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AVERAGING DISTANCES IN CERTAIN BANACH SPACES

REINHARD WOLF

Let E be a Banach space. The averaging interval $AI(E)$ is defined as the set of positive real numbers α, with the following property: For each $n \in \mathbb{N}$ and for all (not necessarily distinct) $x_1, x_2, \ldots, x_n \in E$ with $\|x_1\| = \|x_2\| = \ldots = \|x_n\| = 1$, there is an $x \in E$, $\|x\| = 1$, such that

$$\frac{1}{n}\sum_{i=1}^{n}\|x_i - x\| = \alpha.$$

It follows immediately, that $AI(E)$ is a (perhaps empty) interval included in the closed interval $[1, 2]$. For example in this paper it is shown that $AI(E) = \{\alpha\}$ for some $1 < \alpha < 2$, if E has finite dimension. Furthermore a complete discussion of $AI(C(X))$ is given, where $C(X)$ denotes the Banach space of real valued continuous functions on a compact Hausdorff space X. Also a Banach space E is found, such that $AI(E) = [1, 2]$.

1. INTRODUCTION

Let E be a Banach space. We ask for positive real numbers α, with the following property: For each $n \in \mathbb{N}$ and for all (not necessarily distinct) $x_1, x_2, \ldots, x_n \in E$ with $\|x_1\| = \|x_2\| = \ldots = \|x_n\| = 1$, there is an $x \in E$, $\|x\| = 1$, such that

$$\frac{1}{n}\sum_{i=1}^{n}\|x_i - x\| = \alpha.$$

Since the unit sphere $S = \{x \in E, \|x\| = 1\}$ of E is connected, and for each choice $x_1, \ldots, x_n$ in S (not necessarily distinct) the function $F(x_1, \ldots, x_n)$ on S defined by $F(x_1, \ldots, x_n)(x) := 1/n \sum_{i=1}^{n} \|x_i - x\|$ for all $x \in S$, is continuous, we get:

$F(x_1, \ldots, x_n)(S) \subseteq \mathbb{R}^+$ is a nonempty interval (closed, open, half closed - half open). So $\alpha \in \mathbb{R}^+$ has the desired property if and only if

$$\alpha \in \bigcap_{\substack{n \in \mathbb{N} \\ x_1, \ldots, x_n \in S}} F(x_1, \ldots, x_n)(S).$$

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We define

$$AI(E) := \bigcap_{\substack{n \in \mathbb{N} \\ x_1, \dots, x_n \in S}} F(x_1, \dots, x_n)(S)$$

as the averaging distance interval of E.

Since $\|x - y\| \leqslant 2$, $(\|x - y\| + \|x + y\|)/2 \geqslant 1$, for all $x, y \in S$, it follows that $AI(E)$ is an interval (closed, open, half closed - half open, or consisting of exactly one number, or the empty set) included in the closed interval $[1,2]$.

In this paper we discuss $AI(E)$ for certain real Banach spaces E.

2. Basic definitions and notation

All Banach spaces E in this paper are considered real and of dimension at least two. By $S = \{x \in E, \|x\| = 1\}$ we denote the unit sphere of E. For $n \in \mathbb{N}$, $1 \leqslant p \leqslant \infty$, let $l^p(n)$ denote $\mathbb{R}^n$ with the usual p-norm. Recall that a sequence of elements $x_1, x_2, \dots$ in E is called a basic sequence if for each x in the closed linear span $\overline{[(x_n)_{n \geqslant 1}]}$ generated by $x_1, x_2, \dots$ there exist a unique sequence of real numbers $\alpha_1, \alpha_2, \dots$ such that

$$x = \lim_{n \to \infty} \sum_{i=1}^{n} \alpha_i x_i.$$

Further recall that a topological space X is completely regular if X is a Hausdorff space with the following property: For each closed subset A of X and for each $x \notin A$, there exists a continuous function f on X such that $0 \leqslant f(y) \leqslant 1$ for all y in X, $f(x) = 1$ and $f(a) = 0$ for all a in A.

A subset B of X is called a G_δ-set if B is the countable intersection of open subsets of X.

A completely regular space X is called a P-space if every G_δ-set in X is open. (See [**1**, p.63].)

3. The results

When E is of finite dimension the following Theorem of Gross describes the averaging interval $AI(E)$:

Theorem. [3] *Let (X, d) be a compact connected metric space. There is a unique positive real number $r(X, d)$, $D(X)/2 \leqslant r(X, d) < D(X)$, with the following property: For each positive integer n and for all (not necessarily distinct) $x_1, x_2, \dots, x_n$ in X, there exists an x in X such that*

$$\frac{1}{n} \sum_{i=1}^{n} d(x_i, x) = r(X, d).$$

$r(X,d)$ is called the rendezvous number of X and $D(X)$ denotes the diameter of X.

For a proof see [**3**].

From this we obtain:

PROPOSITION 1. *Let E be a finite dimensional Banach space. Then we have $AI(E) = \{\alpha\}$, for some $1 < \alpha < 2$.*

For example, in [**5**] Morris and Nickolas proved that this unique α is $\left(2^{n-1}\left[\Gamma(n/2)\right]^2\right)\Big/(\sqrt{\pi}\Gamma((2n-1)/2))$, for all $n \geqslant 2$ in the case $E = l^2(n)$, where Γ denotes the Gamma function.

In [**6**] it is shown that $\alpha = 3/2$ for $E = l^\infty(n)$, $\alpha = 2 - 1/n$ for $E = l^1(n)$ and in [**7**] we give a proof that $\alpha \leqslant 2 - 1/n$, if E is an n-dimensional real Banach space with a 1-unconditional basis and equality holds if and only if E is isometrically isomorphic to $l^1(n)$. (In both papers the unique positive real number α is denoted by $r(E)$.)

In the case E is of infinite dimension it is shown in [**6**] that $AI(l^2) = \{\sqrt{2}\}$, $AI(l^\infty) = \{3/2\}$ and $AI(l^1) = \emptyset$, where l^p $(1 \leqslant p \leqslant \infty)$ denotes the real sequence space with the usual p-norm. Recently Pei-Kee Lin (private communication, see [4]) showed that $AI(l^p) \subseteq \{2^{1/p}\}$ if $1 \leqslant p < \infty$, $AI(l^p) = \emptyset$ if $1 \leqslant p < 2$ and $\lim_{n\to\infty} r(l^p(n)) = 2^{1/p}$ for all $1 \leqslant p < \infty$ $(AI(l^p(n)) = \{r(l^p(n))\})$. By looking at the proofs of Proposition 1, 4, 5 in [**6**] and since $1 \notin AI(c_0)$ (notice that there is no x in c_0, $\|x\| = 1$ such that $\|x_1 - x\| + \|x_1 + x\| + \|x_2 - x\| + \|x_2 + x\| = 4$, where $x_1 = (1,0,0,\dots)$ and $x_2 = (1,1/2,1/3,\dots)$) it follows that $AI(c_0) = (1,3/2]$, where c_0 denotes the subspace of l^∞ consisting of all zero tending sequences.

Notice that Proposition 1 implies that the numbers 1 and 2 are forbidden values for α, if E is finite dimensional.

If E has infinite dimension the fact 1, respectively 2, are elements of $AI(E)$ implies that c_0 (under an added condition), respectively l^1, are in E:

PROPOSITION 2. *Let E be a Banach space of infinite dimension. Then we get*

1. *$1 \in AI(E)$ and E having a two dimensional subspace isometrically isomorphic to $l^\infty(2)$ implies that E contains a closed subspace isometrically isomorphic to c_0.*
2. *$2 \in AI(E)$ implies that E contains a closed subspace isometrically isomorphic to l^1.*

Since a Banach space is reflexive if and only if all its closed subspaces are reflexive, we obtain

COROLLARY. *Let E be a reflexive Banach space. Then $2 \notin AI(E)$. In addition, if E does not contain a two dimensional subspace isometrically isomorphic to $l^\infty(2)$,* *then* $1 \notin AI(E)$ *also.*

For a compact Hausdorff space X let $C(X)$ denote the Banach space of all real valued continuous functions on X with the supremum norm. The following result gives a complete discussion of $AI(C(X))$:

Proposition 3. *Let X be a compact Hausdorff space with at least two points. Then*

1. $AI(C(X)) = \{3/2\}$ *if X has at least one isolated point.*
2. $AI(C(X)) = [3/2, 2)$ *if X has no isolated points and has at least one point with a countable neighbourhood basis.*
3. $AI(C(X)) = [3/2, 2]$ *if no point in X has a countable neighbourhood basis.*

Therefore, for example, we get $AI(C[0,1]) = [3/2, 2)$.

Quite recently Pei-Kee Lin (private communication, see [4]) generalised Proposition 3, part 1 to $C_b(X)$, the space of all bounded real valued continuous functions on a normal space X, and moreover he showed that $AI(C_b(X)) \supseteq (3/2, 2)$, if X is a normal space without isolated points.

The next result gives an answer to the question: What is the maximal size of $AI(E)$?

Proposition 4. *Let X be a P-space without isolated points and let E be the Banach space of all bounded continuous real valued functions on X vanishing at one point x_0 in X, with the supremum norm. Then we have*

$$AI(E) = [1, 2].$$

Remark. Of course each discrete space is a P-space. An example of a P-space with exactly one non isolated point is the following: Let S be an uncountable space in which all points are isolated except for a distinguished point s_0, a neighbourhood of s_0 being any set containing s_0 whose complement is countable. (See [**1**, 4 N.1, p.64].)

The existence of a P-space without isolated points is not trivial. A construction of such a space is given in [**1**, Chapter 13], in particular see 13 P. 1, page 193.

Summing up, we notice that if a Banach space E is finite dimensional with dimension at least two then $AI(E) = \{\alpha\}$ for some unique positive real number α. If E has infinite dimension then all extreme cases for $AI(E)$ are possible: For example $AI(l^1) = \emptyset$, $AI(l^2) = \{\sqrt{2}\}$, and $AI(E) = [1, 2]$ for E the Banach space given in Proposition 4.

4. The proofs

Proof of Proposition 1: By assumption the unit sphere S of E equipped with the norm induced metric is a compact connected metric space. Applying Gross's

Theorem and since the diameter of S is two, we get $AI(E) = \{\alpha\}$, for some $\alpha \in [1,2)$. It remains to show that $\alpha > 1$. Assume $\alpha = 1$.

Let $n \in \mathbb{N}$. By compactness of S find a $1/n$-net $x_1, \ldots, x_N$ of S, $N = N(n)$. Since $AI(E) = \{1\}$ we find some y_n in S such that

$$\frac{1}{2N} \sum_{i=1}^{N} \|x_i - y_n\| + \|x_i + y_n\| = 1.$$

It follows that $\|x_i - y_n\| + \|x_i + y_n\| = 2$ for all $1 \leqslant i \leqslant N$. So for each x in S we obtain

$$2 \leqslant \|x - y_n\| + \|x + y_n\| \leqslant 2 + \frac{2}{n}.$$

Compactness of S again implies that a subsequence of $(y_n)_{n \geqslant 1}$ converges to some y in S. Therefore we have

$$(\star) \qquad \|x - y\| + \|x + y\| = 2 \quad \text{for all } x \text{ in } S.$$

Now choose some y_0 in S with $\|y - y_0\| = 1$. By formula $(\star)$ we get $\|y + y_0\| = 1$.

Therefore $y + y_0$ and $y - y_0$ are elements of S. Applying formula $(\star)$ to $y + y_0$ and $y - y_0$ we get $\|2y - y_0\| = \|2y + y_0\| = 1$. But $4 = \|4y\| \leqslant \|2y - y_0\| + \|2y + y_0\| = 2$ leads to a contradiction. $\square$

For proving Proposition 2 we need a well known criterion for basic sequences:

Lemma 1. *Let $x_1, x_2, \ldots$ be a sequence of nonzero elements in a Banach space E. Then in order that $x_1, x_2, \ldots$ be a basic sequence, it is both necessary and sufficient that there be a finite constant $K > 0$ so that for any choice of scalars $(\alpha_n)_{n \geqslant 1}$ and any integers m, n with $m < n$ we have*

$$\left\| \sum_{i=1}^{m} \alpha_i x_i \right\| \leqslant K \cdot \left\| \sum_{i=1}^{n} \alpha_i x_i \right\|.$$

For a proof see [**2**, Theorem 1, p.36].

Proof of Proposition 2: (1): Since $l^\infty(2)$ is isometrically included in E we find x_1, x_2 in S such that $\|\alpha_1 x_1 + \alpha_2 x_2\| = \max(|\alpha_1|, |\alpha_2|)$ for all α_1, α_2 in $\mathbb{R}$.

We inductively construct a sequence $x_1, x_2, \ldots$ of elements in S such that $\|\sigma_1 x_1 + \ldots + \sigma_n x_n\| = 1$ for all $\sigma_1, \ldots, \sigma_n$ in $\{1, -1\}$ and all $n \geqslant 2$.

Now let $n \geqslant 3$ and assume that we have found $x_1, \ldots, x_{n-1}$ in S such that $\|\sigma_1 x_1 + \ldots + \sigma_{n-1} x_{n-1}\| = 1$ for all $\sigma_1, \ldots, \sigma_{n-1}$ in $\{-1, 1\}$. Since $1 \in AI(E)$ we get some x_n in S such that

$$\frac{1}{2^n} \sum_{\sigma_1, \ldots, \sigma_{n-1} \in \{1,-1\}} \|\sigma_1 x_1 + \ldots + \sigma_{n-1} x_{n-1} + x_n\| + \|\sigma_1 x_1 + \ldots + \sigma_{n-1} x_{n-1} - x_n\| = 1.$$

Therefore we get

$$(\star)\qquad \frac{1}{2^n}\sum_{\sigma_1,\dots,\sigma_n\in\{1,-1\}}\|\sigma_1x_1+\dots+\sigma_nx_n\|=1.$$

Since $\|x+x_i\|+\|x-x_i\|\geqslant 2\|x_i\|=2$ for all $1\leqslant i\leqslant n$ and all $x\in E$ and applying formula $(\star)$ we get

$$\Big\|\sum_{\substack{r=1\\ r\neq i}}^{n}\sigma_rx_r+x_i\Big\|+\Big\|\sum_{\substack{r=1\\ r\neq i}}^{n}\sigma_rx_r-x_i\Big\|=2$$

for all $1\leqslant i\leqslant n$ and for all $\sigma_1,\dots,\sigma_{i-1},\sigma_{i+1},\dots,\sigma_n$ in $\{1,-1\}$.

For $0\leqslant j\leqslant n$ define

$$A_j=\{\sigma_1x_1+\dots+\sigma_nx_n,\ \sigma_1,\dots,\sigma_n\in\{1,-1\}\text{ and}$$
$$\text{exactly } j \text{ elements in } \sigma_1,\dots,\sigma_n \text{ are } -1\}.$$

Now it follows that $\|x\|=s_j$ for all x in A_j for some $0\leqslant s_j\leqslant 2$, $0\leqslant j\leqslant n$ and $s_0+s_1=s_1+s_2\dots=s_{n-1}+s_n=2$.

Since $\|x+\sigma_1x_1+\sigma_2x_2\|+\|x-\sigma_1x_1-\sigma_2x_2\|\geqslant 2\|\sigma_1x_1+\sigma_2x_2\|=2$ for all σ_1,σ_2 in $\{1,-1\}$ and all $x\in E$, by again applying formula $(\star)$ we get $s_0+s_2=2$ and therefore $s_0=s_1=\dots=s_n=1$. So $\|\sigma_1x_1+\dots+\sigma_nx_n\|=1$ for all $\sigma_1,\dots,\sigma_n$ in $\{1,-1\}$.

Now let $n\geqslant 3$ and take $\alpha_1,\dots,\alpha_n$ in $\mathbb{R}$ with $\max_{1\leqslant i\leqslant n}|\alpha_i|=1$.

Since the set $A=\{\sigma=(\sigma_1,\dots,\sigma_n),\ \sigma_1,\dots,\sigma_n\in\{1,-1\}\}$ is the set of extreme points of the unit ball in $l^\infty(n)$ we get some $0\leqslant b_\sigma\leqslant 1$, $\sum_{\sigma\in A}b_\sigma=1$ such that $(\alpha_1,\dots,\alpha_n)=\sum_{\sigma\in A}b_\sigma\cdot\sigma$. So $\alpha_1x_1+\dots+\alpha_nx_n=\sum_{\sigma\in A}b_\sigma\cdot(\sigma_1x_1+\dots+\sigma_nx_n)$ and hence $\|\alpha_1x_1+\dots+\alpha_nx_n\|\leqslant 1$. On the other hand choose $x'_1,\dots,x'_n\in E'$ with $x'_i(x_i)=1$, $\|x'_i\|=1$ for all $1\leqslant i\leqslant n$. Fix some $1\leqslant j\leqslant n$ and take $\sigma_1,\dots,\sigma_{j-1},\sigma_{j+1},\dots,\sigma_n$ in $\{1,-1\}$. Since $x_j=1/2\Big(\big(\sum_{\substack{r=1\\ r\neq j}}^{n}\sigma_rx_r+x_j\big)+\big(\sum_{\substack{r=1\\ r\neq j}}^{n}-\sigma_rx_r+x_j\big)\Big)$ and $x'_j(x_j)=1$ we get

$$x'_j\Big(\sum_{\substack{r=1\\ r\neq j}}^{n}\sigma_rx_r\Big)=0.$$

It is easy to check that

$$\alpha_1x_1+\dots+\alpha_nx_n=1/(2^n)\sum_{\sigma\in A}(\sigma_1\alpha_1+\dots+\sigma_n\alpha_n)(\sigma_1x_1+\dots+\sigma_nx_n)$$

and therefore we get

$$x_j'(\alpha_1 x_1 + \ldots + \alpha_n x_n) = \alpha_j.$$

So

$$\|\alpha_1 x_1 + \ldots + \alpha_n x_n\| \geqslant \max_{1 \leqslant j \leqslant n} |\alpha_j| = 1.$$

So it follows that

$$\|\alpha_1 x_1 + \ldots + \alpha_n x_n\| = \max_{1 \leqslant i \leqslant n} |\alpha_i|, \quad \text{for all } \alpha_1, \ldots, \alpha_n \in \mathbb{R}.$$

Lemma 1 now guarantees that $x_1, x_2, \ldots$ is a basic sequence in E (take $K = 1$). Let $F = \overline{[(x_n)_{n \geqslant 1}]}$ be the closed linear span of $x_1, x_2, \ldots$ and define

$$T : c_0 \to F, \quad T\big((\alpha_1, \alpha_2, \ldots)\big) = \sum_{n=1}^{\infty} \alpha_n x_n.$$

Then it follows that T is an isometry from c_0 onto F.

(2): Take some x_1 in S. Since $2 \in AI(E)$ we get some x_2 in S such that $1/2(\|x_1 - x_2\| + \|-x_1 - x_2\|) = 2$. Therefore $\|x_1 - x_2\| = \|x_1 + x_2\| = 2$. We inductively construct a sequence $x_1, x_2, \ldots$ of elements of S such that $\|\sigma_1 x_1 + \ldots + \sigma_n x_n\| = n$ for all $n \geqslant 2$ and all $\sigma_1, \ldots, \sigma_n$ in $\{1, -1\}$. Now let $n \geqslant 3$ and assume that we have found $x_1, \ldots, x_{n-1}$ in S such that

$$\|\sigma_1 x_1 + \ldots + \sigma_{n-1} x_{n-1}\| = n - 1 \ \text{ for all } \sigma_1, \ldots, \sigma_{n-1} \text{ in } \{1, -1\}.$$

Since $2 \in AI(E)$ we get some x_n in S such that

$$\frac{1}{2^n} \sum_{\sigma_1, \ldots, \sigma_{n-1} \in \{1,-1\}} \left\| \frac{1}{n-1}(\sigma_1 x_1 + \ldots + \sigma_{n-1} x_{n-1}) - x_n \right\| + \left\| \frac{1}{n-1}(\sigma_1 x_1 + \ldots + \sigma_{n-1} x_{n-1}) + x_n \right\| = 2.$$

This implies

$$\|(\sigma_1 x_1 + \ldots + \sigma_{n-1} x_{n-1}) - (n-1) x_n\| = \|(\sigma_1 x_1 + \ldots + \sigma_{n-1} x_{n-1}) + (n-1) x_n\| = 2(n-1),$$

for all $\sigma_1, \ldots, \sigma_{n-1}$ in $\{1, -1\}$. Hence we get

$$\begin{aligned} 2(n-1) &= \|\sigma_1 x_1 + \ldots + \sigma_{n-1} x_{n-1} + \sigma_n (n-1) x_n\| \\ &\leqslant \|\sigma_1 x_1 + \ldots + \sigma_{n-1} x_{n-1} + \sigma_n x_n\| + (n-2)\|x_n\| \leqslant 2n - 2, \end{aligned}$$

for all $\sigma_1, \dots, \sigma_n$ in $\{1, -1\}$ and therefore

$$\|\sigma_1 x_1 + \dots + \sigma_n x_n\| = n \quad \text{for all } \sigma_1, \dots, \sigma_n \text{ in } \{1, -1\}.$$

Now let $n \geqslant 2$ and take $\alpha_1, \dots, \alpha_n \in \mathbb{R}$. Define $\tau_i = 1$ if $\alpha_i \geqslant 0$ and $\tau_i = -1$ if $\alpha_i < 0$ for all $1 \leqslant i \leqslant n$. Since $\|\tau_1 x_1 + \dots + \tau_n x_n\| = n$ choose some x' in E', $\|x'\| = 1$ and $x'(\tau_1 x_1 + \dots \tau_n x_n) = n$. Hence $x'(x_i) = \tau_i$ for all $1 \leqslant i \leqslant n$ and therefore

$$\|\alpha_1 x_1 + \dots + \alpha_n x_n\| \geqslant x'(\alpha_1 x_1 + \dots + \alpha_n x_n) = |\alpha_1| + \dots + |\alpha_n|.$$

So we have

$$\|\alpha_1 x_1 + \dots + \alpha_n x_n\| = |\alpha_1| + \dots + |\alpha_n| \text{ for all } n \geqslant 2 \text{ and all } \alpha_1, \dots, \alpha_n \in \mathbb{R}.$$

Lemma 1 again guarantees that the sequence $x_1, x_2, \dots$ is a basic sequence in E (take $K = 1$). Let $F = \overline{[(x_n)_{n \geqslant 1}]}$ be the closed linear span of $x_1, x_2, \dots$ and define

$$T : l^1 \to F, \quad T((\alpha_1, \alpha_2, \dots ,)) = \sum_{n=1}^{\infty} \alpha_n x_n.$$

Then it follows that T is an isometry from l^1 onto F and so we are done. □

In order to prove Proposition 3 we need the following lemmata:

Lemma 2. *Let X be a compact Hausdorff space with at least two points. For each $n \in \mathbb{N}$ and $f_1, \dots, f_n$ in $C(X)$ with $\|f_1\| = \dots = \|f_n\| = 1$ there are f_0 and g_0 in $C(X)$ with $\|f_0\| = \|g_0\| = 1$ such that*

$$\frac{1}{n} \sum_{i=1}^{n} \|f_i - f_0\| \leqslant \frac{3}{2} \leqslant \frac{1}{n} \sum_{i=1}^{n} \|f_i - g_0\| .$$

Proof: Let $A = \{f_1, \dots, f_n\}$ and choose some x in X. Define

$$A^0 = \{f \in A,\ f(x) = 0\},\ A^+ = \{f \in A,\ f(x) > 0\},\ A^- = \{f \in A,\ f(x) < 0\}.$$

Find an open neighbourhood U of x such that:

For all y in U we get $|f(y)| < 1/2$ for all f in A^0, $f(y) > 0$ for all f in A^+, $f(y) < 0$ for all f in A^-.

Since X is completely regular we get f_0 in $C(X)$ with $0 \leqslant f_0(y) \leqslant 1$ for all y in X, $f_0(x) = 1$ and $f_0(y) = 0$ for all y in $X \backslash U$.

Now it follows that

$$\|f-f_0\| \leqslant \frac{3}{2},\quad \|f+f_0\| \leqslant \frac{3}{2} \quad \text{for all } f \text{ in } A^0,$$
$$\|f-f_0\| \leqslant 1,\quad \|f+f_0\| \leqslant 2 \quad \text{for all } f \text{ in } A^+,$$
$$\|f-f_0\| \leqslant 2,\quad \|f+f_0\| \leqslant 1 \quad \text{for all } f \text{ in } A^-.$$

Therefore if $|A^+| \geqslant |A^-|$ we get

$$\frac{1}{n}\sum_{i=1}^{n}\|f_i - f_0\| \leqslant \frac{3}{2};$$

and

$$\frac{1}{n}\sum_{i=1}^{n}\|f_i + f_0\| \leqslant \frac{3}{2}$$

if $|A^+| \leqslant |A^-|$.

For the remaining inequality in Lemma 2, find a finite subset Y of X with at least two elements, such that for each $1 \leqslant i \leqslant n$, there is some y in Y with $|f_i(y)| = 1$.

Let k in $\mathbb{N}$ be the order of Y. Define

$$y_i = (f_i(y))_{y\in Y} \qquad \text{for each } 1 \leqslant i \leqslant n.$$

By definition of Y it follows that $y_1, y_2, \ldots, y_n$ are elements of the unit sphere of $l^\infty(k)$. Since $AI(l^\infty(k)) = \{3/2\}$ (see [6]) we find some z in the unit sphere of $l^\infty(k)$ such that

$$\frac{1}{n}\sum_{i=1}^{n}\|y_i - z\|_\infty = \frac{3}{2}.$$

Let $z = (\lambda_y)_{y\in Y}$. Since X is normal the Tietze Extension Theorem applies giving some g_0 in $C(X)$ such that $g_0(y) = \lambda_y$ for all y in Y and $\|g_0\| = \|z\|_\infty = 1$. Hence we get

$$\frac{1}{n}\sum_{i=1}^{n}\|f_i - g_0\| \geqslant \frac{1}{n}\sum_{i=1}^{n}\max_{y\in Y}|f_i(y) - g_0(y)| = \frac{1}{n}\sum_{i=1}^{n}\|y_i - z\|_\infty = \frac{3}{2}. \qquad \square$$

Lemma 3. *Let X be an infinite compact Hausdorff space and let $\varepsilon > 0$. Then there is a finite subset A of norm one elements in $C(X)$ such that*

$$\frac{1}{|A|}\sum_{f\in A}\|f-h\| > \frac{3}{2} - \varepsilon \quad \textit{for all } h \textit{ in } C(X) \textit{ with } \|h\| = 1.$$

Proof: Let $n \in \mathbb{N}$ and choose a finite subset Y of X order n. Furthermore find open neighbourhoods U_y for all y in Y such that $U_y \cap U_{y'} = \emptyset$ for all $y \neq y'$ in Y. By the complete regularity of X, find for each y in Y some f_y in $C(X)$ such that $-1 \leqslant f_y(x) \leqslant 1$ for all x in X, $f_y(y) = 1$ and $f_y(x) = -1$ for all x in $X \backslash U_y$. Now take h in $C(X)$, $\|h\| = 1$. Choose some x_0 in X with $|h(x_0)| = 1$.

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CASE 1. $h(x_0) = 1$. If $x_0 \in U_{y_0}$ for some y_0 in Y we get

$$\|f_y - h\| \geqslant |f_y(x_0) - h(x_0)| = |-1-1| = 2,$$

for all $y \neq y_0$ in Y.

$$\begin{aligned}\|f_y + h\| + \|f_{y'} + h\| &\geqslant |f_y(y) + h(y)| + |f_{y'}(y) + h(y)| \\ &= |1 + h(y)| + |-1 + h(y)| = 2,\end{aligned}$$

for all $y \neq y'$ in Y. Hence we get

$$\frac{1}{2n}\sum_{y\in Y}(\|f_y - h\| + \|f_y + h\|) \geqslant \frac{1}{2n}\Big(2(n-1) + 2\cdot\frac{n-1}{2}\Big) = \frac{3}{2} - \frac{3}{2n}.$$

If $x_0 \notin \bigcup_{y\in Y} U_y$ we get

$$\|f_y - h\| \geqslant |f_y(x_0) - h(x_0)| = |-1-1| = 2,$$

for all $y \in Y$. Hence we get

$$\frac{1}{2n}\sum_{y\in Y}(\|f_y - h\| + \|f_y + h\|) \geqslant \frac{1}{2n}\Big(2n + 2\cdot\frac{n-1}{2}\Big) = \frac{3}{2} - \frac{1}{2n}.$$

Summing up, we get

$$\frac{1}{2n}\sum_{y\in Y}(\|f_y - h\| + \|f_y + h\|) \geqslant \frac{3}{2} - \frac{3}{2n}.$$

CASE 2. $h(x_0) = -1$. Take $-h$ for h and look at Case 1. Now take $A = \{f_y, -f_y;\ y \in Y\}$ and choose n big enough. ∎

LEMMA 4. *Let X be a compact Hausdorff space without isolated points and let $\varepsilon > 0$. For each $n \in \mathbb{N}$ and $f_1, \dots, f_n$ in $C(X)$ with $\|f_1\| = \cdots = \|f_n\| = 1$ there is some f_0 in $C(X)$ with $\|f_0\| = 1$ such that*

$$\frac{1}{n}\sum_{i=1}^{n}\|f_i - f_0\| > 2 - \varepsilon.$$

PROOF: Let $A = \{f_1, \dots, f_n\}$. Choose some finite subset Y of X such that for each f in A there is some y in Y with $|f(y)| = 1$. Furthermore take open neighbourhoods U_y for each y in Y such that $U_y \cap U_{y'} = \emptyset$ for $y \neq y'$ in Y. Let

$$B_y = \{f \in A,\ f(y) = 1\},\quad C_y = \{f \in A,\ f(y) = -1\},$$

for y in Y. For each y in Y take open neighbourhoods V_y of y such that $f(x) > 1-\varepsilon$ for all x in V_y and all f in B_y. Let

$$W_y = U_y \cap V_y, \quad y \text{ in } Y.$$

Since X has no isolated points, take some z_y in W_y, $z_y \neq y$. The Tietze extension theorem provides f_0 in $C(X)$ such that $-1 \leqslant f_0(x) \leqslant 1$ for all x in X, $f_0(y) = 1$, $f_0(z_y) = -1$, for all y in Y.

Let y in Y. If f is in B_y we get $\|f - f_0\| \geqslant |f(z_y) - f_0(z_y)| > 2-\varepsilon$. If f is in C_y we get $\|f - f_0\| \geqslant |f(y) - f_0(y)| = 2$. Hence

$$\frac{1}{n}\sum_{i=1}^{n} \|f_i - f_0\| > 2 - \varepsilon.$$

□

LEMMA 5. *Let X be a compact Hausdorff space and let x_0 be a point in X without a countable neighbourhood basis and let A be a closed neighbourhood of x_0. Furthermore let $n \in \mathbb{N}$ and take $f_1, \dots, f_n$ in $C(X)$ with $f_i(x) \geqslant 0$ for all x in A, $f_i(x_0) = 0$, for all $1 \leqslant i \leqslant n$.*

Then there are $y_1, \dots, y_n$ in A such that $y_i \neq x_0$, $y_i \neq y_j$ and $f_i(y_i) = 0$ for all $1 \leqslant i \neq j \leqslant n$.

PROOF: By induction on n. Let $n = 1$. Assume $f_1(x) > 0$ for all x in A, $x \neq x_0$. Take an open neighbourhood U of x_0, $U \subseteq A$ and take $k \in \mathbb{N}$. Let $U_k = f_1^{-1}[0, 1/k) \cap U$. We claim that $(U_k)_{k \geqslant 1}$ is a neighbourhood basis of x_0, consisting of open neighbourhoods U_k of x. Let V be an open neighbourhood of x_0. If $A \backslash V = \emptyset$ we get $A \subseteq V$ and therefore $U_k \subseteq V$ for all $k \in \mathbb{N}$. Now let $A \backslash V \neq \emptyset$. Since $A \backslash V$ is compact we get some $s \in A \backslash V$ such that $f_1(y) \geqslant f_1(s)$ for all y in $A \backslash V$. Since $s \in A$, $s \notin V$, we get $f_1(s) > 0$. Choose some $k_0 \in \mathbb{N}$ with $f_1(s) > 1/k_0$. Assume there is some y in $U_{k_0} \backslash V$. Then we get $f_1(y) < 1/k_0$ and by $U_{k_0} \subseteq U \subseteq A$, $f_1(y) > 1/k_0$. Therefore $U_{k_0} \subseteq V$.

So $(U_k)_{k \geqslant 1}$ build a countable neighbourhood basis of x_0 which leads to a contradiction. Therefore we get some y_1 in A, $y_1 \neq x_0$ and $f_1(y_1) = 0$. Now let $f_1, \dots, f_{n+1}$ be in $C(X)$ with $f_i(x) \geqslant 0$ for all x in X, $f_i(x_0) = 0$, for all $1 \leqslant i \leqslant n+1$. Assume that we have found $y_1, \dots, y_n$ in A such that $y_i \neq x_0$, $y_i \neq y_j$ and $f_i(y_i) = 0$ for all $1 \leqslant i \neq j \leqslant n$.

Now choose some closed neighbourhood A_n of x_0 such that $y_1, \dots, y_n \notin A_n$. The case $n = 1$ leads to some $y_{n+1} \in A_n \cap A$ such that $y_{n+1} \neq x_0$ and $f_{n+1}(y_{n+1}) = 0$. □

Now we verify Proposition 3:

PROOF OF PROPOSITION 3: (1): If X is finite we get $C(X) \cong l^\infty(n)$ for some $n \geqslant 2$. Since $AI(l^\infty(n)) = \{3/2\}$ (see [6]) we are done. So assume that X is infinite.

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By Lemma 2 and 3 and the Intermediate Value Theorem we get $3/2 \in AI(C(X))$ and $AI(C(X)) \subseteq [3/2, 2]$. Now take some isolated point x_0 in X. Define f_0, by $f_0(x_0) = 1$ and $f_0(x) = 0$ for all $x \neq x_0$ in X. Hence we get $f_0 \in C(X)$, $\|f_0\| = 1$. It is easy to check that $\|f_0 - f\| + \|f_0 + f\| \leqslant 3$ for all $f \in C(X)$ with $\|f\| = 1$ and therefore $AI(C(X)) = \{3/2\}$.

(2): By Lemma 2, 3, 4 and the Intermediate Value Theorem we get $[3/2, 2) \subseteq AI(C(X))$ and $AI(C(X)) \subseteq [3/2, 2]$. It remains to show that $2 \notin AI(C(X))$. Take some x_0 in X with a countable neighbourhood basis $(U_n)_{n\geqslant 1}$. Without loss of generality let U_n be open neighbourhoods of x_0 for all $n \geqslant 1$. Since X is completely regular take some $(g_n)_{n\geqslant 1}$ in $C(X)$ such that $0 \leqslant g_n(x) \leqslant 1$ for all x in X, $g_n(x_0) = 0$ and $g_n(x) = 1$ for all $x \in X \backslash U_n$, for all $n \geqslant 1$.

It is easy to check that f defined by $f = 1 - \sum_{n\geqslant 1} 1/2^n g_n$ is in $C(X)$ and is such that $0 \leqslant f(x) \leqslant 1$ for all x in X, $f(x_0) = 1$ and $f(x) < 1$ for all $x \neq x_0$ in X. Assume that we have $2 \in AI(C(X))$. Then there must be some $h \in C(X)$, $\|h\| = 1$ and $\|f - h\| + \|f + h\| = 4$. Hence we get $h(x_0) = 1$ and $h(x_0) = -1$, which is a contradiction.

(3): It remains to show that $2 \in AI(C(X))$. Let $n \in \mathbb{N}$ and $f_1, \dots, f_n$ in $C(X)$ with $\|f_i\| = 1$ for all $1 \leqslant i \leqslant n$. Let $A = \{f_1, \dots, f_n\}$. Choose a finite subset Y of X such that for each f in A there is some y in Y with $|f(y)| = 1$. Furthermore let C_y be closed neighbourhoods of y such that $C_y \cap C_{y'} = \emptyset$ for all $y \neq y'$ in Y.

Let $A_y = \{f \in A,\ f(y) = 1\}$ and $B_y = \{f \in A,\ f(y) = -1\}$ for all y in Y. Now fix some y in Y. By Lemma 5 there are $(z_{y,f})_{f\in A_y}$ in C_y such that $z_{y,f} \neq y$, $z_{y,f} \neq z_{y,f'}$ and $(1 - f)(z_{y,f}) = 0$ for all $f \neq f'$ in A_y.

By the Tietze Extension Theorem we get some g in $C(X)$ such that $-1 \leqslant g(x) \leqslant 1$ for all x in X, $g(y) = 1$ and $g(z_{y,f}) = -1$ for all y in Y and all f in A_y.

Now let f in A.

CASE (a). $f \in A_{y_0}$ for some y_0 in Y. It follows that

$$\|f - g\| \geqslant |f(z_{y_0,f}) - g(z_{y_0,f})| = |1 - (-1)| = 2.$$

CASE (b). $f \in B_{y_0}$ for some y_0 in Y. It follows that

$$\|f - g\| \geqslant |f(y_0) - g(y_0)| = |-1 - 1| = 2.$$

So $\|f - g\| = 2$ for all f in A and therefore

$$\frac{1}{n} \sum_{i=1}^{n} \|f_i - g\| = 2. \qquad \square$$

The next two lemmata lead to Proposition 4.

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Lemma 6. *Let X be a P-space. Then we have*

1. *For every continuous function f on X the zero set Nf of f, $Nf = \{x \in X,\ f(x) = 0\}$ is an open and closed subset of X.*
2. *Let A be a countable subset of X. It follows that A is closed in X and for each a in A there are open neighbourhoods U_a of a such that $U_a \cap U_{a'} = \emptyset$ for all $a \neq a'$ in A.*
3. *Let A be a countable subset of X and f a function defined on A. (By (2) f is continuous on A.) Then there exists some continuous function $\tilde{f}$ on X such that $\tilde{f}(a) = f(a)$ for all a in A.*

Proof: For (1) and (3) see exercise 4J$\cdot$(3) and 4K$\cdot$(2) in [**1**, p.63].

(2): A is closed by exercise 4K$\cdot$(1) in [**1**, p.63].

Let $A = \{x_1, x_2, \dots\}$ and $n \in \mathbb{N}$. Since $A\backslash\{x_n\}$ is countable we get $A\backslash\{x_n\}$ is closed in X. Since X is completely regular there are f_n in $C(X)$ such that $f_n(x_n) = 0$ and $f_n(x_m) = 1$ for all $n \neq m$ in $\mathbb{N}$.

Let $Nf_n = \{x \in X,\ f_n(x) = 0\}$ for $n \in \mathbb{N}$. By (1) each Nf_n is an open subset of X. For each $n \in \mathbb{N}$ define $U_n = Nf_n \cap \Big[X\backslash \bigcup\limits_{k\neq n} Nf_k\Big]$.

Note that $X\backslash \bigcup\limits_{k\neq n} Nf_k = \bigcap\limits_{k\neq n} (X\backslash Nf_k)$ is a G_δ-set in X and therefore open since X is a P-space. It is easy to check that $(U_n)_{n\geqslant 1}$ are open neighbourhoods of x_n and $U_n \cap U_m = \emptyset$ for all $n \neq m$ in $\mathbb{N}$. $\square$

Lemma 7. *Let X be a P-space without isolated points and let $f_1, \dots, f_n \in C_b(X)$ with $\|f_1\| = \dots = \|f_n\| = 1$. Then there exist countable subsets $A_1, \dots, A_n$ of X such that $\sup\limits_{x\in A_i} |f_i(x)| = 1$ and $A_i \cap A_j = \emptyset$ for all $1 \leqslant i \neq j \leqslant n$.*

Proof: By induction on n. The case $n = 1$ is trivial. Now let $f_1, \dots, f_{n+1} \in C_b(X)$ with $\|f_1\| = \dots = \|f_{n+1}\| = 1$ and assume that we have found countable subsets $A_1, \dots, A_n$ of X such that $\sup\limits_{x\in A_i} |f_i(x)| = 1$ and $A_i \cap A_j = \emptyset$ for all $1 \leqslant i \neq j \leqslant n$. Choose some countable subset B_{n+1} of X such that $\sup\limits_{x\in B_{n+1}} |f_{n+1}(x)| = 1$. Let $A = A_1 \cup \dots \cup A_n \cup B_{n+1}$. By Lemma 6, part (2), find some open neighbourhoods U_a for each a in A such that $U_a \cap U_{a'} = \emptyset$ for all $a \neq a'$ in A. If $B_{n+1} \cap (A_1 \cup \dots \cup A_n) = \emptyset$, let $A_{n+1} = B_{n+1}$. So assume that $B_{n+1} \cap (A_1 \cup \dots \cup A_n) \neq \emptyset$. Take $y \in B_{n+1} \cap (A_1 \cup \dots \cup A_n)$. By Lemma 6, part (1), find some open neighbourhood V_y of y such that $V_y \subseteq U_y$ and $|f_{n+1}(x)| = |f_{n+1}(y)|$ for all x in V_y. Since X has no isolated points take some z_y in V_y, $z_y \neq y$ for all y in $B_{n+1} \cap (A_1 \cup \dots \cup A_n)$. Now let $A_{n+1} = [B_{n+1}\backslash(A_1 \cup \dots \cup A_n)] \cup \bigcup\limits_{y\in B_{n+1}\cap(A_1\cup\dots\cup A_n)} \{z_y\}$. By definition of A_{n+1} we get $A_{n+1} \cap A_1 = \dots = A_{n+1} \cap A_n = \emptyset$, A_{n+1} countable and $\sup\limits_{x\in A_{n+1}} |f_{n+1}(x)| =$

$\sup_{x \in B_{n+1}} |f_{n+1}(x)| = 1$. □

Proof of Proposition 4: Let $f_1, \dots, f_n \in E$, $\|f_1\| = \dots = \|f_n\| = 1$. Let $Nf_i = \{x \in X,\ f_i(x) = 0\}$ for all $1 \leqslant i \leqslant n$. Let $U = Nf_1 \cap \dots \cap Nf_n$. By Lemma 6, part (1), U is an open neighbourhood of x_0. Since X has no isolated points, there is $x_1 \in U$, $x_1 \neq x_0$. Take some open neighbourhood V of x_1 such that $x_0 \notin V$ and $V \subseteq U$. Since X is completely regular we can find f in $C_b(X)$ such that $0 \leqslant f(x) \leqslant 1$ for all x in X, $f(x_1) = 1$ and $f(x) = 0$ for all x in $X \backslash V$. Therefore $f \in E$ and $\|f\| = 1$. Now it follows that $\|f - f_1\| = \dots = \|f - f_n\| = 1$ and therefore $1 \in AI(E)$. It remains to show that $2 \in AI(E)$. By Lemma 7 there are some countable subsets $A_1, \dots, A_n$ of X such that $\sup_{x \in A_i} |f_i(x)| = 1$ and $A_i \cap A_j = \emptyset$ for all $1 \leqslant i \neq j \leqslant n$. Without loss of generality let $x_0 \notin A_1 \cup \dots \cup A_n$. We define a function g on $A = \{x_0\} \cup A_1 \cup \dots \cup A_n$. Put $g(x_0) = 0$, $g(x) = -f_i(x)$ for all x in A_i and all $1 \leqslant i \leqslant n$. It follows that $\sup_{x \in A} |g(x)| = \max_{1 \leqslant i \leqslant n} \|f_i\| = 1$. By Lemma 6, part (3), we get some continuous function $\widetilde{g}$ on X such that $\widetilde{g}(x) = g(x)$ for all x in A. Let $h(x) = \min(\max(-1, \widetilde{g}(x)), 1)$ for all x in X. It follows that $h \in E$, $\|h\| = 1$ and $h(x) = g(x)$ for all x in A. Now let $1 \leqslant i \leqslant n$. We get $\|f_i - h\| \geqslant \sup_{x \in A_i} |f_i(x) - h(x)| = 2 \sup_{x \in A_i} |f_i(x)| = 2$, hence $2 \in AI(E)$. By the Intermediate Value Theorem it follows that $AI(E) = [1, 2]$. □

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Institut für Mathematik
Universität Salzburg
Hellbrunnerstraße 34
A-5020 Salzburg
Austria