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SUPPLEMENTATION IN GROUPS

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Dedicated to D.J.S. Robinson on the occasion of his 60th birthday

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Abstract. In this paper, groups are investigated in which all subgroups, all normal subgroups, or all characteristic subgroups have a proper supplement. This supplement can be either an arbitrary subgroup, a normal or a characteristic subgroup, resulting in nine classes of groups. Properties of these classes are studied such as containment and closure properties, and characterizations for several of these classes are given.

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1. Introduction. A subgroup H of a group G is *supplemented* in G if there is a subgroup K of G such that $G = HK$. If $H \cap K = \{1\}$, then H is *complemented* in G by K. While groups which satisfy certain complementation properties have been extensively studied, little has been done to investigate groups which satisfy certain supplementation properties.

The topic of this paper is a comprehensive investigation of supplementation in general, as well as in the case of finite groups. To make our notions more precise, we make the following definition, using notation due to Christensen [**5**].

Definition 1.1. A group G is an *xP-group* if every nontrivial x-subgroup satisfies condition P, where x and P can have the following values:

$x = a$ (arbitrary subgroup);
$= n$ (normal subgroup);
$= c$ (characteristic subgroup);

$P = D$ (is a direct factor);
$= C$ (has a complement);
$= S$ (has a proper supplement);
$= PNS$ (has a proper normal supplement);
$= CS$ (has a proper characteristic supplement).

The class of groups which have a certain property will be denoted by xP. There are extensive studies of xD- and xC-groups. In [**12**], Kertész classified aD-groups, groups in which every subgroup is a direct factor. Wiegold [**17**] studied nD-groups, groups in which every normal subgroup is a direct factor. In his paper, Wiegold does classify finite nD-groups. Christensen [**5**] showed that the class of finite nD-groups equals the class of finite cD-groups. The work done by Wiegold [**17**] was extended by Head [**9**], where he characterized infinite $nPNS$-groups. Finite aC-groups, groups in which every subgroup is complemented, have been classified by P. Hall [**8**]. Baeva [**1**], Chernikova [**4**], and Sheriev [**16**] extended these results to infinite aC-groups. The class of finite nC-groups, groups in which every normal subgroup is complemented, have been investigated in numerous papers (Bechtell [**2**], Christensen [**5**] and [**6**], and Wright [**18**] to name a few).

The goal of this paper is to investigate supplementation in groups by studying the properties of xS-, $xPNS$-, and xCS-groups. The notation used is standard (e.g. see Robinson [**14**]). When property P is mentioned, P will always refer to S, PNS, or CS unless otherwise indicated. If a nontrivial subgroup H of a group G satisfies property P, we will say that H has a *P-supplement*. If H is characteristic in G, this will be denoted by H char G. For any subset S of G, the subgroup generated by S will be denoted by $\langle S\rangle$. The normal closure of S in G will be denoted by S^G, and the characteristic closure of S in G by $S^{A(G)}$. The unrestricted direct product will be denoted by Π and the restricted direct product or direct sum by Σ. Finally, π will denote the set of all primes.

First, the containment relation between the nine different classes of xP-groups is given. These results are straightforward and given without proof.

Proposition 1.2. *$aP \subseteq nP \subseteq cP$ for $P = S$, PNS, and CS.*

Proposition 1.3. *$xCS \subseteq xPNS \subseteq xS$ for $x = a, n$, and c.*

Propositions 1.2 and 1.3 result in the diagram below that presents the containment relations between the nine classes of groups to be studied.

$$\begin{array}{ccccc} aS & \subseteq & nS & \subseteq & cS \\ \cup\!| & & \cup\!| & & \cup\!| \\ aPNS & \subseteq & nPNS & \subseteq & cPNS \\ \cup\!| & & \cup\!| & & \cup\!| \\ aCS & \subseteq & nCS & \subseteq & cCS \end{array}$$

Diagram 1.

The question arises whether each of these containments is proper or not. As it turns out, the containments are all proper in the general case, but not when we restrict the classes to only consist of finite groups. This restriction to finite groups has important ramifications. The proofs of the following theorems are given in Section 6.

Theorem 1.4. *In the general case, all the class containments in Diagram 1 are proper.*

$$\begin{array}{ccccc} aS & \subset & nS & \subset & cS \\ \cup & & \cup & & \cup \\ aPNS & \subset & nPNS & \subset & cPNS \\ \cup & & \cup & & \cup \\ aCS & \subset & nCS & \subset & cCS \end{array}$$

Diagram 2.

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THEOREM 1.5. *When only finite groups are considered, the following class containments result.*

$$\begin{array}{ccccc} aS & \subset & nS & = & cS \\ \cup & & \cup & & \cup \\ aPNS & \subset & nPNS & = & cPNS \\ \cup & & \cup & & \| \\ aCS & \subset & nCS & \subset & cCS \end{array}$$

Diagram 3.

When studying groups which satisfy specific supplementation properties, maximal subgroups become an integral part of any investigation. Given the three different collections of subgroups and supplements (arbitrary, normal, and characteristic), the maximal, maximal normal, and maximal characteristic subgroups of a group G are important. Definitions and results concerning these classes of subgroups, along with some other preliminaries, are established in Section 2. These results are then used to characterize xS-, $xPNS$-, and xCS-groups in Sections 3, 4 and 5 respectively, where $x = a$, n, and c. In certain instances, finiteness conditions are imposed to obtain special results. The classifications and characterizations established in these sections are then used to prove Theorems 1.4 and 1.5 in Section 6 and to obtain closure (subgroup and homomorphic image) properties for the various classes in Section 7.

2. Preliminary results. In this section we give three lemmas which reduce showing that a group G is in aP, nP, or cP to establishing that every cyclic subgroup, every normal closure or every characteristic closure of an element, respectively, has a P-supplement. In addition, we give definitions and some results concerning the Frattini subgroup and some of its analogues which play an important role in the characterization of xP-groups. These results can be found in [**11**].

LEMMA 2.1. *A nontrivial group G is an aP-group ($P = S$, PNS, CS) if and only if, for every nontrivial $x \in G$, $\langle x \rangle$ has a P-supplement in G.*

Proof. The necessity of the condition follows from Definition 1.1. Conversely, let H be a nontrivial subgroup of G and h nontrivial in H. Since $\langle h \rangle$ is a nontrivial subgroup of G, there is a P-supplement K in G such that $G = \langle h \rangle K$. But $\langle h \rangle \subseteq H$ implies $G = HK$. □

LEMMA 2.2. *A nontrivial group G is an nP-group ($P = S$, PNS, CS) if and only if, for every nontrivial $x \in G$, the normal closure x^G has a P-supplement in G.*

Proof. The proof here is the same as the proof of Lemma 2.1, except that $\langle x \rangle$ is replaced by x^G. □

LEMMA 2.3. *A nontrivial group G is a cP-group ($P = S$, PNS, CS) if and only if, for every nontrivial $x \in G$, the characteristic closure $x^{A(G)}$ has a P-supplement in G.*

Proof. The proof here is the same as the proof of Lemma 2.1, except that $\langle x \rangle$ is replaced by $x^{A(G)}$. □

Now we turn to the Frattini subgroup and some of its analogues needed in the context of supplementation.

Definition 2.4. (1.1 in [**11**].) Let G be a group; then

(i) $\mathcal{M} = \{M \leq G;\ M \neq G,\ M \leq L \leq G \Rightarrow M = L \text{ or } L = G\}$;
(ii) $\mathcal{N} = \{N \trianglelefteq G;\ N \neq G,\ N \trianglelefteq L \trianglelefteq G \Rightarrow N = L \text{ or } L = G\}$;
(iii) $\mathcal{K} = \{K \text{ char } G;\ K \neq G,\ K \text{ char } L \text{ char } G \Rightarrow K = L \text{ or } L = G\}$.

Definition 2.5. (1.3 in [**11**].) For a group G we define the following subgroups:

(i) $\mathrm{Frat}(G) = \bigcap_{M \in \mathcal{M}} M$ if $\mathcal{M} \neq \emptyset$, and $\mathrm{Frat}(G) = G$ if $\mathcal{M} = \emptyset$;
(ii) $n\mathrm{Frat}(G) = \bigcap_{N \in \mathcal{N}} N$ if $\mathcal{N} \neq \emptyset$, and $n\mathrm{Frat}(G) = G$ if $\mathcal{N} = \emptyset$;
(iii) $c\mathrm{Frat}(G) = \bigcap_{K \in \mathcal{K}} K$ if $\mathcal{K} \neq \emptyset$, and $c\mathrm{Frat}(G) = G$ if $\mathcal{K} = \emptyset$.

Note that $\mathrm{Frat}(G)$, $n\mathrm{Frat}(G)$, and $c\mathrm{Frat}(G)$ are all characteristic in G.

Definition 2.6. (3.3 in [**11**].) A normal subgroup H of G is *finitely n-generated over G* if there are elements $x_1, \ldots, x_n$ in G such that $H = \langle x_1, \ldots, x_n \rangle^G$.

Lemma 2.7. (3.4 in [**11**].) *If* $n\mathrm{Frat}(G)$ *is finitely n-generated over a nontrivial group G, then* $n\mathrm{Frat}(G)$ *is a proper subgroup of G.*

More information on the properties of $n\mathrm{Frat}(G)$ and $c\mathrm{Frat}(G)$, and the relationship between $\mathrm{Frat}(G)$, $n\mathrm{Frat}(G)$, and $c\mathrm{Frat}(G)$ can be found in [**11**].

3. Characterizations of xS-groups. In this section, the structure of xS-groups is investigated. As we shall see, the Frattini subgroup plays an important role here. In the case of finite groups, the class of aS-groups is identical to the class of aC-groups, and the class of nS-groups is identical to the class of cS-groups.

Proposition 3.1. *If* $\mathrm{Frat}(G) = \{1\}$*, then G is an nS-group, and there exists a group T in nS such that* $\mathrm{Frat}(T) \neq \{1\}$.

Proof. Since $\mathrm{Frat}(G) = \{1\}$, G admits maximal subgroups. Suppose that there is a nontrivial normal subgroup N of G that has no proper supplement. Then, for every maximal subgroup $M \in \mathcal{M}$, $NM \neq G$. Since $NM \neq G$ and M is maximal in G, $NM = M$. Thus $N \subseteq M$, for all $M \in \mathcal{M}$. Consequently, $N \subseteq \mathrm{Frat}(G)$, a contradiction. Thus G is an nS-group.

According to [**13**], there exists an infinite simple group T with $\mathrm{Frat}(T) = T$. Since T is simple, it is trivially an nS-group. However, $\mathrm{Frat}(T) = T \neq \{1\}$. □

Proposition 3.2. *Let G be a group with* $\mathrm{Frat}(G)$ *finitely generated. Then G is a cS-group if and only if* $\mathrm{Frat}(G) = \{1\}$*. Furthermore, there exists a cS-group F with* $\mathrm{Frat}(F)$ *not finitely generated.*

Proof. Suppose that G is a cS-group with $\mathrm{Frat}(G) \neq \{1\}$. Since $\mathrm{Frat}(G)$ char G, there exists a proper subgroup K of G such that $G = \mathrm{Frat}(G)K$. Given that $\mathrm{Frat}(G)$ is finitely generated, $G = K$ (7.3.8 of [**15**]). This contradiction implies that $\mathrm{Frat}(G) = \{1\}$.

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Conversely, let $\mathrm{Frat}(G) = \{1\}$. By Proposition 3.1, we have $G \in nS$. It follows by Proposition 1.2 that $G \in cS$.

To show that the Frattini subgroup being finitely generated is a necessary assumption, consider $F = \mathbb{Q} \times A_5$, the direct product of the rationals under addition and the alternating group on 5 letters. We have $F \in cS$, but $F \notin nS$, and $\mathrm{Frat}(F) \cong \mathbb{Q}$, which is not finitely generated. □

The following corollary is now an immediate consequence of the above propositions.

Corollary 3.3. *In the class of groups in which* Frat(G) *is finitely generated, the collection of nS-groups is identical to the collection of cS-groups.*

Proposition 3.4. *If G is an aS-group, then* $\mathrm{Frat}(G) = \{1\}$*, and there exists a finite group G with* $\mathrm{Frat}(G) = \{1\}$*, but* $G \notin aS$.

Proof. Suppose that $\mathrm{Frat}(G) \neq \{1\}$. Let $x \in \mathrm{Frat}(G)$, $x \neq 1$. Then there is a proper subgroup H of G such that $G = \langle x \rangle H$. Consequently, $G = \langle x, H \rangle = \langle H \rangle$, a contradiction. Thus $\mathrm{Frat}(G) = \{1\}$.

Consider $A_4 = \langle a, b, c \mid a^2 = b^2 = c^3 = 1, ab = ba, ac = cb \rangle$ and $\langle a \rangle \leq A_4$. Now $\mathrm{Frat}(A_4) = \{1\}$, but $\langle a \rangle$ has no proper supplement as the only proper subgroups of A_4 that do not contain $\langle a \rangle$ are of order 2 or 3. □

The closure properties of xP-groups will be studied in detail in Section 7, but the following result is presented here to help prove Theorem 3.6.

Proposition 3.5. *Every subgroup of an aS-group is an aS-group.*

Proof. Let G be an aS-group and $H \leq G$. If $H = \{1\}$ or G, then the result follows. Suppose that H is nontrivial and proper in G.

Let K be a nontrivial subgroup of H. Since K is a nontrivial subgroup of G, there exists a proper subgroup L of G such that $G = KL$. By the modular identity, $H = G \cap H = KL \cap H = K(L \cap H)$. If $L \cap H = H$, then $H \leq L$. This would then imply that $K \leq L$ and $G = L$, a contradiction. Thus $L \cap H$ is a proper subgroup of H, and H is an aS-group. □

Theorem 3.6. *If G is an aS-group which satisfies the descending chain condition on subgroups, then G is an aC-group.*

Proof. Let $H \leq G$. Let K be minimal among subgroups which supplement H in G. Let $H_1 = H \cap K$, and suppose that $H_1 \neq \{1\}$. By Proposition 3.5, H_1 has a proper supplement K_1 in K. Thus $G = HK = H(H_1K_1) = (HH_1)K_1 = HK_1$. This contradicts the minimality of K. Consequently, $H_1 = \{1\}$ and H is complemented in G. □

The following corollary is now obvious.

Corollary 3.7. *The class of finite aS-groups is identical to the class of finite aC-groups.*

Given Corollary 3.7 and the work done by P. Hall [**8**] on finite aC-groups, a finite aS-group has all of its Sylow subgroups elementary abelian, all of its chief factors cyclic, and is isomorphic to a subgroup of the direct product of a certain number of groups of square free order. A result analogous to Corollary 3.7 cannot be obtained in the general case. The infinite dihedral group is an aS-group, but is not an aC-group as all aC-groups are torsion groups. Infinite aC-groups have been studied by Baeva [**1**] and Chernikova [**4**].

4. Characterizations of $xPNS$-groups. In this section, the structure of $xPNS$-groups is investigated. Here, the role played by the Frattini subgroup in the context with xS-groups is taken over by the n-Frattini subgroup. We shall see that finite $nPNS$-groups coincide with finite nD-groups as well as finite $cPNS$-groups. Finally, we characterize $aPNS$-groups. Our first theorem characterizes $nPNS$-groups, extending a result by Head [**9**].

THEOREM 4.1. *The following are equivalent for a group G.*

(a) *G is an $nPNS$-group.*
(b) *G is the subdirect product of simple groups.*
(c) $n\mathrm{Frat}(G) = \{1\}$.

Proof. The equivalence of (a) and (b) follows from Theorem 2 in [**9**].

To show that (b) implies (c), let G be the subdirect product of simple groups. Then G admits maximal normal subgroups. Thus $n\mathrm{Frat}(G) \neq G$. Suppose that $n\mathrm{Frat}(G) \neq \{1\}$. Since G is an $nPNS$-group, $G = n\mathrm{Frat}(G)N$, for some proper normal subgroup N of G. Then $N \leq M$, where M is a maximal normal subgroup in G and $G = n\mathrm{Frat}(G)M$. This contradiction implies that $n\mathrm{Frat}(G) = \{1\}$.

To show that (c) implies (a), let $n\mathrm{Frat}(G) = \{1\}$. Then G admits maximal normal subgroups. Suppose there is a nontrivial normal subgroup N of G that has no proper normal supplement. Then $G \neq NM$ for all $M \in \mathcal{N}$. Since $NM \lneq G$, $NM = M$. This implies that $N \leq M$ for all $M \in \mathcal{N}$. Thus $N \leq n\mathrm{Frat}(G)$, a contradiction. Thus G is an $nPNS$-group. □

Corollary 3.3 indicates that for the class of groups whose Frattini subgroups are finitely generated, the collection of nS-groups is identical to the collection of cS-groups. Given that $nPNS \subseteq nS$ and $cPNS \subseteq cS$, it is natural to try to extend this result to the classes of $nPNS$- and $cPNS$-groups. This can be done when $\mathrm{Frat}(G)$ is replaced by $n\mathrm{Frat}(G)$ and finitely generated is replaced by finitely n-generated. (See Definition 2.6.)

THEOREM 4.2. *In the class of groups in which* $n\mathrm{Frat}(G)$ *is finitely n-generated, the collection of $nPNS$-groups is identical to the collection of $cPNS$-groups. Furthermore, there exists a group F in $cPNS$ with a not finitely generated n-Frattini subgroup and F is not in $nPNS$.*

Proof. Since every $nPNS$-group is a $cPNS$-group, all we need to show is that $cPNS \subseteq nPNS$. Let G be a $cPNS$-group. Since $n\mathrm{Frat}(G)$ is finitely generated, $n\mathrm{Frat}(G) \neq G$, by Lemma 2.7. Let $n\mathrm{Frat}(G) = \langle x_1, \ldots, x_n \rangle^G$.

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If G is simple or trivial, the result follows. Suppose G is nontrivial, not simple, and not an $nPNS$-group. Then some nontrivial normal subgroup N of G has no proper normal supplement.

Since $n\mathrm{Frat}(G) \neq G$, by Lemma 2.7, G admits maximal normal subgroups. Thus, for each $M \in \mathcal{N}$, $G \neq NM$. Since $NM \trianglelefteq G$, $NM = M$ and $N \subseteq M$, for all $M \in \mathcal{N}$. Thus $N \subseteq n\mathrm{Frat}(G)$ and $n\mathrm{Frat}(G)$ is nontrivial in G. Since $n\mathrm{Frat}(G)$ char G, there exists a proper normal subgroup L of G such that $G = n\mathrm{Frat}(G)L$, so $G = \langle x_1, \ldots, x_n\rangle^G L = \langle x_1, \ldots, x_n, L\rangle^G$. By Theorem 2.6 of [**11**], $G = \langle x_2, \ldots, x_n, L\rangle^G$. Continuing in this manner, $G = \langle L\rangle^G = L$, a contradiction. Thus G is an $nPNS$-group.

Finally, consider the group $F = \mathbb{Q} \times A_5$, which clearly is a $cPNS$-group and has $n\mathrm{Frat}(F) \cong \mathbb{Q}$. Thus F is not in $nPNS$ and $n\mathrm{Frat}(F)$ is not finitely generated. □

The classification of finite $nPNS$-groups first appeared in [**3**]. The following corollary is now an immediate consequence of Theorems 4.1 and 4.2.

Corollary 4.3. *The classes of finite nPNS-groups, nD-groups, cD-groups, and cPNS-groups are identical and any group in this class is the direct product of simple groups.*

Proof. Clearly, a finite nD-group is a finite $nPNS$-group. Let G be a finite $nPNS$-group. By Theorem 4.1, G is the direct product of simple groups. Thus, by Theorem 4.4 of [**17**], G is an nD-group. The class of finite nD-groups is identical to the class of finite cD-groups by Theorem 3.1 of [**5**]. The fact that the class of $cPNS$-groups is identical to the class of $nPNS$-groups follows from Theorem 4.2. The second part of our claim is an immediate consequence of Theorem 4.1. □

The last result in this section presents a classification of $aPNS$-groups.

Theorem 4.4. *The following are equivalent for a nontrivial group G:*

(a) *G is an aPNS-group;*
(b) *G is the subdirect product of a family of cyclic groups of prime order;*
(c) *G is abelian with $\bigcap_{p\in\pi} G^p = \{1\}$, where π is the set of all primes.*

Proof. To show that (a) implies (b), let G be an $aPNS$-group, and let $x \in G$, $x \neq 1$. If $\langle x\rangle = G$, then G is isomorphic to an infinite cyclic group, which is the subdirect product of $\prod_{p\in\pi} C_p$, or G is isomorphic to a finite cyclic group of square free order, by Corollary 4.3.

Suppose $\langle x\rangle \neq G$. Then there exists a proper normal subgroup N of G such that $G = \langle x\rangle N$. Let M be a normal subgroup of G maximal with respect to $x \notin M$ and $N \subseteq M$. Such an M exists by Zorn's Lemma. We claim that G/M is cyclic of prime order.

Since $G = \langle x\rangle M$, $G/M \cong \langle x\rangle M/M \cong \langle x\rangle/\langle x\rangle \cap M$. Thus G/M is cyclic. If G/M is not simple, then there is a nontrivial proper subgroup $K/M \triangleleft G/M$. Consequently, K is a proper normal subgroup of G with $M \subseteq K$. If $x \in K$, then $\langle x\rangle M = G \subseteq K$, a contradiction. Thus $x \notin K$, which contradicts the maximality of M. This implies that G/M is simple. Since G/M is cyclic, G/M has prime order.

Hence, for each nontrivial $x \in G$, there is a maximal normal subgroup M_x of G such that $x \notin M_x$ and G/M_x is cyclic of prime order. Thus G is the subdirect product of a family of groups which are simple of prime order.

To show (b) implies (c), let G be a group satisfying the condition stated in (b). Then G is abelian. Furthermore, by Theorem 4.1, G is an *nPNS*-group with $n\mathrm{Frat}(G) = \{1\}$. But since G is abelian, $n\mathrm{Frat}(G) = \mathrm{Frat}(G) = \bigcap_{p\in\pi} G^p$.

To show that (c) implies (a), let G be an abelian group with $\bigcap_{p\in\pi} G^p = \{1\}$. Let $g \in G$, $g \neq 1$. Choose a prime p such that $g \notin G^p$. Since G/G^p is an elementary abelian p-group, there is a subgroup M, containing G^p, which complements $\langle g\rangle$ modulo G^p. Thus $G = \langle g\rangle M$ and all nontrivial cyclic subgroups have a proper normal supplement. By Lemma 2.1, G is an *aPNS*-group. □

5. Characterizations of *xCS*-groups. In this section, the structure of *xCS*-groups is investigated. The role played by the Frattini and the n-Frattini subgroup in the preceding two sections is taken over by the c-Frattini subgroup. Because of the weaker closure properties of characteristic subgroups, results as strong as in the preceding section cannot be expected. However under certain finiteness conditions, a picture for *xCS*-groups emerges which is similar to the one for *xPNS*-groups. Since *aCS*-groups are abelian, by Theorem 4.4 and Proposition 1.3, additive notation will be used in Theorem 5.1.

THEOREM 5.1. *A nontrivial torsion group G is an aCS-group if and only if it is the direct sum of cyclic groups of prime order for distinct primes p.*

Proof. Suppose that G is an *aCS*-group. Then G is an *aPNS*-group, by Proposition 1.3. Thus, by Theorem 4.4, each Sylow p-subgroup of G is elementary abelian.

Let G_p be a Sylow p-subgroup of G and suppose that $|G_p| > p$. Let $a_1 \in G_p$ with $|a_1| = p$. Since G is an *aCS*-group, there is a proper subgroup H char G such that $G = \langle a_1\rangle + H$. Since $|a_1| = p$, $G = \langle a_1\rangle \oplus H$. Given that $|G_p| > p$, there is an $a_2 \in H$ such that $|a_2| = p$. Again, since G is an *aCS*-group, $G = \langle a_2\rangle \oplus K$, for a proper subgroup K char G. Thus, by the modular identity, $H = G \cap H = (\langle a_2\rangle \oplus K) \cap H = \langle a_2\rangle + (K \cap H)$. Since $a_2 \notin K \cap H$, we have $H = \langle a_2\rangle \oplus (H \cap K)$. Thus $G = \langle a_1\rangle \oplus \langle a_2\rangle \oplus (H \cap K)$, where $H \cap K$ char G.

Define a map $\phi : G \to G$ by $\phi(a_1) = a_2$, $\phi(a_2) = a_1$, and $\phi(l) = l$, for all $l \in H \cap K$. This is an automorphism of G, yet $\phi(H) \neq H$, a contradiction. Thus, for each Sylow p-subgroup G_p, $|G_p| = p$. Consequently, G is the direct sum of cyclic groups for distinct primes p.

Conversely, suppose that $G = \Sigma_{p\in\pi'} C_p$ is the direct sum of cyclic subgroups C_p for distinct primes p, where π' is a collection of primes. Let H be a nontrivial subgroup of G. Then, for some prime $q \in \pi'$, the Sylow q-subgroup G_q of G is contained in H. Since $G = \Sigma_{p\in\pi'} C_p$, there is a subgroup K in G which has index q in G. Furthermore, K char G. Given that $G_q \subseteq H$ and $G = G_q + K$, we have $G = H + K$, and H has a proper characteristic supplement. □

THEOREM 5.2. *A finite group G is an nCS-group if and only if it is the direct product of distinct simple groups.*

Proof. Let G be an *nCS*-group. Since G is a finite *nPNS*-group, Corollary 4.3 implies that $G = S_1 \times \cdots \times S_n$, where each S_i, $1 \leq i \leq n$, is simple.

Suppose, without loss of generality, that $S_1 \cong S_2$. Since S_1 is a nontrivial normal subgroup of G, there exists a proper characteristic subgroup K of G, such that

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$G = S_1K$. Now $S_1 \cap K \trianglelefteq S_1$ implies that $S_1 \cap K = \{1\}$. Thus $K \cong G/S_1 \cong S_2 \times \cdots \times S_n$. Consequently, $K = T_2 \times \cdots \times T_n$, where $T_j \cong S_j$ for $2 \leq j \leq n$.

Since $G = S_1 \times K$, we have $G = S_1 \times T_2 \times \cdots \times T_n$, where $S_1 \cong S_2 \cong T_2$. Thus there is a nontrivial automorphism ϕ of G such that $\phi(S_1) = T_2$, $\phi(T_2) = S_1$, and $\phi(T_k) = T_k$ for $3 \leq k \leq n$. This implies K is not characteristic in G, a contradiction. Thus $S_i \ncong S_j$, for $1 \leq i \neq j \leq n$.

Conversely, let $G = S_1 \times \cdots \times S_n$, where each S_i, $1 \leq i \leq n$, is simple and $S_i \ncong S_j$ for $1 \leq i \neq j \leq n$. Let H be a nontrivial normal subgroup of G. If $H = G$, then H clearly has a proper characteristic supplement. If H is proper in G, then without loss of generality, $H = S_1 \times \cdots \times S_t$, where $1 \leq t \leq n-1$. Let $K = S_{t+1} \times \cdots \times S_n$. Then K is characteristic in G and $G = HK$. Thus G is an nCS-group. □

The results from Theorem 4.1 motivate the following theorems concerning cCS-groups. Unfortunately, results for cCS-groups are not as strong as those for $nPNS$-groups, as characteristic subgroups do not satisfy the same closure properties as normal subgroups do.

THEOREM 5.3. *A group G is a cCS-group if and only if* $c\mathrm{Frat}(G) = \{1\}$.

Proof. Let G be a cCS-group and suppose that $c\mathrm{Frat}(G) \neq \{1\}$. We first show that $c\mathrm{Frat}(G) \neq G$. Suppose that $c\mathrm{Frat}(G) = G$. Then G has a nontrivial proper characteristic subgroup H. Let $x \in H$, $x \neq 1$. Then $x^{A(G)} \subseteq H$. It follows that there exists a proper subgroup K char G such that $G = x^{A(G)}K$. Let M be maximal with respect to $x \notin M$ and $K \subseteq M$ (Zorn's Lemma). We claim that M is maximal characteristic in G. Suppose that M is not. Then there exists a subgroup L char G such that $M \lneq L \lneq G$. If $x \in L$, then $x^{A(G)}K \subseteq L$, a contradiction. Thus $x \notin L$. But this contradicts the maximality of M. Thus M is a maximal characteristic subgroup of G. This contradiction implies that $c\mathrm{Frat}(G) \neq$ G.

Given that $c\mathrm{Frat}(G)$ is nontrivial and characteristic in G, there exists a maximal characteristic subgroup M^* of G such that $G = c\mathrm{Frat}(G)M^*$. Since $c\mathrm{Frat}(G) \subseteq M^*$, $G = M^*$, a contradiction. Thus $c\mathrm{Frat}(G) = \{1\}$.

Conversely, let $c\mathrm{Frat}(G) = \{1\}$. If G is characteristically simple or trivial, then G is clearly a cCS-group. Suppose otherwise. If G is not a cCS-group, then for some nontrivial characteristic subgroup H of G, $G \neq HK$, for all subgroups K char G. Then $G \neq HM$ for all $M \in \mathcal{K}$. Thus $HM = M$ and $H \subseteq M$, for all $M \in \mathcal{K}$.

Therefore $H \subseteq c\mathrm{Frat}(G)$, a contradiction. Thus G is a cCS-group. □

THEOREM 5.4. *If G is a cCS-group, then G is the subdirect product of characteristically simple groups. Furthermore, there exists a group W which is the subdirect product of characteristically simple groups, but is not a cCS-group.*

Proof. If G is trivial, the result follows. Let $x \in G$, $x \neq 1$, and consider $x^{A(G)}$.

First consider the case in which $x^{A(G)} = G$. If there are no proper characteristic subgroups in G, then G is characteristically simple. Suppose this is not the case. By Theorem 5.3, $c\mathrm{Frat}(G) = \{1\}$ and G admits maximal characteristic subgroups. Thus there is a maximal characteristic subgroup M of G such that $G = x^{A(G)}M$ and $x \notin M$.

Now suppose $x^{A(G)} \neq G$. By an argument identical to the one given in the proof of Theorem 5.3, there is a maximal characteristic subgroup M of G such that $G = x^{A(G)}M$ and $x \notin M$. Consequently, for each nontrivial element $x \in G$, regardless of whether $x^{A(G)} = G$ or $x^{A(G)} \neq G$, there exists a maximal characteristic subgroup M_x of G such that $x \notin M_x$ and $G = x^{A(G)} M_x$.

Each G/M_x is characteristically simple. Create the map $\phi : G \to \Pi_{x \in G} G/M_x$ defined by $\phi(g) = \Pi_{x \in G} gM_x$. Since $c\mathrm{Frat}(G) = \{1\}$, $\ker(\phi) = \{1\}$. Thus ϕ is an isomorphism into $\Pi_{x \in G} G/M_x$.

Finally, consider the group $W = \mathbb{Z} \times \mathbb{Z}_2 = \langle a, b \mid b^2 = 1, ab = ba \rangle$, which is the subdirect product of characteristically simple groups. Consider the torsion subgroup $\langle b \rangle$, which is characteristic in W. For any proper subgroup M of G which supplements $\langle b \rangle$, $M \cap \langle b \rangle = \{1\}$. Thus $G = M \times \langle b \rangle$ and $M = \langle a \rangle$ or $\langle ab \rangle$. Since neither $\langle a \rangle$ or $\langle ab \rangle$ is characteristic in W, W is not a cCS-group. □

6. Class containments. In this section, we shall prove Theorems 1.4 and 1.5.

Proof of Theorem 1.4. To do this, we need to establish all the proper containments listed in Diagram 2.

We start with those proper containments which will be established by using finite counterexamples. Consider S_3, the symmetric group on three letters. Obviously S_3 is in aS, hence in nS and cS, by Proposition 1.2. On the other hand, S_3 is not a $cPNS$-group, since its commutator subgroup does not have a proper normal supplement. By Proposition 1.2, it follows that S_3 is not in $nPNS$ and $aPNS$ either. We conclude that $xPNS \subset xS$ for $x = a$, n, and c.

Next consider A_5, the alternating group on five letters. By Theorem 5.2, A_5 is an nCS-group. Proposition 1.3 implies that A_5 is an $nPNS$- and nS-group. On the other hand, A_5 is not in aS since its Sylow 2-subgroups have no proper supplement. Thus, by Proposition 1.3, A_5 is not in $aPNS$ and aCS. We conclude that $aP \subset nP$ for $P = S$, PNS, and CS.

To show that $xCS \subset xPNS$ for $x = a$, n, and c, consider the group $W = \mathbb{Z} \times \mathbb{Z}_2$. Since $\mathbb{Z}$ is the subdirect product of $\prod_{p \in \pi} C_p$, W is an $aPNS$-group, by Theorem 4.4. Consequently, by Proposition 1.2, it is an $nPNS$- and $cPNS$-group. On the other hand, W is not a cCS-group, by Theorem 5.4. Proposition 1.2 implies that W is not an aCS- and nCS-group. We conclude that $xC \subset xPNS$, for $x = a$, n, and c.

To complete the proof, consider $F = \mathbb{Q} \times A_5$, where $\mathbb{Q}$ is the group of rational numbers under addition. Since $c\mathrm{Frat}(F) = \{1\}$, it follows from Theorem 5.3 that F is a cCS-group. Proposition 1.3 implies that F is a $cPNS$- and cS-group. On the other hand, it was shown in the proof of Proposition 3.2 that F is not an nS-group. Therefore, Proposition 1.3 implies that F is not an $nPNS$- and nCS-group. We conclude that $nP \subset cP$ for $P = S$, PNS, and CS. □

Proof of Theorem 1.5. Recall that in Theorem 1.5, all groups considered are finite. To prove the theorem, we need to establish all the proper containments and equalities of Diagram 3.

All of the proper containments, except that $nCS \subset cCS$, $aCS \subset aPNS$, and $nCS \subset nPNS$, follow from Theorem 1.4, since they were established using finite counterexamples. The fact that $nS = cS$ follows from Corollary 3.3, and $nPNS = cPNS$ follows from Corollary 4.3.

To show that $cCS = nPNS$, let G be a finite cCS-group. By Theorem 5.4, G is the direct product of characteristically simple groups. However, a finite characteristically simple group is either simple or the direct product of isomorphic simple groups (3.3.15 of [**14**]). Consequently, G is the direct product of simple groups. By Theorem 4.1, G is an $nPNS$-group.

Conversely, let G be a finite $nPNS$-group. By Corollary 4.3, G is the direct product of simple groups. By 3.3.15 in [**14**], G can be written as $G = K_1 \times \cdots \times K_n$, where each K_i, $(1 \le i \le n)$, is characteristically simple and characteristic in G. If $n = 1$, then the maximal characteristic subgroup of G is $\{1\}$ and $c\text{Frat}(G) = \{1\}$. If $n \ge 2$, then $M_j = K_1 \times \cdots \times K_{j-1} \times K_{j+1} \times \cdots \times K_n$ is maximal characteristic in G for $j = 1, \ldots, n$. Therefore $c\text{Frat}(G) \subseteq \bigcap_{j=1}^{n} M_j = \{1\}$. Thus, by Theorem 5.3, G is a cCS-group.

To show that $xCS \subset xPNS$ for $x = a$ and n, consider the Klein four group K_4. By Theorem 4.4 and Proposition 1.2, K_4 is an $aPNS$- and $nPNS$-group. By Theorem 5.2 and Proposition 1.2, K_4 is not an aCS- and nCS-group. Thus $xCS \subset xPNS$ for $x = a$ and n.

Finally, given that $cCS = nPNS$ and $nPNS = cPNS$, we have $cCS = cPNS$. Furthermore, since $nCS \subset nPNS$ and $nPNS = cCS$, we conclude that $nCS \subset cCS$. □

Theorem 1.5 indicates that the class of finite nS-groups is identical to the class of finite cS-groups. Thus for a finite group G, every nontrivial characteristic subgroup having a proper supplement implies that every nontrivial normal subgroup does. While this is surprising, it is not unexpected given the following parallel result, established independently by N. T. Dinerstein [**7**] and M. Hofmann [**10**].

Theorem 6.1. *If each characteristic subgroup of a finite group G is complemented in G, then each normal subgroup of G is complemented in G.*

In addition, one might be led to believe that a finite cS-group (which is also an nS-group) would have to have all of its normal subgroups characteristic. This is not the case as indicated by the Klein four group.

7. Closure properties. In this section, we will examine the subgroup and homomorphic image closure properties of xP-groups for $P = S$, PNS, and CS. The closure properties of xD- and xC-groups have been studied by Christensen in [**5**]. For convenience, the closure results stated in [**5**] are listed here in one result. In his paper, x could also equal f (fully invariant).

Proposition 7.1. (Christensen [**5**]). *The following statements hold.*

(i) *Every x-subgroup of an xD-group is an xD-group.*

(ii) *Every x-subgroup of an xC-group is an xC-group for $x = a, c, f$.*

(iii) *For any x and $P = D$ or C, let G be an xP-group and ϕ a homomorphism of G whose kernel is an x-subgroup. Then G^{ϕ} is an xP-group.*

(iv) *Every complement of an x-subgroup of an xP-group, where $P = D$ or C, is also an xP-group.*

(v) *The direct product of two xP-groups, where $P = D$ or C, is an xP-group.*

In this section, subgroup and homomorphic image closure properties of xS, $xPNS$, and xCS will be studied to the extent possible. We first show that every subgroup of an aP-group is an aP-group for $P = S$ or PNS. This also holds for an

aCS-group when the restriction is made to torsion groups. We then show that a normal subgroup N of an nP-group is an nP-group, when $P = S$ and Frat(N) is finitely generated, when $P = PNS$, and when $P = CS$ and the nCS-group is finite. In addition, it is proven that every characteristic subgroup of a cP-group is a cP-group when $P = S$ or PNS. This also holds for finite cCS-groups. Finally, we show that the homomorphic image of a finite xP-group is an xP-group, except for $xP = nS$ and cS.

Theorem 7.2. *The following statements hold.*

(i) *Every subgroup of an aP-group is an aP-group for $P = S$ or PNS.*
(ii) *Every subgroup of a torsion aCS-group is an aCS-group.*

Proof. In the case that $P = S$, this follows from Proposition 3.5. The proof that every subgroup of an $aPNS$-group is an $aPNS$-group is almost identical to the proof presented in Proposition 3.5. Statement (ii) follows from Theorem 5.1. □

Next, the subgroup closure properties of nP-groups are studied.

Theorem 7.3. *The following statements hold.*

(i) *Every normal subgroup N of an nS-group with* Frat*(N) finitely generated is an nS-group, and there exists a group H in nS with $N \lhd H$ and* Frat*(N) not finitely generated such that N is not an nS-group.*
(ii) *Every normal subgroup of an $nPNS$-group is an $nPNS$-group.*
(iii) *Every normal subgroup of a finite nCS-group is an nCS-group.*

Proof. To prove (i), let N be a normal subgroup of an nS-group G with Frat(N) finitely generated. We observe that N is clearly an nS-group if $N = \{1\}$ or G. Assume N is nontrivial and proper in G and consider Frat(N). We shall show that Frat(N) $= \{1\}$.

Suppose Frat(N) $\neq \{1\}$. Then Frat(N) is a nontrivial normal subgroup of G. Thus there exists a proper subgroup L of G such that $G =$ Frat(N)L. Since Frat(N) is finitely generated, Frat(N) $\subseteq$ Frat(G). This implies that $G = L$, a contradiction. Thus Frat(N) $= \{1\}$. By Proposition 3.1, N is an nS-group.

The following example shows that the condition imposed on Frat(N) is necessary. Consider the group $H = \mathrm{Hol}(\mathbb{Q}) = NS$ the semidirect product of N, the rationals under addition, and $S = \mathrm{Aut}(\mathbb{Q})$, the multiplicative group of rationals, isomorphic to $\mathbb{Z}_2 \times F$, where F is a free abelian group of countable rank. We note that for $n = (a, 1) \in N$, $a \neq 0$, $n^H = N$, and that $N = H'$. Let M be a nontrivial normal subgroup in H with $m = (u, v) \in M$, $v \neq 1$. Then there is a $g \in H$ such that the commutator $[g, m] \neq 1$. Hence $[m, g]^H = N$ and $N \subseteq M$. It follows that any normal subgroup is of the form $M = NA$, where A is a subgroup of S. But S is an $aPNS$-group, and hence there exists a supplement B of A such that $AB = S$. Thus $H = MB$ and H is an nS-group. However, N is a normal subgroup of H, not finitely generated and not in nS.

To prove (ii), let N be a nontrivial normal subgroup of an $nPNS$-group G. By Theorem 4.1, G is the subdirect product of simple groups. Thus there exist mappings $\phi : G \to \Pi_{i\in\Lambda} S_i$, where Λ is an index set and S_i is simple for all $i \in \Lambda$, and $\rho_j : \Pi_{i\in\Lambda} S_i \to S_j$ (projection map), such that ϕ is an isomorphism into $\Pi_{i\in\Lambda} S_i$ and $\phi\rho_j$ is onto.

Now consider $N^{\phi\rho_j}$. Since $N^{\phi\rho_j} \trianglelefteq G^{\phi\rho_j} = S_j$, $N^{\phi\rho_j} = \{1\}$ or S_j. Let Λ' be the collection of those $l \in \Lambda$ such that $N^{\phi\rho_l} = S_l$. Then N is the subdirect product of $\Pi_{l\in\Lambda'} S_l$ and is an $nPNS$-group, by Theorem 4.1.

Statement (iii) follows from Theorem 5.2. □

Now subgroup closure properties for cP-groups will be investigated.

THEOREM 7.4. *The following statements hold.*

(i) *Every characteristic subgroup of a cP-group is a cP-group for $P = S$ or PNS.*
(ii) *Every characteristic subgroup of a finite cCS-group is a cCS-group.*

Proof. To prove statement (i), let C be a characteristic subgroup of a cP-group G. If $C = \{1\}$ or G, then C is clearly a cP-group. Assume that C is nontrivial and proper in G. Let K be a nontrivial characteristic subgroup of C.

If $K = C$, then K has a proper P-supplement ($P = S$ or PNS) in C. Assume that $K \neq C$. Since K char G, there is a proper P-supplement L in G such that $G = KL$. By the modular identity, $C = G \cap C = KL \cap C = K(L \cap C)$. If $L \cap C = \{1\}$, then $C = K$, a contradiction. If $L \cap C = C$, then $C \subseteq L$. This would then imply that $G = L$, another contradiction. Thus $L \cap C$ is a nontrivial, proper subgroup of C.

If $P = S$, then $C = K(L \cap C)$ and thus C is a cS-group. If $P = PNS$, we need to show that $(L \cap C) \triangleleft C$. However, since $L \triangleleft G$, we have $(L \cap C) \triangleleft C$, and C is a $cPNS$-group.

For statement (ii), let H be a characteristic subgroup of a finite cCS-group G. Since H is normal in G and G is an $nPNS$-group (Theorem 1.5), Theorem 7.3 implies that H is an $nPNS$-group. By Theorem 1.5, H is a cCS-group. □

Next, homomorphic image properties are investigated.

THEOREM 7.5. *Every homomorphic image of a finite aP-group is an aP-group for $P = S$, PNS, or CS.*

Proof. Let G be a finite aP-group. If $P = PNS$ or CS, the result follows from Theorems 4.4 and 5.1 respectively.

Suppose $P = S$. By Corollary 3.7, G is an aC-group. By Theorem 2 of [**8**], a group is an aC-group if and only if all of its chief factors are cyclic and all of its Sylow subgroups are elementary abelian. Since G is an aC-group having the above properties, every homomorphic image of G also has the properties above. Thus every homomorphic image of G is an aC-group and hence, by Corollary 3.7, an aS-group. □

THEOREM 7.6. *Every homomorphic image of a finite nP-group is an nP-group with $P = PNS$ or CS.*

Proof. If G is an $nPNS$-group, the result follows from Lemma 3 of [**3**]. If G is an nCS-group, the result follows from Theorem 5.2. □

THEOREM 7.7. *Every homomorphic image of a finite cP-group is a cP-group with $P = PNS$ or CS.*

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Proof. This follows from Theorem 7.6 and the fact that every finite $cPNS$ and cCS-group is an $nPNS$-group (Theorem 1.5). □

Theorems like 7.6 and 7.7 do not hold for finite nS or cS-groups. Consider the group $G = \langle a, b \mid a^5 = b^4 = 1,\ b^{-1}ab = a^2\rangle$. The normal and characteristic subgroups of G are $\langle a\rangle$ and $\langle a, b^2\rangle$, each of which has $\langle b\rangle$ as a supplement. However, $G/\langle a\rangle \cong \langle b\rangle$ and $\mathrm{Frat}(\langle b\rangle) = \langle b^2\rangle \neq \{1\}$. Thus $G/\langle a\rangle$ is not an nS or cS-group.

In addition, making statements similar to Theorems 7.5, 7.6, and 7.7 does not seem to be possible in the general case. Let $G = \Pi_{p\in\pi} C_p$. By Theorem 4.4, G is an $aPNS$-group, and thus by Propositions 1.2 and 1.3, an aS and $nPNS$-group. Consider $T(G) = \Sigma_{p\in\pi} C_p$, the torsion subgroup of G.

It can be easily seen that G/T is a torsion free divisible group, and hence the direct sum of rationals under addition. We observe that $n\mathrm{Frat}(G/T) = \mathrm{Frat}(G/T) = G/T$. By Theorem 4.4 and 4.1 it follows that G/T is not an $aPNS$-group nor an $nPNS$-group. Proposition 3.4 implies that G/T is not an aS-group.

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