal{F}(E)$. Then there exists a topology on A with respect to which it is a topological algebra if and only if E is a Banach space.*

Proof. Let $\| \cdot \|$ and p_α be as above, and define

$$\tilde{p}(x) = \sup_\alpha p_\alpha(x) \quad (x \in E).$$

Then $\tilde{p}(x) \leq \|x\|$, and, since (p_a) is a separating family of seminorms, $\tilde{p}$ is a norm on E.

Define $\tilde{B} = \{x \in E : \tilde{p}(x) \leq 1\}$. Then $\tilde{B}$ is an absolutely convex, absorbing set. Since $\tilde{B} = \bigcap_\alpha \{x \in E : p_\alpha(x) \leq 1\}$, clearly $\tilde{B}$ is τ-closed. This shows that $\tilde{B}$ is a barrel. Since (E, τ) is a barrelled space, $\tilde{B}$ contains a neighbourhood of the origin. Hence there exists α such that

$$\tilde{p}(x) \leq c_\alpha p_\alpha(x) \quad (x \in E)$$

for some $c_\alpha > 0$. Consequently the identity map *id*: $(E, \tau) \to (E, \tilde{p})$, is continuous.

The definition of $\tilde{p}$ shows that *id*: $(E, \tilde{p}) \to (E, \tau)$ is also continuous. Therefore the topology τ of E as a Fréchet space can be defined by the norm $\tilde{p}$, and so E is a Banach space.

The converse is immediate.

This proposition shows that for a non-normable, Fréchet space E, any subalgebra of $\mathcal{B}(E)$ containing the ideal of continuous, finite-rank operators cannot be topologized as a topological algebra.

For a related result to ours, see [**5**].

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