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DIFFERENTIAL FORMS AND RESOLUTIONS ON CERTAIN ANALYTIC SPACES II. FLAT RESOLUTIONS

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CONTENTS

Introduction. 1. This paper gives another construction of $(0,p)$-forms on a complex analytic space and of the $\bar{\partial}$ operator. This construction is independent of the one in [1] and apart from the general result of Section 1 of [1], it can be read independently. As in [1], the hypotheses on S are the following: S has normal singularities, its singular locus X is smooth, the exceptional divisor $\tilde{X}$ in a desingularization of S is irreducible. But now, we do not assume that S is a relatively compact open set of a complex space or is itself

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compact. The price to pay is that we have to impose on the forms an infinite number of conditions on the exceptional divisor.

2. As a result of our construction, the functions or forms that we construct define flat modules over the sheaf of holomorphic functions. We could not obtain such results in [1]. As a consequence, it seems that this resolution is more canonical than that of [1], although it is also more complicated.

3. In Section 1, we define infinite jet of forms along the exceptional divisor. We also prove that, in some sense, we can define infinite jet of forms along the singular locus X in S. In Section 2, we define the complex of $(0,p)$-forms on S, together with the $\bar{\partial}$ operator and we prove that it is a resolution of O_S. However, the proof is more complicated than in [1], because we cannot compare this resolution directly to the Dolbeault resolution of the desingularization $\tilde{S}$. We have to construct a special resolution of $O_{\tilde{S}}$ which can be compared to the resolution of O_S. Sections 3 and 4 prove that these sheaves of forms on S are flat O_S-modules. This is rather technical and can be skipped at first: our method is to reduce the situation to the global theorems of Malgrange about ideals of differentiable functions ([5]).

1. **Jets of forms along an irreducible exceptional divisor.**

1.1 *The formal neighborhood of an exceptional divisor.*

a) Notation: We shall assume the properties of Section 2.1 of [1]. Namely, S is a complex analytic space, X is its singular locus, which is supposed to be smooth and the singularities of S are normal. We call $\varphi\colon \tilde{S} \longrightarrow S$ a *desingularization* of S and $\tilde{X} = \varphi^{-1}(X)$ the exceptional divisor lying above X and we assume that $\tilde{X}$ is irreductible.

b) The formal neighborhood of $\tilde{X}$: $O_{\tilde{S}}$ is the sheaf of holomorphic functions on $\tilde{S}$ and we shall denote $\hat{O}_{\tilde{S}}$ the formal completion of $O_{\tilde{S}}$ along $\tilde{X}$, so that if $I_{\tilde{X}}$ is the ideal of definition of $\tilde{X}$, we have

$$\hat{O}_{\tilde{S}} = \lim_n \operatorname{proj} O_{\tilde{S}}/I^n_{\tilde{X}}.$$

c) The formal neighborhood of X in S: Let O_S be the sheaf of holomorphic functions on S, $\widehat{O_S}$ the formal completion of O_S along X. It is known that for all $i \geq 0$, the natural morphisms

$$[R^i\varphi_*(O_{\tilde{S}})]^\wedge \longrightarrow R^i\varphi_*(\hat{O}_{\tilde{S}}) \tag{1-1}$$

are isomorphisms (see [3] p. 21, without the ^ this is the relative form of the comparison theorem of Grothendieck [2]).

1.2 *Jets of C^∞ functions along $\tilde{X}$.* We shall adopt conventions different from [1] concerning jets of functions and forms along $\tilde{X}$.

a) Coordinates along $\tilde{X}$: Let V be an open set in $\tilde{S}$ with $V \cap \tilde{X}$ nonempty. We can cover V by a finite number of coordinates charts $(V_i)_{i\in I}$ such that $\tilde{X} \cap V_i$ has a local equation $\zeta^{(i)} = 0$, and such that we have local coordinates $(\zeta^{(i)}, z^{(i)})$ on V_i where the $z^{(i)}$ define holomorphic coordinates along $\tilde{X} \cap V_i$. The change of coordinates in $V_i \cap V_j$ is given by

$$\begin{aligned} z^{(j)} &= \varphi_{ij}(z^{(i)}) \\ \zeta^{(j)} &= \zeta^{(i)}\psi_{ij}(z^{(i)}, \zeta^{(i)}). \end{aligned} \tag{1-2}$$

b) Jet of a function along $\tilde{X}$: Let f be a C^∞ function in V. We can take at any point $\tilde{m} \in \tilde{X} \cap V_i$, the complete Taylor expansion of f with respect to $\zeta^{(i)}$ and we call it

$$J_{\tilde{X}}^{(\infty)}(f) = \sum_{k,\ell=0}^{\infty} \frac{\partial^{k+\ell} f}{\partial \zeta^{(i)k} \partial \bar{\zeta}^{(i)\ell}}\bigg|_{\zeta^{(i)}=0} \frac{\zeta^{(i)k}\bar{\zeta}^{(i)\ell}}{k!\,\ell!}. \tag{1-3}$$

Moreover, we define the purely holomorphic part $j_{\tilde{X}}^{(\infty)}(f)$ of the total jet $J_{\tilde{X}}^{(\infty)}(f)$ to be

$$j_{\tilde{X}}^{(\infty)}(f) = \sum_{k=0}^{\infty} \frac{\partial^k f}{\partial \zeta^{(i)k}}\bigg|_{\zeta^{(i)}=0} \frac{\zeta^{(i)k}}{k!}. \tag{1-4}$$

In [1], we had defined the holomorphic jet of order r to be

$$j_{\tilde{X}}^{(r)}(f) = \sum_{k=0}^{r} \frac{\partial^k f}{\partial \zeta^{(i)k}}\bigg|_{\zeta^{(i)}=0} \frac{\zeta^{(i)k}}{k!}$$

and we had proved that $j_{\tilde{X}}^{(r)}(f)$ can be identified to a C^∞ section of a vector bundle $E_{\tilde{X}}^{(r)} \longrightarrow \tilde{X}$ and is thus intrinsically defined.

Then, $j_{\tilde{X}}^{(\infty)}(f)$ is intrinsically defined on $\tilde{X}$ as an element of $\lim_r \operatorname{proj} C^\infty(V \cap \tilde{X}, E_{\tilde{X}}^{(r)})$ where $C^\infty(\tilde{X} \cap V, E_{\tilde{X}}^{(r)})$ is the space of C^∞ sections of $E_{\tilde{X}}^{(r)}$ over $\tilde{X} \cap V$.

In particular, the equality

$$J_{\tilde{X}}^{(\infty)}(f) = j_{\tilde{X}}^{(\infty)}(f) \equiv \lim_r \operatorname{proj} j_{\tilde{X}}^{(r)}(f) \tag{1-5}$$

has an intrinsic meaning.

Definition. We shall say that the jet $J_{\tilde{X}}^{(\infty)}(f)$ is purely holomorphic along $\tilde{X}$ if the equation (1-5) holds. In this case we say that f has a purely holomorphic jet along $\tilde{X}$.

1.3 *Jets of* $(0,p)$*-forms along* $\tilde{X}$.

a) Splitting of $(0,p)$-form along $\tilde{X}$: Let π be a $C^\infty(0,p)$ form on V. In the coordinate chart V_i, we have a natural splitting of π

$$\pi = \sum_{|J|=p} \pi_J^{(i)}\, d\bar{z}^{(i)J} + \sum_{|K|=p-1} \pi_{K,\bar{\zeta}}^{(i)}\, d\bar{z}^{(i)K} \wedge d\bar{\zeta}^{(i)} \tag{1-6}$$

in components containing $d\bar{\zeta}^{(i)}$ and components which do not contain $d\bar{\zeta}^{(i)}$. Here the sets J (resp. K) are multi-indices of length p (resp. $p-1$).

b) Jet of π along $\tilde{X}$: In V_i, we can take the formal Taylor expansion in $\zeta^{(i)}$ and $\bar{\zeta}^{(i)}$ of the components of the form π

$$J_{\tilde{X}}^{(\infty)}(\pi_I^{(i)}) = \sum_{k,\ell=0}^{\infty} \frac{\partial^{k+\ell} \pi_I^{(i)}}{\partial \zeta^{(i)k} \partial \bar{\zeta}^{(i)\ell}}\bigg|_{\zeta^{(i)}=0} \frac{\zeta^{(i)k}\bar{\zeta}^{(i)\ell}}{k!\,\ell!} \tag{1-7}$$

$$J_{\tilde{X}}^{(\infty)}(\pi_{K,\bar{\zeta}}^{(i)}) = \sum_{k,\ell=0}^{\infty} \frac{\partial^{k+\ell} \pi_{K,\bar{\zeta}}^{(i)}}{\partial \zeta^{(i)k} \partial \bar{\zeta}^{(i)\ell}}\bigg|_{\zeta^{(i)}=0} \frac{\zeta^{(i)k}\bar{\zeta}^{(i)\ell}}{k!\,\ell!}.$$

When we change coordinates according to formulas (1-2) between V_i and V_j, all these jets get mixed up *a priori*. Nevertheless, we have the following lemma.

LEMMA 1.1. *The fact that all $J^{(\infty)}_{\tilde X}(\pi^{(i)}_{K,\bar\zeta}) = 0$ for all multi-indices K of length $p-1$, does not depend on the coordinate chart V_i chosen. Moreover, if this condition is fulfilled, the fact that all the jets $J^{(\infty)}_{\tilde X}(\pi^{(i)}_I)$ are equal to their purely holomorphic part $j^{(\infty)}_{\tilde X}(\pi^{(i)}_I)$ for all multi-indices I of length p, does not depend on the coordinate chart V_i chosen.*

PROOF. When we do the coordinate change (1-2), we see that

$$d\bar z^{(j)I} = \ell(\{d\bar z^{(i),J}\})$$

where ℓ is a linear function and that

$$d\bar z^{(j)K} \wedge d\bar\zeta^{(j)} = \ell'\big(\{d\bar z^{(i),J}\}_{|J|=p}, \{d\bar z^{(i)L} \Lambda\, d\bar\zeta^{(i)}\}_{|L|=p-1}\big)$$

where ℓ' is a linear function. It is clear that if the $\pi^{(j)}_{K,\bar\zeta}$ vanish to infinite order on $\tilde X$, the $\pi^{(i)}_{K,\bar\zeta}$ vanish also to infinite order on $\tilde X$, because they are combinations of the $\pi^{(j)}_{K,\bar\zeta}$ with C^∞ coefficients. If this condition is fulfilled, the jet of the $\pi^{(i)}_I$ along $\tilde X$ becomes linear combinations of the jets of the $\pi^{(j)}_K$ along $\tilde X$ with coefficients which are antiholomorphic of the variables $z^{(i)}$. This means that if the $J^{(\infty)}_{\tilde X}(\pi^{(j)}_K)$ depend only on $\zeta^{(j)}$ (and not on $\bar\zeta^{(j)}$), the same happens with the $J^{(\infty)}_{\tilde X}(\pi^{(j)}_L)$.

DEFINITION. We shall say that a $(0,p)$ form π, C^∞ in V has a purely holomorphic jet on $\tilde X$, if

$$\text{(1-8)} \qquad \begin{cases} J^{(\infty)}_{\tilde X}(\pi^{(i)}_{K,\bar\zeta}) = 0 & \text{for all } |K| = p-1, \text{ all } i \in I \\ J^{(\infty)}_{\tilde X}(\pi^{(i)}_L) = j^{(\infty)}_{\tilde X}(\pi^{(i)}_L) & \text{for all } |L| = p, \text{ all } i \in I \end{cases}.$$

NOTATIONS. Let π be a $(0,p)$-form on V, which has a purely holomorphic jet on $\tilde X$: we define

$$\text{(1-9)} \qquad j^{(\infty)}_{\tilde X}(\pi) = \sum_{|J|=p} j^{(\infty)}_{\tilde X}(\pi^{(i)}_J)\, d\bar z^{(i),J} \text{ in } V_i \cap V.$$

This is clearly an element in $\lim_r \operatorname{proj} C^\infty(V \cap \tilde X, E^{(r)}_{\tilde X} \otimes_{\mathbb C} \Lambda^{0,p}_{\tilde X})$.

c) Action of $\bar\partial_{\tilde S}$:

LEMMA 1.2. *Let π be a $C^\infty(0,p)$-form on V in $\tilde S$ which has a purely holomorphic jet in $\tilde X$. Then $\bar\partial_{\tilde S}\pi$ is a $(0,p)$-form on V which has also a purely holomorphic jet on $\tilde X$ and we have*

$$\text{(1-10)} \qquad j^{(\infty)}_{\tilde X}(\bar\partial_{\tilde S}\pi) = \lim_r \operatorname{proj} \bar\partial_{\tilde X}\big(j^{(r)}_{\tilde X}(\pi)\big)$$

where $j^{(r)}_{\tilde X}(\pi)$ is the usual jet of order r of π (defined as in formula (2-4) of [1]) and $\bar\partial_{\tilde X}$ denotes the usual $\bar\partial$ with respect to $\tilde X$ acting on C^∞ section of holomorphic vector bundles over $\tilde X$.

PROOF. We fix a chart V_i and compute $\bar\partial_{\tilde S}\pi$ using the splitting (1-6) of π. Then the $d\bar\zeta$ part of $\bar\partial_{\tilde S}\pi$ comes from $\frac{\partial}{\partial\bar\zeta^{(i)}}\pi^{(i)}_I$ and the jets of these terms $\tilde X$ are 0 because π has a purely holomorphic jet on $\tilde{X}$. The pure $d\bar{z}^{(i)}$-terms of $\bar{\partial}_{\tilde{S}}\pi$ comes from $\frac{\partial}{\partial \bar{z}_k}\pi_J^{(i)}$ only and the jet of these terms on $\tilde{X}$ are obviously purely holomorphic. We thus see that

$$\begin{aligned} j_{\tilde{X}}^{(\infty)}(\bar{\partial}_{\tilde{S}}\pi) &= j_{\tilde{X}}^{(\infty)}\Big(\bar{\partial}_{\bar{z}}\Big(\sum_{|J|=p}\pi_J^{(i)}\,d\bar{z}^{(i)J}\Big)\Big) \\ &= \lim_r \operatorname{proj} j_{\tilde{X}}^{(r)}\Big(\bar{\partial}_{\bar{z}}\Big(\sum_{|J|=p}\pi_J^{(i)}\,d\bar{z}^{(i)J}\Big)\Big) \\ &= \lim_r \operatorname{proj}\sum_{k=0}^{r}\frac{\zeta^{(i)k}}{k!}\bar{\partial}_{\bar{z}}\Big(\sum_{|J|=p}\frac{\partial^k\pi_J^{(i)}}{\partial\zeta^{(i)k}}\Big|_{\zeta^{(i)}=0}d\bar{z}^{(i)J}\Big) \\ &= \lim_r \operatorname{proj}\bar{\partial}_{\tilde{X}}\big(j_{\tilde{X}}^{(r)}(\pi)\big). \end{aligned}$$

1.4 *The sheaf* $\hat{O}_{\tilde{S}}\otimes_{O_{\tilde{X}}}\Lambda_{\tilde{X}}^{(0,p)}$.

a) $\hat{O}_{\tilde{S}}$ is a $O_{\tilde{X}}$-module: We know that $O_{\tilde{S}}/I_{\tilde{X}}^r$ are all $O_{\tilde{X}}$-modules (as sheaf of holomorphic sections of the bundle $E_{\tilde{X}}^{(r)}$ (see [1] Section 2.3). Moreover the natural mappings $O_{\tilde{S}}/I_{\tilde{X}}^{r+1}\longrightarrow O_{\tilde{S}}/I_{\tilde{X}}^r$ are morphisms of $O_{\tilde{X}}$-modules, so that $\hat{O}_{\tilde{S}}$ is an $O_{\tilde{X}}$-module.

In particular, we can define the tensor product over $O_{\tilde{X}}$, $\hat{O}_{\tilde{S}}\otimes_{O_{\tilde{X}}}\Lambda_{\tilde{X}}^{(0,p)}$.

b) Identification of $\hat{O}_{\tilde{S}}\otimes_{O_{\tilde{X}}}\Lambda_{\tilde{X}}^{(0,p)}$ as jet of forms: Let us consider an element $\omega\in\Gamma(V,\hat{O}_{\tilde{S}}\otimes_{O_{\tilde{X}}}\Lambda_{\tilde{X}}^{(0,p)})$. Around any point $m\in\tilde{X}\cap V$, we can say that

$$\omega=\sum_{j=1}^{N}\theta_j\otimes_{O_{\tilde{X}}}\alpha_j \tag{1-11}$$

where θ_j are a finite number of elements of $\hat{O}_{\tilde{S}}$ and α_j are in $\Lambda_{\tilde{X}}^{(0,p)}$. We consider the mapping

$$\omega\longrightarrow\sum_{j=1}^{N}\theta_j.\,\alpha_j \tag{1-12}$$

which has as image a $(0,p)$ form with coefficient in $\hat{O}_{\tilde{S}}$ and can be identified to

$$\sum_{k=0}^{\infty}\frac{\zeta^{(i)k}}{k!}\sum_{j=1}^{N}\theta_j^{(k)}\alpha_j$$

where we have written

$$\theta_j=\sum_{k=0}^{\infty}\frac{\zeta^{(i)k}}{k!}\theta_j^{(k)}.$$

LEMMA 1.3. *The mapping (1-12) is injective.*

PROOF. Because $\Lambda_{\tilde{X}}^{(0,p)}$ is a direct sum of the $\Lambda_{\tilde{X}}^{(0,0)}$, it is sufficient to prove the lemma for $p=0$.

The first fact we need is that the natural map

$$(O_{\tilde{S}}/I_{\tilde{X}}^{r+1})_{\tilde{x}}\otimes_{O_{\tilde{X},\tilde{x}}}C_{\tilde{X},\tilde{x}}^{\infty}\longrightarrow(O_{\tilde{S}}/I_{\tilde{X}}^{r+1}).\,C_{\tilde{X},\tilde{x}}^{\infty} \tag{1-13}$$

is injective, which is obvious because $O_{\tilde{S}}/I_{\tilde{X}}^{r+1}$ is the sheaf of holomorphic sections of a vector bundle over $\tilde{X}$. Moreover it is an isomorphism.

Let us now consider

$$\hat{O}_{\tilde{S},\tilde{x}} = \lim_{r} \operatorname{proj}(O_{\tilde{S}}/I_{\tilde{X}}^{r})_{\tilde{x}}$$

and in $\hat{O}_{\tilde{S},\tilde{x}}$ let us denote by $\tilde{O}_{\tilde{S},\tilde{x}}$ the submodule of the projective systems $(f_r)_{r\geq 0} \in \lim_r \operatorname{proj}(O_{\tilde{S}}/I_{\tilde{X}}^{r})_{\tilde{x}}$ such that $f_r = f_{r_0}$ for all $r \geq r_0$ for a certain r_0. Then $\tilde{O}_{\tilde{S},\tilde{x}}$ is dense for $I_{\tilde{X}}$-adic topology in $\hat{O}_{\tilde{S},\tilde{x}}$ and from the injectivity of the mapping (1-13) we deduce the injectivity of

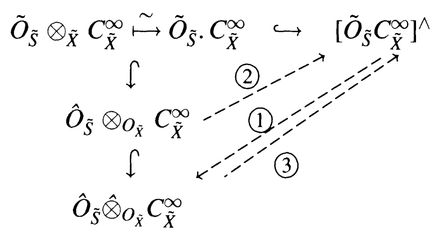
(1-14) $$\tilde{O}_{\tilde{S},\tilde{x}} \otimes_{O_{\tilde{X},\tilde{x}}} C^{\infty}_{\tilde{X},\tilde{x}} \hookrightarrow \tilde{O}_{\tilde{S},\tilde{x}} C^{\infty}_{\tilde{X},\tilde{x}}$$

and in fact it is an isomorphism.

From the injectivity of $\tilde{O}_{\tilde{S},\tilde{x}} \hookrightarrow \hat{O}_{\tilde{S},\tilde{x}}$, we deduce the injectivity of $\tilde{O}_{\tilde{S},\tilde{x}} C^{\infty}_{\tilde{X},\tilde{x}} \hookrightarrow [\tilde{O}_{\tilde{S},\tilde{x}} C^{\infty}_{\tilde{X},\tilde{x}}]^{\wedge}$ which is the space of formal series in ζ with coefficient in $C^{\infty}_{\tilde{X},\tilde{x}}$ and because $C^{\infty}_{\tilde{X},\tilde{x}}$ is flat over $O_{\tilde{X},\tilde{x}}$, we deduce also the injectivity of

(1-15) $$\tilde{O}_{\tilde{S}} \otimes_{O_{\tilde{X},\tilde{x}}} C^{\infty}_{\tilde{X},\tilde{x}} \hookrightarrow \hat{O}_{\tilde{S},\tilde{x}} \otimes_{O_{\tilde{X},\tilde{x}}} C^{\infty}_{\tilde{X},\tilde{x}}.$$

Now, we have the following diagram (where the dotted arrows will be constructed below) and where $\hat{\otimes}_{O_{\tilde{X}}}$ is the completion of the tensor product.



The arrow ② exists because of the universal property of the tensor product.

The arrow ① exists because $[\tilde{O}_{\tilde{S}} C^{\infty}_{\tilde{X}}]^{\wedge}$ is the completion of $\tilde{O}_{\tilde{S}} C^{\infty}_{\tilde{X}}$, because $\hat{O}_{\tilde{S}} \hat{\otimes}_{O_{\tilde{X}}} C^{\infty}_{\tilde{X}}$ is complete and because of the universal property of the completion $[\tilde{O}_{\tilde{S}} C^{\infty}_{\tilde{X}}]^{\wedge}$.

The arrow ③ exists because ② exists and because of the universal property of the completions $\hat{\otimes}_{O_{\tilde{X}}}$. These arrows ① and ③ are then reciprocal isomorphisms, and ② is injective. But ② is the natural mapping (1-12) (for $p = 0$).

1.5 *The sheaf* $\Lambda^{(0,0)}_{\tilde{S},\tilde{X}}$.

a) Definition: For $\tilde{x} \in \tilde{X} - \tilde{S}$, we define simply

$$\Lambda^{0,p}_{\tilde{S},\tilde{X},\tilde{x}} = \Lambda^{0,p}_{\tilde{S},\tilde{x}}.$$

For $\tilde{x} \in \tilde{X}$, we call $\Lambda^{0,p}_{\tilde{S},\tilde{X},\tilde{x}}$ the space of germs of $C^{\infty}(0,p)$-forms π in a neighborhood V of $\tilde{x}$ which have a purely holomorphic jet on $\tilde{X}$ such that

(1-16) $$j^{(\infty)}_{\tilde{X}}(\pi) \in \Gamma(V \cap \tilde{X}, \hat{O}_{\tilde{S}} \otimes_{O_{\tilde{X}}} \Lambda^{(0,p)}_{\tilde{X}}).$$

This last definition has a meaning because of the injectivity of the mapping (1-12).

b) The $\tilde{\partial}_{\tilde{S}}$:

LEMMA 1.4. *Let $\tilde{x} \in \tilde{X}$, and an element of $\Lambda^{0,p}_{\tilde{S},\tilde{X},\tilde{x}}$. Then $\bar{\partial}_{\tilde{S}}\pi$ is in $\Lambda^{0,p}_{\tilde{S},\tilde{X},\tilde{x}}$ and moreover if we consider $j^{(\infty)}_{\tilde{X}}(\pi)$ as an element in $\hat{O}_{\tilde{S},\tilde{x}} \otimes_{O_{\tilde{X},\tilde{x}}} \Lambda^{0,p}_{\tilde{X},\tilde{x}}$ then*

$$j^{(\infty)}_{\tilde{X}}(\bar{\partial}_{\tilde{S}}\pi) = \bar{\partial}_{\tilde{X}} j^{(\infty)}_{\tilde{X}}(\pi) \tag{1-17}$$

where $\bar{\partial}_{\tilde{X}}$ is the $\bar{\partial}$ defined naturally as

$$\bar{\partial}_{\tilde{X}}\colon \hat{O}_{\tilde{S},\tilde{x}} \otimes_{O_{\tilde{X},\tilde{x}}} \Lambda^{0,p}_{\tilde{X},\tilde{x}} \xrightarrow{I\otimes\bar{\partial}_X} \hat{O}_{\tilde{S},\tilde{x}} \otimes_{O_{\tilde{X},\tilde{x}}} \Lambda^{0,p+1}_{\tilde{X},\tilde{x}}. \tag{1-18}$$

PROOF. By Lemma 1.2 we know that $\bar{\partial}_{\tilde{S}}\pi$ has a purely holomorphic jet on $\tilde{X}, j^{(\infty)}_{\tilde{X}}(\bar{\partial}_{\tilde{S}}\pi)$ and we know how to compute it. More precisely, in our case

$$j^{(\infty)}_{\tilde{X}}(\pi) = \sum_{j=1}^{N} \theta_j \alpha_j$$

where $\theta_j \in \hat{O}_{\tilde{S},\tilde{x}}$, $\alpha_j \in \Lambda^{0,p}_{\tilde{X},\tilde{x}}$, and then

$$j^{(\infty)}_{\tilde{X}}(\bar{\partial}_{\tilde{S}}\pi) = \sum_{j=1}^{N} \theta_j \bar{\partial}_{\tilde{X}} \alpha_j.$$

But because of the injectivity of the mapping (1-12) (now for forms of type $(0, p+1)$), we see that up to an identification

$$j^{(\infty)}_{\tilde{X}}(\bar{\partial}_{\tilde{S}}\pi) = \sum_{j=1}^{N} \theta_j \otimes_{O_{\tilde{X}}} \bar{\partial}_{\tilde{X}} \alpha_j$$

which is exactly what we want.

1.6 *The sheaf* $\varphi_*(\hat{O}_{\tilde{S}}) \otimes_{O_X} \Lambda^{0,p}_X$.

a) $\varphi_*(\hat{O}_{\tilde{S}})$ is an O_X-module: This statement is obvious because $\hat{O}_{\tilde{S}}$ is an $O_{\tilde{X}}$-module (see Section 1.4). In particular, we can take the tensor product over O_X with $\Lambda^{0,p}_X$.

b) Identification of $\varphi_*(\hat{O}_{\tilde{S}}) \otimes_{O_X} \Lambda^{0,p}_X$ as jet of form along $\tilde{X}$: Take a point $m \in X$, an open neighborhood U of x in S and $\omega \in \Gamma\big(U, \varphi_*(\hat{O}_{\tilde{S}}) \otimes_{O_X} \Lambda^{0,p}_X\big)$. We have

$$\omega = \sum_{j=1}^{N} \theta_j \otimes_{O_X} \alpha_j$$

where θ_j are in $\Gamma(\varphi^{-1}(U), \hat{O}_{\tilde{S}})$ and α_j are in $\Gamma(U \cap X, \Lambda^{0,p}_X)$. We can consider the mapping

$$\omega \longrightarrow (\varphi|_{\tilde{X}})^* \omega \tag{1-19}$$

where

$$(\varphi|_{\tilde{X}})^* \omega = \sum_{j=1}^{N} \theta_j (\varphi|_{\tilde{X}})^* \alpha_j. \tag{1-20}$$

This is a $(0, p)$ form with coefficient in $\hat{O}_{\tilde{S}}$.

THEOREM 1.1. *The mapping* $\omega \longrightarrow (\varphi|_{\tilde X})^* \omega$ *defined by (1-19) and (1-20) is injective.*

c) Proof of Theorem 1.1: We shall refer mainly to the proof and notations of the analogue Theorem 2.2 given in [1] Section 2.4. We fix a point $x \in X$, and $\tilde x \in \varphi^{-1}(x)$ such that we can choose around $\tilde x$ the coordinate $(z_1 \cdots z_{n-m+1}, w_1, \ldots, w_m)$ where the w's are coordinates on X around x. We also denote by $j^{(r)}\colon \hat O_{\tilde S} \longrightarrow O_{\tilde S}/I_{\tilde X}^{r+1}$ the natural projection. We have seen that for all r, the mappings

$$(O_{\tilde S}/I_{\tilde X}^{r+1})_{\tilde X} \otimes_{O_{X,x}} C^\infty_{X,x} \longrightarrow (O_{\tilde S}/I_{\tilde X}^{r+1})_{\tilde x} \otimes_{O_{\tilde X,\tilde x}} C^\infty_{\tilde X,\tilde x} \simeq (O_{\tilde S}/I_{\tilde X}^{r+1})_{\tilde x}. C^\infty_{\tilde X,\tilde x} \tag{1-21}$$

are both injective (see Lemma 2.3 of [1]) and we can identify

$$(O_{\tilde S}/I_{\tilde X}^{r+1})_{\tilde x} \otimes_{O_{X,x}} C^\infty_{X,x} \simeq (O_{\tilde S}/I_{\tilde X}^{r+1})_{\tilde x}. C^\infty_{X,x}. \tag{1-22}$$

We have

LEMMA 1.4. *The mapping*

$$\hat O_{\tilde S,\tilde x} \otimes_{O_{X,x}} C^\infty_{X,x} \longrightarrow \hat O_{\tilde S,\tilde x}. C^\infty_{\tilde X,\tilde x} \tag{1-23}$$

is injective.

End of proof of Theorem 1.1: The mapping,

$$\varphi_*(\hat O_{\tilde S})_x \longrightarrow \hat O_{\tilde S,\tilde x} \quad \big(\tilde x \in \varphi^{-1}(x)\big)$$

is injective (Zariski's main theorem on normal singularities). Because $C^\infty_{X,x}$ is a flat $O_{X,x}$-module

$$\varphi_*(\hat O_{\tilde S})_x \otimes_{O_{X,x}} C^\infty_{X,x} \longrightarrow \hat O_{\tilde S,\tilde x} \otimes_{O_{X,x}} C^\infty_{X,x}$$

is injective. The composition with the injection mapping of Lemma 1.4 gives the result.

d) Proof of Lemma 1.4: This is a slight modification of the proof of Lemma 1.3. We introduce as in Lemma 1.3, the module $\tilde O_{\tilde S,\tilde x}$. Because of the injectivity of (1-21) and the identification (1-22), we have an identification

$$\tilde O_{\tilde S,\tilde x} \otimes_{O_{X,x}} C^\infty_{X,x} \simeq \tilde O_{\tilde S,\tilde x}. C^\infty_{X,x}. \tag{1-24}$$

Now

$$\begin{array}{ccc}
\tilde O_{\tilde S,\tilde x} \otimes_{O_{X,x}} C^\infty_{X,x} \simeq \tilde O_{\tilde S,\tilde x}. C^\infty_{X,x} & \hookrightarrow \tilde O_{\tilde S,\tilde x} \quad (O_z \hat\otimes_{\mathbb C} C^\infty_w) & \hookrightarrow \quad \tilde O_{\tilde S,\tilde s} \otimes \quad {}_{\mathbb C} C^\infty_w]^\wedge \\
\downarrow \quad ② & \dashrightarrow \quad ① & \dashrightarrow \\
\hat O_{\tilde S,\tilde x} \otimes_{O_{X,x}} C^\infty_{X,x} & & \\
\downarrow & \dashleftarrow \quad ③ & \\
\hat O_{\tilde S,\tilde x} \hat\otimes_{X,x} C^\infty_{X,x} & &
\end{array}$$

The horizontal arrows are, apart from the identification (1-24), a completion of $O_{z,w} \otimes_{\mathbb C} C^\infty_w$ (recall that the coordinates of X are the w's) and then a completion with respect to the $I_{\tilde X}$-adic topology. The vertical arrows are first an injective mapping

$$\tilde O_{\tilde S} \otimes_{O_{X,x}} C^\infty_X \hookrightarrow \hat O_{\tilde S} \otimes_{O_{X,x}} C^\infty_X$$

obtained by tensorisation of the injection $\tilde{O}_{\tilde{S}} \hookrightarrow \hat{O}_{\tilde{S}}$ by the flat O_X-module C^∞_X and secondly the completion $\hat{\otimes}_{O_{X,x}}$. Then the diagonal dotted arrows are constructed exactly as in Lemma 1.3 using universal properties of the completion and of the tensor product. From this the natural arrow ② is injective. Finally we know that

$$O_z \hat{\otimes}_{\mathbb{C}} C^\infty_w \hookrightarrow C^\infty_z \hat{\otimes}_{\mathbb{C}} C^\infty_w \simeq C^\infty_{z,w}$$

which achieves the proof.

2. **Differential forms and resolution of** S.

2.1 *A fine resolution of* $\tilde{S}$.

THEOREM 2.1. *The following complex is a fine resolution of* $O_{\tilde{S}}$.

$$\mathbb{R}(O_{\tilde{S}}): 0 \longrightarrow O_{\tilde{S}} \longrightarrow \Lambda^{0,0}_{\tilde{S},\tilde{X}} \xrightarrow{\bar{\partial}_{\tilde{S}}} \Lambda^{0,1}_{\tilde{S},\tilde{X}} \longrightarrow \cdots \longrightarrow \Lambda^{0,p}_{\tilde{S},\tilde{X}} \xrightarrow{\bar{\partial}_{\tilde{S}}} \Lambda^{0,p+1}_{\tilde{S},\tilde{X}} \longrightarrow \cdots$$

PROOF. The sheaves $\Lambda^{0,p}_{\tilde{S},\tilde{X}}$ have been defined in Section 1.5. It is easy to see that they are fine sheaves (using special partitions of unity as in Section 3.1, Lemma 3.2 of [1]). Moreover outside $\tilde{X}$, $\Lambda^{0,p}_{\tilde{S},\tilde{X}}$ is $\Lambda^{0,p}_{\tilde{S}}$, and the local exactness follows from the Dolbeault resolution.

Let us now choose a point $\tilde{x} \in \tilde{X}$ and $\pi \in \Lambda^{0,p}_{\tilde{S},\tilde{X},\tilde{x}}$ on a small open neighborhood V of $\tilde{x}$ in $\tilde{S}$ which is $\bar{\partial}_{\tilde{S}}$-closed. By Dolbeault theorem, we know that

$$\pi = \bar{\partial}_{\tilde{S}}\alpha \tag{2-1}$$

on a smaller V, where $\alpha \in \Lambda^{0,p}_{\tilde{S},\tilde{x}}$. Moreover we know that $j^{(\infty)}_{\tilde{X}}(\pi)$ is in $\hat{O}_{\tilde{S},\tilde{x}} \otimes_{O_{\tilde{X},\tilde{x}}} \Lambda^{0,p}_{\tilde{S},\tilde{x}}$ and by Lemma 1.4, this is $\bar{\partial}_{\tilde{S}}$-closed. Now, because $\Lambda^{0,p}_{\tilde{X}}$ is a flat $O_{\tilde{X}}$-module, we have a Dolbeault resolution of $\hat{O}_{\tilde{S},\tilde{x}}$.

$$O \longrightarrow \hat{O}_{\tilde{S}} \longrightarrow \hat{O}_{\tilde{S}} \otimes_{O_{\tilde{X}}} \Lambda^{0,0}_{\tilde{X}} \longrightarrow \hat{O}_{\tilde{S}} \otimes_{O_{\tilde{X}}} \Lambda^{0,1}_{\tilde{X}} \longrightarrow \cdots$$

$\mathbb{R}_D(\hat{O}_{\tilde{S}})$

$$\cdots \longrightarrow \hat{O}_{\tilde{S}} \otimes_{O_{\tilde{X}}} \Lambda^{0,p}_{\tilde{X}} \xrightarrow{\bar{\partial}_{\tilde{X}}} \hat{O}_{\tilde{S}} \otimes_{O_{\tilde{X}}} \Lambda^{0,p+1}_{\tilde{X}} \cdots\cdots$$

so that we can solve

$$\begin{cases} \rho \in \hat{O}_{\tilde{S},\tilde{x}} \otimes_{O_{\tilde{X},\tilde{x}}} \Lambda^{0,p-1}_{\tilde{X},\tilde{x}} \\ j^{(\infty)}_{\tilde{X}}(\pi) = \bar{\partial}_{\tilde{X}}\rho \end{cases} . \tag{2-2}$$

Let us now assume $p \geq 2$.

It remains to modify α so that $\alpha \in \Lambda^{0,p-1}_{\tilde{S},\tilde{X},\tilde{x}}$. To do this, we shall use the splitting (1-6) for π and α. With the same notation as in Sections 1.2 and 1.3, we write

$$\begin{aligned} \pi &= \pi_{\bar{z}} + \pi_{\bar{\zeta}} \\ \alpha &= \alpha_{\bar{z}} + \alpha_{\bar{\zeta}} \end{aligned} \tag{2-3}$$

where $\pi_{\bar{z}}$ (resp. $\alpha_{\bar{z}}$) is the part of π (resp. α) which does not contain any $d\bar{\zeta}$ and $\pi_{\bar{\zeta}}$ (resp. $\alpha_{\bar{\zeta}}$) is the part π (resp. α) which is divisible by $d\bar{\zeta}$. We have from (2-1) and (2-3)

$$j_{\tilde{X}}^{(\infty)}(\pi) = J_{\tilde{X}}^{(\infty)}\bar{\partial}_{\tilde{X}}(\alpha_{\bar{z}}) = \bar{\partial}_{\tilde{X}}(J_{\tilde{X}}^{(\infty)}\alpha_{\bar{z}})$$

so that in the Taylor expansion of $J_{\tilde{X}}^{(\infty)}\alpha_{\bar{z}}$ the coefficients $\alpha_{\bar{z}}^{(k,\bar{\ell})}$ of any monomial $\zeta^k\bar{\zeta}^\ell$ with $\ell \geq 1$ are $\bar{\partial}_{\tilde{X}}$-closed, and as a consequence they are $\bar{\partial}_{\tilde{X}}$-exact, so that if $p \geq 2$

$$\alpha_{\bar{z}}^{(k,\bar{\ell})} = \bar{\partial}_{\tilde{X}}\beta_{\bar{z}}^{(k,\bar{\ell})}.$$

Now let us choose a C^∞ form $\tilde{\beta}_{\bar{z}}$ such that $J_{\tilde{X}}^{(\infty)}(\tilde{\beta}_{\bar{z}}) = \sum_{k\geq 0,\bar{\ell}\geq 1} \beta_{\bar{z}}^{k\bar{\ell}}\zeta^k\bar{\zeta}^\ell$ and let us change α in $\alpha' = \alpha - \bar{\partial}_{\tilde{S}}\tilde{\beta}_{\bar{z}}$. Then

$$\begin{cases} \pi = \bar{\partial}_{\tilde{S}}\alpha' \\ J_{\tilde{X}}^{\infty}(\alpha'_{\bar{z}}) = j_{\tilde{X}}^{(\infty)}(\alpha'_{\bar{z}}) \end{cases}. \tag{2-4}$$

From the first equation of (2-4), because of the choice of $\pi \in \Lambda_{\tilde{S},\tilde{X}}^{0,p}$, we have that

$$0 = \bar{\partial}_{\tilde{X}}J_{\tilde{X}}^{(\infty)}(\alpha'_{\bar{\zeta}}) \pm \frac{\partial}{\partial\bar{\zeta}}J_{\tilde{X}}^{(\infty)}(\alpha'_{\bar{z}})\Lambda\, d\bar{\zeta}.$$

The last term of this equation is 0 because of the second line of (2-4), so that

$$0 = \bar{\partial}_{\tilde{X}}J_{\tilde{X}}^{(\infty)}(\alpha'_{\bar{\zeta}}).$$

This means that any coefficient of the Taylor expansion in $\zeta^k\bar{\zeta}^\ell$ of $\alpha'_{\bar{\zeta}}$ is $\bar{\partial}_{\tilde{X}}$ closed, so it is $\bar{\partial}_{\tilde{X}}$-exact, and by the same trick as in (2-4), we arrive at a form α'' such that

$$\begin{cases} \pi = \bar{\partial}_{\tilde{S}}\alpha'' \\ \alpha'' \text{ is has a purely holomorphic jet on } \tilde{X} \end{cases}. \tag{2-5}$$

We compare (2-5) to (2-2) by taking $j_{\tilde{X}}^{(\infty)}$ of both sides of equation (2-5). Then

$$j_{\tilde{X}}^{(\infty)}(\pi) = \bar{\partial}_{\tilde{X}}j_{\tilde{X}}^{(\infty)}(\alpha'')$$

so that $\bar{\partial}_{\tilde{X}}\big(\rho - j_{\tilde{X}}^{(\infty)}(\alpha'')\big) = 0$. This means that in the formal Taylor expansion in ζ^k of $\rho - j_{\tilde{X}}^{(\infty)}(\alpha'')$ the coefficient γ_k is $\bar{\partial}_{\tilde{X}}$ closed and so it is $\bar{\partial}_{\tilde{X}}$-exact.

$$\gamma_k = \bar{\partial}_{\tilde{X}}\sigma_k$$

Let us now extend $\sum_{k\geq 0}\zeta^k\sigma_k$ in a C^∞ form $\tilde{\sigma}$ around $\tilde{x}$, with a purely holomorphic jet $j_{\tilde{X}}^{(\infty)}(\tilde{\sigma}) = \sum_{k\geq 0}\zeta^k\sigma_k$. Then define

$$\alpha''' = \alpha'' + \bar{\partial}_{\tilde{S}}\tilde{\sigma}.$$

We have by construction,

$$j_{\tilde{X}}^{(\infty)}(\bar{\partial}_{\tilde{S}}\tilde{\sigma}) = \rho - j_{\tilde{X}}^{(\infty)}(\alpha''')$$

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so that finally

$$j_{\tilde{X}}^{(\infty)}(\alpha''') = \rho \in \hat{O}_{\tilde{S},\tilde{x}} \otimes_{O_{\tilde{X},\tilde{x}}} \Lambda^{0,p-1}_{\tilde{X},\tilde{x}}$$
$$\bar{\partial}_{\tilde{S}}\alpha''' = \pi.$$

This means that α''' is in $\Lambda^{0,p-1}_{\tilde{S},\tilde{X},\tilde{x}}$.

For $p = 1$, α is in $\Lambda^{0,0}_{\tilde{S},\tilde{x}}$ and

$$\pi_{\bar{\zeta}} = \frac{\partial \alpha}{\partial \bar{\zeta}}\, d\bar{\zeta}.$$

But $\pi_{\bar{\zeta}}$ vanishes to infinite order on $\zeta = 0$, so that α has a purely holomorphic jet along $\tilde{X}$. We have from (2-1)

$$j_{\tilde{X}}^{(\infty)}(\pi) = \bar{\partial}_{\tilde{X}} j_{\tilde{X}}^{(\infty)}(\alpha) = \bar{\partial}_{\tilde{X}}\rho.$$

This means that all the coefficients γ_k of ζ^k in the Taylor expansion of $\rho - j_{\tilde{X}}^{(\infty)}(\alpha)$ are holomorphic along $\tilde{X}$, so that $\rho - j_{\tilde{X}}^{(\infty)}(\alpha)$ is in $\hat{O}_{\tilde{S},\tilde{x}}$, and so

$$j_{\tilde{X}}^{(\infty)}(\alpha) = \rho + \hat{O}_{\tilde{S},\tilde{x}} \in \hat{O}_{\tilde{S},\tilde{x}} \otimes_{O_{\tilde{X},\tilde{x}}} \Lambda^{0,0}_{\tilde{X},\tilde{x}}$$

and α as in $\Lambda^{0,0}_{\tilde{S},\tilde{X},\tilde{x}}$.

2.2 *Differential forms on S and the $\bar{\partial}$ operator.*

a) Definition of the sheaves $\Lambda^{0,p}_{S,x}$: If x is a point in $S - X$, we define $\Lambda^{0,p}_{S,x}$ to be the germs of C^∞ forms of type $(0,p)$ at x on S (or what is the same, at $\varphi^{-1}(x)$ on $\tilde{S}$).

Now let x be a point in X, U be an open neighborhood of x in S. We shall say that

$$\pi \in \Gamma(U, \Lambda^{0,p}_S)$$

if π is a C^∞ form of type $(0,p)$ on $\varphi^{-1}(U) \subset \tilde{S}$ which has a purely holomorphic jet on $\tilde{X}$, such that

$$j_{\tilde{X}}^{(\infty)}(\pi) \in \Gamma\big(U \cap X, \varphi_*(\hat{O}_{\tilde{S}}) \otimes_{O_X} \Lambda^{0,p}_X\big). \tag{2-6}$$

This definition has a meaning because of Theorem 1.1 of Section 1.6 which identify $\varphi_*(\hat{O}_{\tilde{S}}) \otimes_{O_X} \Lambda^{0,p}_X$ to purely holomorphic jets on $\tilde{X}$ through the mapping $(\varphi|_{\tilde{X}})^*$.

We have exactly as in [1] Lemma 3.1.

LEMMA 2.1. *The sheaves $\Lambda^{0,p}_S$ are fine sheaves*

b) The $\bar{\partial}_S$ operator.

LEMMA 2.2. *$\bar{\partial}_{\tilde{S}}$ preserves $\Lambda^{0,p}_S$ and we have a commutative diagram*

$$\begin{array}{ccc} \Lambda^{0,p}_S & \xrightarrow{\bar{\partial}_{\tilde{S}}} & \Lambda^{0,p+1} \\ {\scriptstyle j_{\tilde{X}}^{(\infty)}}\downarrow & & \downarrow{\scriptstyle j_{\tilde{X}}^{(\infty)}} \\ \varphi_*(\hat{O}_{\tilde{S}}) \otimes_{O_X} \Lambda^{0,p}_X & \xrightarrow{I\otimes\bar{\partial}_X} & \varphi_*(\hat{O}_{\tilde{S}}) \otimes_{O_X} \Lambda^{0,p+1}_X \end{array} . \tag{2-7}$$

PROOF. *This is exactly the same proof as in Lemma 3.3 of [1] provided we use Lemma 1.2 and the injectivity of* $(\varphi|_{\tilde{X}})^*$.

NOTATION. We shall denote simply

$$\bar{\partial}_S\colon \Lambda_S^{0,p} \longrightarrow \Lambda_S^{0,p+1}.$$

2.3 *Local exactness of the complex* $(\Lambda_S^{0,p}, \bar{\partial}_S)$.

THEOREM 2.2. *The following complex is a fine resolution of* O_S.

$$\mathbb{R}(O_S)\colon O \longrightarrow O_S \longrightarrow \Lambda_S^{0,0} \xrightarrow{\bar{\partial}_S} \Lambda_S^{0,1} \longrightarrow \cdots \longrightarrow \Lambda_S^{0,p} \xrightarrow{\bar{\partial}_S} \Lambda_S^{0,p+1} \longrightarrow \cdots$$

PROOF. Exactness at step O: This results from the fact that the singularity is normal as in [1] Section 3.2.

Exactness at step p: The proof is basically the same as the one in [1] Section 3.2 except for a major modification due to the use of the resolution $\mathbb{R}(O_{\tilde{S}})$ of Theorem 2.1 above instead of the Dolbeault resolution of $O_{\tilde{S}}$. First of all, we have a resolution of $\varphi_*\hat{O}_{\tilde{S}}$ by tensoring this sheaf with the flat O_X-modules $\Lambda_X^{0,p}$

$$\mathbb{R}(O_S)\colon O \longrightarrow \hat{O}_S \otimes_{O_X} \Lambda_X^{0,0} \longrightarrow \hat{O}_S \otimes_{O_X} \Lambda_X^{0,1} \longrightarrow \cdots \longrightarrow \hat{O}_S \otimes_{O_X} \Lambda_X^{0,p} \longrightarrow \hat{O}_S \otimes_{O_X} \Lambda_X^{0,p+1} \longrightarrow$$

where the morphism is $I \otimes \bar{\partial}_X$ which we simply denote $\bar{\partial}_X$. (We have used (1-1) to identify $\varphi_*(\hat{O}_{\tilde{S}})$ with $\hat{O}_S$.

Let us start with $\pi \in \Gamma(U, \Lambda_S^{0,p})$ which is $\bar{\partial}_S$-closed. We know from Lemma 2.2 that $j_{\tilde{X}}^{(\infty)}(\pi)$ is $\bar{\partial}_X$-closed in $\hat{O}_S \otimes_{O_X} \Lambda_X^{0,p}$, and because $U \cap X$ can be taken a Stein open set,

$$j_{\tilde{X}}^{(\infty)}(\pi) = \bar{\partial}_X \omega^{(\infty)} \tag{2-8}$$

where $\omega^{(\infty)}$ is in $\hat{O}_S \otimes_{O_X} \Lambda_X^{0,p-1}$ on $U \cap X$.

Now, we can also consider $j_{\tilde{X}}^{(\infty)}(\pi)$ to be an element of $\Gamma\big(\varphi^{-1}(U), \hat{O}_{\tilde{S}} \otimes_{O_{\tilde{X}}} \Lambda_{\tilde{X}}^{0,p}\big)$, and so it is also $\bar{\partial}_{\tilde{X}}$-exact in $\varphi^{-1}(U)$ (by considering that $\omega^{(\infty)} \in \Gamma\big(\varphi^{-1}(U), \hat{O}_{\tilde{S}} \otimes_{O_{\tilde{X}}} \Lambda_{\tilde{X}}^{0,p-1}\big)$) which we rewrite

$$\begin{cases} j_{\tilde{X}}^{(\infty)}(\pi) = \bar{\partial}_{\tilde{X}} \omega^{(\infty)} \\ \omega^{(\infty)} \in \Gamma\big(\varphi^{-1}(U), \hat{O}_{\tilde{S}} \otimes_{O_{\tilde{X}}} \Lambda_{\tilde{X}}^{0,p-1}\big) \end{cases}. \tag{2-9}$$

(Here we have used the various identifications which are possible because of Lemma 1.3 and Theorem 1.1). But because of the resolution $\mathbb{R}(\hat{O}_{\tilde{S}})$, this means that the cohomology class $[j_{\tilde{X}}^{(\infty)}(\pi)]$ in $H^p\big(\varphi^{-1}(U), \hat{O}_{\tilde{S}}\big)$ is 0.

Now by definition, we have a natural morphism of resolutions.

$$\mathbb{R}^\bullet(O_{\tilde{S}}) \xrightarrow{j_{\tilde{X}}^{(\infty)}} \mathbb{R}^\bullet(\hat{O}_{\tilde{S}}) \tag{2-10}$$

which is compatible with $j_{\tilde{X}}^{(\infty)}\colon O_{\tilde{S}} \longrightarrow \hat{O}_{\tilde{S}}$. We also know that this morphism of sheaves induces an isomorphism in cohomologies in $\varphi^{-1}(U)$

$$H^p\big(\varphi^{-1}(U), O_{\tilde{S}}\big) \simeq H^p\big(\varphi^{-1}(U), \hat{O}_{\tilde{S}}\big) \quad (p \geq 1)$$

so that by comparison of cohomologies, (2-10) induces an isomorphism between the Dolbeault cohomologies computed with the resolutions $\mathbb{R}^\bullet(O_{\tilde{S}})$ and $\mathbb{R}^\bullet(\hat{O}_{\tilde{S}})$.

In particular, our π (which is in $\Lambda^{0,p}_{\tilde{S},\tilde{X}}$ on $\varphi^{-1}(U)$) is O in cohomology of $O_{\tilde{S}}$ on $\varphi^{-1}(U)$, and so we can solve

$$\text{(2-11)} \qquad \begin{cases} \pi = \bar{\partial}_{\tilde{S}}\alpha \\ \alpha \in \Gamma\big(\varphi^{-1}(U), \Lambda^{0,p-1}_{\tilde{S},\tilde{X}}\big) \end{cases}.$$

1°) Case $p = 1$: From (2-11) and (2-8), we deduce

$$\text{(2-12)} \qquad \bar{\partial}_{\tilde{X}}\big(j_{\tilde{X}}^{(\infty)}(\pi) - \omega^{(\infty)}\big) = 0$$

so that $j_{\tilde{X}}^{(\infty)}(\alpha) = \omega^{(\infty)} + \Gamma\big(\varphi^{-1}(U), \hat{O}_{\tilde{S}}\big)$ and is thus in $\Gamma\big(U, \hat{O}_S \otimes_{O_X} \Lambda^{0,0}_X\big)$ because $\omega^{(\infty)}$ is in this space. This means that α is in $\Gamma(U, \Lambda^{0,0}_S)$ by definition.

2°) Case $p > 1$: We still have equation (2-12), so that $j_{\tilde{X}}^{(\infty)}(\alpha) - \omega^{(\infty)}$ is a $\bar{\partial}_{\tilde{X}}$-closed element of $\Gamma\big(\varphi^{-1}(U), \hat{O}_{\tilde{S}} \otimes_{O_{\tilde{X}}} \Lambda^{0,p-1}_{\tilde{X}}\big)$ (because of (2-11), α is in $\Gamma\big(\varphi^{-1}(U), \Lambda^{0,p-1}_{\tilde{S},\tilde{X}}\big)$ so that its jet along $\tilde{X}$ is in $\hat{O}_{\tilde{S}} \otimes_{O_{\tilde{X}}} \Lambda^{0,p-1}_{\tilde{X}}$). In particular, $j_{\tilde{X}}^{(\infty)}(\alpha) - \omega^{(\infty)}$ defines a cohomology class in $H^{p-1}\big(\varphi^{-1}(U), \hat{O}_{\tilde{S}}\big)$ because of the resolution $\mathbb{R}^\bullet(\hat{O}_{\tilde{S}})$.

Again by the comparison of cohomologies, this class of cohomology comes from an element β in $\Gamma\big(\varphi^{-1}(U), \Lambda^{0,p-1}_{\tilde{S},\tilde{X}}\big)$ which is $\bar{\partial}_{\tilde{S}}$-closed: this means that

$$j_{\tilde{X}}^{(\infty)}(\alpha) - \omega^{(\infty)} = j_{\tilde{X}}^{(\infty)}(\beta) + \bar{\partial}_{\tilde{X}}\psi^{(\infty)}$$

where $\psi^{(\infty)}$ is in $\Gamma\big(\varphi^{-1}(U), \hat{O}_{\tilde{S}} \otimes_{O_{\tilde{X}}} \Lambda^{0,p-2}_{\tilde{X}}\big)$. Let ψ be a C^∞ form in $\varphi^{-1}(U)$ such that ψ has a purely holomorphic jet along $\tilde{X}$ and

$$j_{\tilde{X}}^{(\infty)}(\psi) = \psi^{(\infty)}.$$

Let us also define

$$\alpha' = \alpha - \beta - \bar{\partial}_{\tilde{S}}\psi.$$

Because $\bar{\partial}_{\tilde{S}}\beta = 0$, we have

$$\bar{\partial}_{\tilde{S}}\alpha' = \bar{\partial}_{\tilde{S}}\alpha = \pi.$$

Moreover, we have that α' has a purely holomorphic jet along $\tilde{X}$, and

$$j_{\tilde{X}}^{(\infty)}(\alpha') = \omega^{(\infty)} \in \Gamma(X \cap U, \hat{O}_S \otimes_{O_X} \Lambda^{0,p-1}_X)$$

so that α' is in $\Gamma(U, \Lambda^{0,p-1}_S)$ by definition.

2.4 *Comparison of these results with the results of [1].* In [1], we have defined other kinds of C^∞ forms and functions on S with conditions on jets of finite order on $\tilde{X}$. To prove the local exactness of the $\bar{\partial}_S$, we had to use a vanishing theorem, namely that there exists r with

$$H^p\big(\varphi^{-1}(U), I^r_{\tilde{X}}\big) = 0 \quad (p \geq 1).$$

The problem is the following: if S is a compact analytic space, we can choose the same r everywhere on X and everything is correct. If S is not compact, we do not know if we can choose the same r all over X, so that our previous definition is not sufficient. Here we do not assume any theorem of vanishing of cohomology, because we are working with the formal completion along X or $\tilde{X}$.

Moreover, with this new definition, we shall obtain flat $O_{S,x}$ modules as we shall see in the next two sections. We do not know how to prove flatness properties for the definitions of forms in [1].

3. **Flatness of the germs of C^∞ functions on S.**

3.1 *Some definitions and results concerning flat modules.*

1. Let A be a commutative ring with unity, E an A-module, $\vec{f} \equiv (f_1, \ldots, f_n)$ n elements in A. We denote by $R(\vec{f}, E)$ the A-module of relations in E among the f_i. An element of $R(\vec{f}, E)$ is a sequence $\vec{e} = (e_1, \ldots, e_n)$ such that

$$\vec{f} \bullet \vec{e} = \sum_{i=1}^{n} f_i e_i = 0.$$

2. If E is an A-module, the following definitions of flatness are equivalent:

(i) for all exact sequences $M' \longrightarrow M \longrightarrow M''$ of A-modules, the sequence

$$E \otimes_A M' \longrightarrow E \otimes_A M \longrightarrow E \otimes_A M''$$

is exact.

(ii) for any ideal $I \subset A$, the natural mapping $I \otimes_A E \longrightarrow E$ is injective.

(iii) for any $\vec{f} = (f_1, \ldots, f_n)$ in A^n,

$$R(\vec{f}, E) = R(\vec{f}, A).E.$$

3. If $O \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow O$ is a short exact sequence of A-modules, then if M and M'' are flat, M' is flat, if M' and M'' are flat, M is flat.

4. If A is a local noetherian ring, then its completion $\hat{A}$ is faithfully flat over A. (An A-module E is said to be faithfully flat if it is flat and if $E \otimes_A M = O$ implies $M = O$ for any A-module M).

5. If $A \subset B \subset C$ are three commutative rings with unity, so that

(i) C is faithfully flat over A

(ii) for any ideal $I \subset A$, $(IC) \cap B = IB$

then B is faithfully flat over A.

6. Let us now consider the space $\mathbb{C}^n$ and the space of germs of holomorphic functions O_n at 0, the space of germs of real-analytic functions at O, $\mathcal{A}_n$, the space of germs of C^∞ functions at 0, C^∞_n and finally the space of formal series at 0, F_n, so that

$$O_n \subset \mathcal{A}_n \subset C^\infty_n \subset F_n.$$

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F_n is faithfully flat over $\mathcal{A}_n$ (completion of a local noetherian ring) and from Malgrange [5], we know that C_n^∞ is also faithfully flat over $\mathcal{A}_n$.

7. Moreover it is possible to prove that $\mathcal{A}_n$ is faithfully flat over O_n and thus C_n^∞ and F_n are faithfully flat over O_n.

8. From now on, we shall freely use the preceding results.

3.2 *Formal functions on S.* Let M be a smooth complex manifold, x a point in M. We shall denote by $F_{M,x}$ the space of formal Taylor series at x on M. If $d = \dim M$, this is clearly isomorphic to $F_{\mathbb{C}^d,0}$. We also know that $F_{M,x}$ is a faithfully flat $O_{M,x}$-module.

Now, let S be an analytic space satisfying the hypothesis of Section 1.1, X its singular locus, $\varphi\colon \tilde{S} \longrightarrow S$ be a desingularization of S, $\tilde{X} = \varphi^{-1}(X)$ the exceptional divisor which is assumed to be irreducible. If x is a point of $S - X$, we can define $F_{S,x}$ as above because S is a smooth complex manifold at x. Now, let x be a point in X. We define

$$F_{S,x} = \varphi_*(\hat{O}_{\tilde{S}})_x \otimes_{O_{X,x}} C^\infty_{X,x}. \tag{3-1}$$

This definition makes sense, because, we know that $\varphi_*(\hat{O}_{\tilde{S}})_x$ is an $O_{X,x}$-module.

We want to prove.

LEMMA 3.1. *For $x \in X$, $F_{S,x}$ is a faithfully flat $\hat{O}_{S,x}$-module.*

This lemma is a direct consequence of the following abstract result.

LEMMA 3.2. *Let A be a ring which is a C-module, B a flat C-module. Then $A \otimes_C B$ is a flat A-module. Moreover if B is faithfully flat over C, then $A \otimes_C B$ is also faithfully flat over A.*

PROOF OF LEMMA 3.2. Let $O \longrightarrow M \longrightarrow N$ be A-modules. We can always write

$$M = M \otimes_A A \otimes_C C$$
$$N = N \otimes_A A \otimes_C C$$

considered as C-modules. We always have an injection, and because B is flat over C, we can tensorize by B over C while keeping injectivity

$$O \longrightarrow (M \otimes_A A \otimes_C C) \otimes_C B \longrightarrow (N \otimes_A A \otimes_C C) \otimes_C B.$$

But $(M \otimes_A A \otimes_C C) \otimes_C B \simeq M \otimes_A (A \otimes_C B)$ and the same with N, so that we still have an injection

$$O \longrightarrow M \otimes_A (A \otimes_C B) \longrightarrow N \otimes_A (A \otimes_C B)$$

which proves that $A \otimes_C B$ is flat over A.

Now let us assume that M is an A-module such that $M \otimes_A (A \otimes_C B) = O$ and let us again consider M as a C-module writing $M \simeq M \otimes_A A \otimes_C C$. We know that $(M \otimes_A A \otimes_C C) \otimes_C B \simeq M \otimes_A (A \otimes_C B) = O$. But if B is faithfully C-flat then $M \otimes_A A \otimes_C C = O$ so that M is O.

PROOF OF LEMMA 3.1. We take $C = O_{X,x}$, $A = \hat{O}_{S,x}$, $B = C^{\infty}_{X,x}$. Then we know that $F_{X,x}$ is faithfully flat over $O_{X,x}$ because X is smooth. Moreover by Grothendieck-Hartshorne proper morphism theorem, we have

$$\hat{O}_{S,x} \simeq \varphi_*(\hat{O}_{\tilde{S}})_x$$

so that $\hat{O}_{S,x}$ is a $O_{X,x}$-moduel and Lemma 2 gives the result of Lemma 1.

LEMMA 3.3. *$F_{S,x}$ is a faithfully flat $O_{S,x}$-module.*

PROOF. $\hat{O}_{S,x}$ is a faithfully flat $O_{S,x}$-module (completion of a local noetherian ring), so that $F_{S,x}$ is flat over $O_{S,x}$ by Lemma 1 and the transitivity of flatness. To prove that $F_{S,x}$ is faithfully flat over $O_{S,x}$, we have to see that for any ideal $\mathcal{J}$ of $O_{S,x}$

$$\mathcal{J}.F_{S,x} \cap O_{S,x} = \mathcal{J}.$$

Now

$$\mathcal{J}.F_{S,x} \cap \hat{O}_{S,x} = \mathcal{J}.(\hat{O}_{S,x} \otimes_{O_{X,x}} C^{\infty}_{X,x}) \cap \hat{O}_{S,x}.$$

But we have seen in Section 1, that

$$\hat{O}_{S,x} \otimes_{O_{X,x}} C^{\infty}_{X,x} \simeq \hat{O}_{S,x} C^{\infty}_{X,x},$$

so that

$$\mathcal{J}.F_{S,x} \simeq \mathcal{J}\hat{O}_{S,x}C^{\infty}_{X,x} \simeq \mathcal{J}.\hat{O}_{S,x}(\hat{O}_{S,x} \otimes_{O_{X,x}} C^{\infty}_{X,x})$$

and

$$\mathcal{J}.F_{S,x} \cap \hat{O}_{S,x} = \mathcal{J}.\hat{O}_{S,x}F_{S,x} \cap \hat{O}_{S,x}$$

and this is $\mathcal{J}.\hat{O}_{S,x}$ because $F_{S,x}$ is faithfully flat over $\hat{O}_{S,x}$ (Lemma 1). Then

$$\mathcal{J}.F_{S,x} \cap O_{S,x} = \mathcal{J}.\hat{O}_{S,x} \cap O_{S,x} = \mathcal{J}$$

because $\hat{O}_{S,x}$ is faithfully over $O_{S,x}$.

3.3 *Germs of formal functions on S.* Let V be a neighborhood of x in S, and let us define

$$\check{F}_S(V) = \prod_{x \in V} F_{S,x}$$

where $F_{S,x}$ is defined in Section 3.2. Let us also define

$$\check{F}_{S,x} = \lim \check{F}_S(V)$$

for the set of neighborhoods V of x in S.

LEMMA 3.4. *$\check{F}_{S,x}$ is a faithfully flat $O_{S,x}$-module.*

PROOF. Let $\overrightarrow{f}_x = (f_{1,x}, \ldots, f_{n,x})$ with $f_{k,x} \in O_{S,x}$ and $\overrightarrow{\varphi} \in R(\overrightarrow{f}, \check{F}_{S,x})$.

Because the sheaf O_S is coherent by Oka theorem, there exist $\overrightarrow{g}^{(i)}$, $i = 1 \cdots \ell$, generating $R(\overrightarrow{f}_x, O_{S,x})$ and defined on a small neighborhood V of x such that their germs $\overrightarrow{g}_y^{(i)}$ generate $R(\overrightarrow{f}_y, O_{S,y})$ for all y in V. If we restrict V, we can assume that $\overrightarrow{\varphi}$ give relations among the $\overrightarrow{f}$ all over V. In particular for all $y \in V$, the germs $\overrightarrow{\varphi}_y \in R(\overrightarrow{f}_y, F_{S,y}) \subset R(\overrightarrow{f}_y, O_{S,y})F_{S,y}$ because $F_{S,y}$ is flat over $O_{S,y}$, so that

$$\overrightarrow{\varphi}_y = \sum_k \Big(\sum_i \overrightarrow{g}_y^{(i)} h_{i,k,y}\Big)\theta_{k,y}$$

where $h_{i,k,y} \in O_{S,y}$, $\theta_{k,y} \in F_{S,y}$ and then

$$\overrightarrow{\varphi}_y = \sum_i \overrightarrow{g}_y^{(i)}\Big(\sum_k h_{i,k,y}\theta_{k,y}\Big).$$

The set of $(\sum_k h_{i,k,y}\theta_{k,y})_{y\in V}$ define an element of $\check{F}_{S,x}$ and we have proved that

$$R(\overrightarrow{f}, \check{F}_{S,x}) \subset R(\overrightarrow{f}, O_{S,x}) \bullet \check{F}_{S,x}$$

which is the flatness result.

Let us prove now that $\check{F}_{S,x}$ is faithfully flat over $O_{S,x}$. Let M be an $O_{S,x}$-module such that

$$M \otimes_{O_{S,x}} \check{F}_{S,x} = 0.$$

Now, we have a surjective mapping

$$\check{F}_{S,x} \longrightarrow F_{S,x} \longrightarrow 0$$

so that

$$M \otimes_{O_{S,x}} \check{F}_{S,x} \longrightarrow M \otimes_{O_{S,x}} F_{S,x} \longrightarrow 0$$

and so $M \otimes_{O_{S,x}} F_{S,x} = 0$. But $F_{S,x}$ is faithfully flat over $O_{S,x}$, so that $M = 0$.

3.4 *Flatness of* $\Lambda^0_{S,x}$. By the definition of $\Lambda^0_{S,x}$, we see that we have injective mappings

(3-2) $$O_{S,x} \hookrightarrow \Lambda^0_{S,x} \hookrightarrow \check{F}_{S,x}$$

for all $x \in S$. To define the last mapping, we take a neighborhood V of x in S and a $f \in \Gamma(V, \Lambda^0_S)$. If $y \in V - X$ we can define $j_y(f) \in F_{S,y}$ in a natural way because f is $C^\infty_{S,y}$. If $y \in V \cap X$, we define $j_y(f)$ as follows: f is a C^∞ function over $\varphi^{-1}(V)$ such that $j_{\tilde{X}}(f)$ can be identified to an element

$$j_{\tilde{X}}(f) \in \Gamma\big(V \cap X, \varphi_*(\hat{O}_S) \otimes_{O_X} C^\infty_X\big)$$

and we take for $j_y(f)$ the germ at y of $j_{\tilde{X}}(f)$

(3-3) $$j_y(f) = \big(j_{\tilde{X}}(f)\big)_y \in F_{S,y}.$$

Then we associate to $f \in \Gamma(V, \Lambda^0_S)$ the collection $\big(j_y(f)\big)_{y\in V} \in \Gamma(V, \check{F}_S)$ and this is clearly injective.

THEOREM 3.5. *$\Lambda^0_{S,x}$ is a faithfully flat $O_{S,x}$-module.*

PROOF. Because of the result recalled in Section 3.1, and because $\check{F}_{S,x}$ is faithfully flat over $O_{S,x}$, it is sufficient to prove the following lemma.

LEMMA 3.6. *If $\mathcal{J}$ is an ideal of $O_{S,x}$, then*

$$\mathcal{J}\check{F}_{S,x} \cap \Lambda^0_{S,x} = \mathcal{J}.\Lambda^0_{S,x}.$$

PROOF OF LEMMA 3.6. Let us start with a $g \in \mathcal{J}.\check{F}_{S,x} \cap \Lambda^0_{S,x}$ for $x \in X$ (the case where $x \in S - X$ is proved by Malgrange). By definition g is in $\Gamma\big(\varphi^{-1}(U), C^\infty_{\tilde{S}}\big)$ for a small neighborhood U of x in S. Let $\overrightarrow{f} = (f_1, \dots, f_n)$ be generators of $\mathcal{J}$ over U.

1ST STEP: DECOMPOSITION OF g IN $\varphi^{-1}(U)$. At any point $\tilde{y} \in \varphi^{-1}(U)$, the jet of g at $\tilde{y}$, $j_{\tilde{y}}(g)$ is in $\mathcal{J}_{\tilde{y}}\check{F}_{\tilde{X},\tilde{y}}$ where

$$\check{F}_{\tilde{S}}(\tilde{V}) = \prod_{\tilde{y}\in\tilde{V}} F_{\tilde{S},\tilde{y}}$$

$$\mathcal{J}_{\tilde{y}} = \big((f_1 \circ \varphi)_{\tilde{y}}, \dots, (f_n \circ \varphi)_{\tilde{y}}\big).O_{\tilde{S},\tilde{y}}.$$

By the global Malgrange theorem, we see that

$$g = \sum_{i=1}^{n} f_i u_i \tag{3-4}$$

where $u_i \in \Gamma\big(\varphi^{-1}(U), C^\infty_{\tilde{S}}\big)$.

Now, taking the jet along $\tilde{X}$, we have

$$j_{\tilde{X}}(g) = \sum_{i=1}^{n} j_{\tilde{X}}(f_i) j_{\tilde{X}}(u_i) \tag{3-5}$$

and we know also that up to identification

$$j_{\tilde{X}}(g) \in \mathcal{J}.\,\varphi_*(\hat{O}_{\tilde{S}})_x \otimes_{O_{X,x}} C^\infty_{X,x} \simeq \mathcal{J}.\,\varphi_*(\hat{O}_{\tilde{S}})_x.\,C^\infty_{X,x}. \tag{3-6}$$

2ND STEP: SUPPRESSION OF THE ANTIHOLOMORPHIC PART OF THE $j_{\tilde{X}}(u_i)$. We can always write locally in $\tilde{S}$

$$j_{\tilde{X}}(u_i) = j^{(h)}_{\tilde{X}}(u_i) + \bar{\zeta}\hat{v}_i \tag{3-7}$$

where $j^{(h)}_{\tilde{X}}(u_i)$ is the holomorphic part of the jet u_i along $\tilde{X}$ (*i.e.* the part of the Taylor series in ζ) and $\zeta = 0$ is a local equation of $\tilde{X}$ in $\tilde{S}$ and $\hat{v}_i$ is a formal series in $\zeta, \bar{\zeta}$. By taking a partition of unity on $\varphi^{-1}(U)$, we can always find C^∞ functions v_i on $\varphi^{-1}(U)$ such that the complete jet $j_{\tilde{X}}(v_i)$ along $\tilde{X}$ is $j_{\tilde{X}}(v_i) = \hat{v}_i$.

Now let us define

$$w = \sum_{i=1}^{n} f_i v_i.$$

Then $j_{\tilde{X}}(f) = \sum_{i=1}^{n} j_{\tilde{X}}(f_i)\bar{\zeta}\hat{v}_i = 0$ because of (3-5), (3-6) and (3-7), so that w is in the ideal $I^\infty\big(\tilde{X}, \varphi^{-1}(U)\big)$ of the C^∞ functions on $\varphi^{-1}(U)$ which are flat at infinite order in $\tilde{X}$. Moreover w is also in $\mathcal{J}_{\hat{y}}, \tilde{F}_{\tilde{S},\hat{y}}$ for all $\hat{y} \in \varphi^{-1}(U)$, so that by Malgrange theorem

$$w \in \mathcal{J} I^\infty\big(\tilde{X}, \varphi^{-1}(U)\big)$$

or

$$w = \sum_{i=1}^{n} f_i v_i'$$

where $v_i' \in I^\infty\big(\tilde{X}, \varphi^{-1}(U)\big)$. Then

$$g = \sum_{i=1}^{n} f_i(u_i - v_i + v_i')$$

with

$$j_{\tilde{X}}(u_i - v_i + v_i') = j_{\tilde{X}}(u_i - v_i) + j_{\tilde{X}} v_i' = j_{\tilde{X}}^{(h)}(u_i).$$

This means that we can write g as a linear combination of type (3-4)

$$g = \sum_{i=1}^{n} f_i u_i \tag{3-8}$$

with

$$j_{\tilde{X}}(u_i) \equiv j_{\tilde{X}}^{(h)}(u_i) \in \Gamma\big(\tilde{X} \cap \varphi^{-1}(U), \hat{O}_{\tilde{S}} \otimes_{O_{\tilde{X}}} C^\infty_{\tilde{X}}\big). \tag{3-9}$$

3RD STEP: CORRECTION OF THE u_i SO THAT THEY BELONG TO $\Lambda^0_{S,x}$. We want to correct further the u_i in equation (3-8), so that they belong to $\Lambda^0_{S,x}$.

By (3-6), we know that $j_{\tilde{X}}(g) \in \mathcal{J}\varphi_*(\hat{O}_{\tilde{S}})_x C^\infty_{X,x}$, so that

$$j_{\tilde{X}}(g) = \sum j_{\tilde{X}}(f_i).\,\hat{w}_i$$

where $\hat{w}_i \in \varphi_*(\hat{O}_{\tilde{S}})_x C^\infty_{X,x}$ (we identify $\hat{w}_i$ with its lift through φ to $\tilde{S}$).

Let us now find $w_i \in C^\infty\big(\varphi^{-1}(U)\big)$ with $j_{\tilde{X}}(w_i) = \hat{w}_i$. Then

$$j_{\tilde{X}}\big(\sum f_i w_i - \sum f_i u_i\big) = 0$$

and $\sum f_i(w_i - u_i)$ is in the ideal $I^\infty\big(\tilde{X}, \varphi^{-1}(U)\big)$. It is also in $\mathcal{J}\check{F}_{\tilde{S},\tilde{x}}$ for all $\tilde{x} \in \varphi^{-1}(U)$, so that by Malgrange's theorem, it is in $\mathcal{J}.I^\infty(\tilde{X}, \varphi^{-1}(U)\big)$

$$\sum_i f_i(w_i - u_i) = \sum_i f_i p_i$$

where $p_i \in I^\infty\big(\tilde{X}, \varphi^{-1}(U)\big)$. Then

$$g = \sum f_i(w_i - p_i) \tag{3-10}$$

and $j_{\tilde{X}}(w_i - p_i) = j_{\tilde{X}}(w_i) = \hat{w}_i \in \varphi * (\hat{O}_{\tilde{S}})_x C^\infty_{X,x}$ which proves Lemma 3.6.

4. **Flatness of the germs of C^∞ forms on S.**

4.1 *C^∞ forms in S vanishing at infinite order on X.*

a) Definition: Let us denote by $I_{S,x}^{(0,p)}$ (for $x \in X$) the subspace of $\Lambda_{S,x}^{(0,p)}$ formed by the $C^\infty(0,p)$-forms π on $\varphi^{-1}(U)$ (for some open neighborhood U of x in S) such that $j_{\tilde{X}}^{(\infty)}(\pi) = 0$ on a smaller $\varphi^{-1}(U')$ $x \in U' \subset U$. This space is exactly the space of $C^\infty(0,p)$-forms on some $\varphi^{-1}(U)$ such that their coefficients (in any coordinate system) vanish to order ∞ on $\tilde{X}$ on a neighborhood of $\varphi^{-1}(x)$ in $\tilde{S}$. It is obviously an $O_{S,x}$-module.

b) Short exact sequence associated to $j_{\tilde{X}}^{(\infty)}$.

LEMMA 4.1. *For any $x \in X$, we have a short exact sequence of $O_{S,x}$-modules*

$$0 \longrightarrow I_{S,x}^{(0,p)} \longrightarrow \Lambda_{S,x}^{(0,p)} \xrightarrow{j_{\tilde{X}}^{(\infty)}} \varphi_*(\hat{O}_{\tilde{S}})_x \otimes_{O_{X,x}} \Lambda_{X,x}^{0,p} \longrightarrow 0. \tag{4-1}$$

(Here $\varphi_*(\hat{O}_{\tilde{S}})_x \otimes_{O_{X,x}} \Lambda_{X,x}^{0,p}$ is considered as an $O_{S,x}$-module because $\varphi_*(\hat{O}_{\tilde{S}})_x = \hat{O}_{S,x}$).

PROOF. We have, as usual identified $\varphi_*(\hat{O}_{\tilde{S}})_x \otimes_{O_X} \Lambda_X^{0,p}$ to jets of forms along $\tilde{X}$. It is clear that $j_{\tilde{X}}^{(\infty)}$ is surjective. It is then clear that the kernel of $\Lambda_{S,x}^{0,p}$ is $I_{S,x}^{(0,p)}$.

c) Flatness of $I_{S,x}^{(0,0)}$.

LEMMA 4.2. *$I_{S,x}^{(0,0)}$ is a flat $O_{S,x}$-module.*

PROOF. We know that $\Lambda_{S,x}^{0,p}$ is a flat $O_{S,x}$-module (Theorem 3.5 of Section 3.4). We also know that $F_{S,x}$ is a flat $O_{S,x}$-module (Lemma 3.3 of Section 3.2). From the general results of Section 3.1 and the short exact sequence (4-1), we know that $I_{S,x}^{(0,0)}$ is a flat $O_{S,x}$-module.

4.2 *Flatness of formal forms on S.*

LEMMA 4.3. *The $O_{S,x}$-module of formal $(0,p)$-forms on S at x*

$$F_{S,x}^{(0,p)} \equiv \varphi_*(\hat{O}_{\tilde{S}})_x \otimes_{O_{X,x}} \Lambda_{X,x}^{0,p}$$

is faithfully flat over $O_{S,x}$.

PROOF. $\Lambda_{X,x}^{0,p}$ is a direct sum of some copies of $\Lambda_{X,x}^{0,p}$ because X is smooth, so $F_{S,x}^{(0,p)}$ is a direct sum of some copies of $F_{S,x}$ which is faithfully flat over $O_{S,x}$ (Lemma 3.3 of Section 3.2).

4.3 *Flatness of $\Lambda_{S,x}^{0,p}$.*

THEOREM 4.4. *$\Lambda_{S,x}^{0,p}$ is a flat $O_{S,x}$-module for all $p \geq 0$.*

PROOF OF THE THEOREM. We already know the result for $p = 0$. Because of the short exact sequence (4-1) of Lemma 4.1, Lemma 4.3 and the general results of Section 3.1, it is sufficient to prove the following lemma:

LEMMA 4.5. *$I^{0,p}_{S,x}$ is a flat $O_{S,x}$-module.*

PROOF OF LEMMA 4.5. Let $\overrightarrow{f} = (f_1, \ldots, f_N)$ in $O_{S,x}$ so that the f_j are holomorphic functions on some neighborhood $\varphi^{-1}(U)$ in $\tilde{S}$ and let

$$\overrightarrow{\pi} \in R(\overrightarrow{f}, I^{(0,p)}_{S,x}). \tag{4-2}$$

Let us cover $\varphi^{-1}(U')$ (for some $U' \subset U$, $x \in U'$, so that $\overrightarrow{\pi}$ vanishes at order ∞ on $\tilde{X} \cap \varphi^{-1}(U')$) by coordinates charts $(V_i)_{i\in I}$ and call $z^{(i)}$ a coordinate system in V_i. Let us also take a partition of unity $(p_i)_{i\in I}$ of $\varphi^{-1}(U')$ subordinated to $(V_i)_{i\in I}$. Then

$$\overrightarrow{\pi} = \sum_{i\in I} p_i \overrightarrow{\pi} \tag{4-3}$$

and it is clear tht $p_i \overrightarrow{\pi}$ is an N-tuple of C^∞-forms of type $(0,p)$ in $\varphi^{-1}(U')$, with compact support in V_i vanishing at order ∞ on $\tilde{X} \cap \varphi^{-1}(U')$ so that they are in $I^{(0,p)}_{S,x}$. Moreover, we have

$$p_i \overrightarrow{\pi} \in R(\overrightarrow{f}, I^{(0,p)}_{S,x}). \tag{4-4}$$

We shall prove that

$$p_i \overrightarrow{\pi} \in R(\overrightarrow{f}, O_{S,x}).\, I^{0,p}_{S,x}. \tag{4-5}$$

From (4-3), we will then have

$$\overrightarrow{\pi} \in R(\overrightarrow{f}, O_{S,x}).\, I^{(0,p)}_{S,x}$$

which is our result.

Now, we are reduced to the case where $\overrightarrow{\pi}$ has a compact support in a coordinate chart V_i with coordinates z, so that

$$\overrightarrow{\pi} = \sum_{|K|=p} \overrightarrow{\pi}_K \, d\bar{z}^K \tag{4-6}$$

and $\overrightarrow{\pi}_K$ are global C^∞ functions on $\varphi^{-1}(U')$ vanishing to order ∞ on $\tilde{X} \cap \varphi^{-1}(U')$ and

$$\overrightarrow{\pi}_K \in R(\overrightarrow{f}, I^{(0,0)}_{S,x})$$

because the $d\bar{z}^K$ are linearly independent. But $I^{(0,0)}_{S,x}$ is $O_{S,x}$-flat by Lemma 4.2. Call $(\overrightarrow{\theta}_j)_{j=1,\ldots,r}$ a basis of relations $R(\overrightarrow{f}, O_{S,x})$. Then

$$\overrightarrow{\pi}_K = \sum_{j=1}^{r} \overrightarrow{\theta}_j \bar{\omega}^j_K \tag{4-7}$$

where $\bar{\omega}^j_K$ are in $I^{(0,0)}_{S,x}$.

Now let χ be a C^∞ function which is 1 on the support of $\overrightarrow{\pi}$ and with compact support in the coordinates chart V_i where $\overrightarrow{\pi}$ has its support. Then

$$\overrightarrow{\pi}_K = \chi \overrightarrow{\pi}_K$$

and so

$$\overrightarrow{\pi}_K = \sum_{j=1}^{r} \overrightarrow{\theta}_j(\chi\bar{\omega}^j_K).$$

Now, $\chi\bar{\omega}^j_K$ are C^∞ functions with support in V_i, vanishing at infinite order on $\tilde{X}$, so that

$$\bar{\omega}^j = \sum_{|K|=p} (\chi\bar{\omega}^j_K)\, d\bar{z}^K$$

is a globally defined $C^\infty(0,p)$ form on $\varphi^{-1}(U'')$ (for some $(U'' \subset U)$ with compact support in V_i and so is in $I^{(0,p)}_{S,x}$, and

$$\overrightarrow{\pi} = \sum_{j=1}^{r} \overrightarrow{\theta}_j\bar{\omega}^j \in R(\overrightarrow{f}, O_{S,x})I^{(0,p)}_{S,x}.$$

References

1. V. Ancona and B. Gaveau, *Differential forms and resolutions on certain analytic spaces: I Irreducible exceptional divisor*, Bull. Sci. Math., to appear.
2. A. Grothendieck, *Eléments de géométrie algébrique*, EGA **III**.
3. R. Hartshorne, *On the de Rham cohomology of algebraic varieties*, Publ. Math. IHES.
4. H. E. Rossi and H. Hironaka, *On the equivalence of imbeddings of exceptional complex spaces*, Math. Annalen, **156**(1964), 313–333.
5. B. Malgrange, *Ideals of differentiable functions*, Oxford University Press, 1966.

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