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Nagoya Math. J.
Vol. 59 (1975), 59–64

ON THE HOPF FIBRATION $S^7 \to S^4$ OVER Z

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§ 1. Statement of the result

Let K be the classical quaternion field over the field $\boldsymbol{Q}$ of rational numbers with the quaternion units $1, i, j, k$, with relations $i^2 = j^2 = -1$, $k = ij = -ji$. For a quaternion $x \in K$, we write its conjugate, trace and norm by $\bar{x}, Tx$ and Nx, respectively. Put

$$A = K \times K\,, \qquad B = \boldsymbol{Q} \times K$$

and consider the map $h: A \to B$ defined by

$$h(z) = (Nx - Ny, 2\bar{x}y)\,, \qquad z = (x, y) \in A\,. \tag{1.1}$$

The map h is the restriction on $\boldsymbol{Q}^8$ of the map $\boldsymbol{R}^8 \to \boldsymbol{R}^5$ which induces the classical Hopf fibration $S^7 \to S^4$ where each fibre is S^3.[1] For a natural number t, put

$$S_A(t) = \{z = (x, y) \in A,\ Nx + Ny = t\}\,, \tag{1.2}$$

$$S_B(t) = \{w = (u, v) \in B,\ u^2 + Nv = t\}\,. \tag{1.3}$$

Then, h induces a map

$$h_t: S_A(t) \to S_B(t^2)\,. \tag{1.4}$$

Now, let $\mathfrak{o}$ be the unique maximal order of K which contains the standard order $\boldsymbol{Z} + \boldsymbol{Z}i + \boldsymbol{Z}j + \boldsymbol{Z}k$. As is well-known, $\mathfrak{o}$ is given by

$$\mathfrak{o} = \boldsymbol{Z}\rho + \boldsymbol{Z}i + \boldsymbol{Z}j + \boldsymbol{Z}k\,, \qquad \rho = \tfrac{1}{2}(1 + i + j + k)\,.$$

The group $\mathfrak{o}^\times$ of units of $\mathfrak{o}$ is a finite group of order 24. The 24 units are: $\pm 1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k)$. We know that the number of quaternions in $\mathfrak{o}$ with norm n is equal to $24 s_0(n)$ where $s_0(n)$ denotes the sum of odd divisors of n.

Received November 6, 1974.

1) H. Hopf, Über die Abbildungen von Sphären auf Sphären niedrigerer Dimension, Fund. Math. 25 (1935) 427–440.

Back to our geometrical situation, put

$$A_Z = \mathfrak{o} \times \mathfrak{o}\ , \qquad B_Z = Z \times \mathfrak{o}$$

and define $S_A(t)_Z, S_B(t)_Z$ by taking z, w in (1.2), (1.3) from A_Z, B_Z, respectively. Then, the map h_t in (1.4) induces a map

$$h_{t,Z}\colon S_A(t)_Z \to S_B(t^2)_Z\ . \tag{1.5}$$

Because of the presence of 2 in (1.1), $h_{t,Z}$ is actually a map $S_A(t)_Z \to S_B(t^2)^*_Z$, where we have put

$$S_B(t^2)^*_Z = \{w = (u, v) \in S_B(t^2)_Z\ , \qquad v \in 2\mathfrak{o}\}\ . \tag{1.6}$$

To each $w \in S_B(t^2)^*_Z$, we shall associate two numbers as follows. First, we denote by a_w the number of $z \in S_A(t)_Z$ such that $h_{t,Z}(z) = w$. Next, we denote by n_w the greatest common divisor of the following six integers:

$$\tfrac{1}{2}(t + u), \tfrac{1}{2}(t - u), \tfrac{1}{2}T(\rho v), \tfrac{1}{2}T(iv), \tfrac{1}{2}T(jv), \tfrac{1}{2}T(kv)\ . \tag{1.7}$$

The purpose of the present paper is to prove the relation:

$$a_w = 24 s_0(n_w)\ , \qquad w \in S_B(t^2)^*_Z\ . \tag{1.8}$$

This is a type of formula which the author has in mind for the algebraic fibration over Z and has proved for Hopf fibrations of type $S^3 \to S^2$.[2)]

For proofs of facts concerning the arithmetic of quaternions the reader is referred to the report by Linnik.[3)]

§ 2. Change of the fibration.

Our problem is to determine the fibre of the map $h_{t,Z}$ in (1.5). To do this, it is convenient to replace the map h by a map f in the following way. Namely, put

$$\begin{aligned}
&\Sigma = \{\sigma = (a, \beta, c) \in \boldsymbol{Q} \times K \times \boldsymbol{Q},\ N\beta = ac\}\ ,\\
&f(z) = (Nx, \bar{x}y, Ny)\ , \qquad z = (x, y) \in A = K \times K\ ,\\
&g(\sigma) = (a - c, 2\beta)\ , \qquad \sigma = (a, \beta, c) \in \Sigma\ ,\\
&\tau(\sigma) = (a, T(\rho\beta), T(i\beta), T(j\beta), T(k\beta), c) \quad \text{and} \quad \phi = \tau f\ .
\end{aligned}$$

2) T. Ono, On the Hopf fibration over Z, Nagoya Math. J. Vol. 56 (1975), 201–207, T. Ono. Quadratic fields and Hopf fibrations (to appear).

3) Yu V. Linnik, Quaternions and Cayley numbers. Some applications of quaternion arithmetic. (Russian), Uspehi Mat. Nauk, IV, 5(33), (1949) 49–98.

$$\begin{tikzcd}
& A \arrow[dl, "h"'] \arrow[dr, "\phi"] \arrow[dd, "f"] & \\
B & & Q^6 \\
& \Sigma \arrow[ul, "g"] \arrow[ur, "\tau"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& A_Z \arrow[dl, "h_Z"'] \arrow[dr, "\phi_Z"] \arrow[dd, "f_Z"] & \\
B_Z & & Z^6 \\
& \Sigma_Z \arrow[ul, "g_Z"] \arrow[ur, "\tau_Z"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& S_A(t)_Z \arrow[dl, "h_{t,Z}"'] \arrow[dr, "\phi_{t,Z}"] \arrow[dd, "f_{t,Z}"] & \\
S_B(t^2)_Z^* & & S(t)_Z \\
& \Sigma(t)_Z \arrow[ul, "g_{t,Z}"] \arrow[ur, "\tau_{t,Z}"'] &
\end{tikzcd}$$

(2.1) (2.2) (2.3)

Clearly, the diagram (2.1) is well-defined and commutative. If we restrict everything on the integral part, we obtain naturally the commutative diagram (2.2), where

$$\Sigma_Z = \Sigma \cap (Z \times \mathfrak{o} \times Z) .$$

Next, consider the portion of (2.2) corresponding to a natural number t as follows. Put

$$\Sigma(t)_Z = \{\sigma = (a, \beta, c) \in \Sigma_Z,\ a + c = t\} ,$$
$$S(t)_Z = \{s = (a, b_1, b_2, b_3, b_4, c) \in Z^6,\ a + c = t\} .$$

Then, f_Z, ϕ_Z induce the maps $f_{t,Z}, \phi_{t,Z}$, respectively. It is almost trivial to check that the diagram (2.3) is well-defined and commutative. The only non-trivial map is $g_{t,Z}$ and it is in fact a bijection: First of all, $g_{t,Z}$ is well-defined, because we have

$$g(\sigma) = (a - c, 2\beta) \quad \text{and} \quad N(g(\sigma)) = (a - c)^2 + 4N\beta = (a + c)^2 = t^2$$

for $\sigma = (a, \beta, c) \in \Sigma(t)_Z$. Next, suppose that $g(\sigma) = g(\sigma')$ with $\sigma = (a, \beta, c), \sigma' = (a', \beta', c') \in \Sigma(t)_Z$. Then we have $\beta = \beta'$ and $a - c = a' - c'$, but, since $a + c = a' + c' = t$, we have $\sigma = \sigma'$, i.e. $g_{t,Z}$ is injective. Finally, take an element $w = (u, v) \in S_B(t^2)_Z^*$, where $u \in Z$ and $v \in 2\mathfrak{o}$ by (1.6). Put $a = \frac{1}{2}(t + u)$, $\beta = \frac{1}{2}v$, $c = \frac{1}{2}(t - u)$. Then $\beta \in \mathfrak{o}$. Substituting $v = 2\beta$ in the relation $u^2 + Nv = t^2$, we see that $a, c \in Z$, $a + c = t$ and $N\beta = ac$, i.e. $\sigma = (a, \beta, c) \in \Sigma(t)_Z$. Furthermore, we have $g(\sigma) = (a - c, 2\beta) = (u, v) = w$, which proves that $g_{t,Z}$ is surjective. Hence, the study of the map $h_{t,Z}$ is reduced to the study of the map $f_{t,Z}$. Now, we can make one more reduction in view of the equality

$$f_{t,Z}^{-1}(\sigma) = f_Z^{-1}(\sigma) , \qquad \sigma \in \Sigma(t)_Z ,$$

which can be verified easily. Therefore, our problem is reduced to the determination of the structure of the fibre

$$X(\sigma) = f_Z^{-1}(\sigma) \qquad \text{for } \sigma = (a, \beta, c) \in \Sigma_Z \text{ with } a + c \geqq 1 .$$

https://doi.org/10.1017/S0027763000016780 Published online by Cambridge University Press

§ 3. Number of solutions

We shall denote by I_K the set of all non-zero fractional right ideals of K with respect to the maximal order $\mathfrak{o}$ and by I_K^+ the subset of I_K consisting of right ideals in $\mathfrak{o}$. For an n-tuple $(a_1, \cdots, a_n) \neq (0, \cdots, 0)$, $a_i \in K$, we denote by $\mathrm{id}_K(a_1, \cdots, a_n)$ the right ideal in I_K generated by $a_1, \cdots, a_n$. As is well-known, every right ideal $\mathfrak{a}$ in I_K is principal: $\mathfrak{a} = \alpha\mathfrak{o}$, $\alpha \in K^\times$. Hence, we may define the norm of $\mathfrak{a}$ by $N\mathfrak{a} = N\alpha$.

LEMMA (3.1) *The following diagram is commutative:*

$$\begin{array}{ccc} A_Z - \{0\} & \xrightarrow{\mathrm{id}_K} & I_K^+ \\ {\scriptstyle \phi_Z}\downarrow & & \downarrow{\scriptstyle N} \\ Z^6 - \{0\} & \xrightarrow{\mathrm{id}_Q} & N . \end{array}$$

Here, the map id_Q *is to take the greatest common divisor of six integers and* $\phi_Z(z) = \tau_Z f_Z(z) = (Nx, T(\rho\bar{x}y), T(i\bar{x}y), T(j\bar{x}y), T(k\bar{x}y), Ny)$.

Proof. Take an element $z = (x, y) \in A_Z - \{0\}$. There is an $\alpha \in \mathfrak{o}$ such that $\mathrm{id}_K(z) = x\mathfrak{o} + y\mathfrak{o} = \alpha\mathfrak{o}$. We must prove that

$$(N\alpha)Z = (Nx)Z + T(\rho\bar{x}y)Z + T(i\bar{x}y)Z + T(j\bar{x}y)Z + T(k\bar{x}y)Z + (Ny)Z . \tag{3.2}$$

Now, since $x\mathfrak{o} + y\mathfrak{o} = \alpha\mathfrak{o}$, we can write $x = \alpha\lambda$, $y = \alpha\mu$ with $\lambda, \mu \in \mathfrak{o}$. Then, $Nx = (N\alpha)(N\lambda) \in (N\alpha)Z$, $Ny = (N\alpha)(N\mu) \in (N\alpha)Z$. Let ε be any one of the four quaternions ρ, i, j, k. Then we have

$$T(\varepsilon\bar{x}y) = T(\varepsilon\bar{\lambda}\bar{\alpha}\alpha\mu) = (N\alpha)T(\varepsilon\bar{\lambda}\mu) \in (N\alpha)Z .$$

From these, we see that the right hand side of (3.2) is contained in the left hand side. To prove the other inclusion, write $\alpha = x\xi + y\eta$ with $\xi, \eta \in \mathfrak{o}$. Then, we have

$$\begin{aligned} N\alpha &= (\bar{\xi}\bar{x} + \bar{\eta}\bar{y})(x\xi + y\eta) \\ &= \bar{\xi}\bar{x}x\xi + \bar{\eta}\bar{y}y\eta + \bar{\xi}\bar{x}y\eta + \bar{\eta}\bar{y}x\xi \\ &= (Nx)(N\xi) + (Ny)(N\eta) + T(\bar{\xi}\bar{x}y\eta) . \end{aligned}$$

Here, obviously, $(Nx)(N\xi) \in (Nx)Z$, $(Ny)(N\eta) \in (Ny)Z$. As for the term $T(\bar{\xi}\bar{x}y\eta)$, we have, first of all, $T(\bar{\xi}\bar{x}y\eta) = T(\eta\bar{\xi}\bar{x}y)$. Next, write $\eta\bar{\xi}$ as

$$\eta\bar{\xi} = a_1\rho + a_2 i + a_3 j + a_4 k \quad \text{with} \quad a_\nu \in Z,\ 1 \leqq \nu \leqq 4 .$$

Then we have

$$T(\eta\bar{\xi}\bar{x}y) = a_1T(\rho\bar{x}y) + a_2T(i\bar{x}y) + a_3T(j\bar{x}y) + a_4T(k\bar{x}y)$$
$$\in T(\rho\bar{x}y)Z + T(i\bar{x}y)Z + T(j\bar{x}y)Z + T(k\bar{x}y)Z ,$$

which proves that the left hand side of (3.2) is contained in the right hand side, q.e.d.

For a natural number n, put

$$I_K^+(n) = \{\mathfrak{j} \in I_K^+,\ N\mathfrak{j} = n\} .$$

This set is non-empty for any n (Lagrange) and contains $s_0(n)$ elements.

Now, take an element $\sigma = (a, \beta, c) \in \Sigma_Z$ with $a + c \geqq 1$ and take a $z = (x, y) \in X(\sigma) = f_Z^{-1}(\sigma)$. Using the same $\alpha \in \mathfrak{o}$ for $z = (x, y)$ as in the proof of (3.1), we have, by (3.1),

$$N(\mathrm{id}_K(z)) = N\alpha = \mathrm{id}_Q(\phi_Z(z)) = \mathrm{id}_Q(\tau_Z f_Z(z)) = \mathrm{id}_Q(\tau_Z(\sigma)) .$$

Hence, if we put

$$n_\sigma = \mathrm{id}_Q(\tau_Z(\sigma)) = \mathrm{id}_Q(a, T(\rho\beta), T(i\beta), T(j\beta), T(k\beta), c) ,$$

we obtain a map

$$d_\sigma : X(\sigma) \to I_K^+(n_\sigma) \quad \text{defined by} \quad d_\sigma(z) = \mathrm{id}_K(z) .$$

Note that $n_\sigma = n_w$ in (1.7) if $w = g_{t,Z}(\sigma)$ for $\sigma \in \Sigma(t)_Z$.

LEMMA (3.3) *The map d_σ is surjective.*

Proof. Take any $\mathfrak{j} \in I_K^+(n_\sigma)$ and write $\mathfrak{j} = \alpha\mathfrak{o}$, $\alpha \in \mathfrak{o}$. Since $a + c \geqq 1$, either $a \neq 0$ or $c \neq 0$. Without loss of generality, we may assume that $a \neq 0$. Take $\omega \in \mathfrak{o}$ such that $\mathrm{id}_K(a, \beta) = a\mathfrak{o} + \beta\mathfrak{o} = \omega\mathfrak{o}$. Then, we have $a = \omega\theta$, $\beta = \omega\psi$ with $\theta, \psi \in \mathfrak{o}$. From (3.1), it follows that

$$\begin{aligned} N\omega &= N(\mathrm{id}_K(a, \beta)) = \mathrm{id}_Q(\phi_Z(a, \beta)) \\ &= \mathrm{id}_Q(Na, T(\rho a\beta), T(ia\beta), T(ja\beta), T(ka\beta), N\beta) \\ &= a\,\mathrm{id}_Q(a, T(\rho\beta), T(i\beta), T(j\beta), T(k\beta), c) = an_\sigma = aN\mathfrak{j} = aN\alpha . \end{aligned}$$

Hence we have $a = N(\omega\alpha^{-1})$. Put $\eta = \omega\alpha^{-1}$, $x = \eta^{-1}a$ and $y = \eta^{-1}\beta$. Since we can also write $x = \alpha\theta$, $y = \alpha\psi$, we see that $z = (x, y) \in A_Z - \{0\}$. We claim that z is an element $\in X(\sigma)$ such that $d_\sigma(z) = \mathfrak{j}$. In fact, firstly, we have

$$\begin{aligned} f(z) &= (Nx, \bar{x}y, Ny) = (N(\eta^{-1}a), a\bar{\eta}^{-1}\eta^{-1}\beta, N(\eta^{-1}\beta)) \\ &= (N\eta)^{-1}(a^2, a\beta, N\beta) = (N\eta)^{-1}a(a, \beta, c) = (a, \beta, c) = \sigma , \end{aligned}$$

which shows that $z \in X(\sigma)$. Next, we have

$$d_\sigma(z) = \mathrm{id}_Q(x, y) = \eta^{-1}\alpha\mathfrak{o} + \eta^{-1}\beta\mathfrak{o} = \eta^{-1}\omega\mathfrak{o} = \alpha\mathfrak{o} = j\ ,$$

which completes the proof of our assertion.

We shall now study the fibre $d_\sigma^{-1}(\mathfrak{j})$ for a fixed $\mathfrak{j} \in I_K^+(n_\sigma)$. Write $\mathfrak{j} = \alpha\mathfrak{o}$ as before, and put $\Gamma_\mathfrak{j} = \alpha\mathfrak{o}^\times\alpha^{-1}$, this being a finite group of order 24 depending only on $\mathfrak{j}$ and not on the choice of the generator α.

LEMMA (3.4) *The group $\Gamma_\mathfrak{j}$ acts on the fibre $d_\sigma^{-1}(\mathfrak{j})$ simply and transitively by $z = (x, y) \mapsto \lambda z = (\lambda x, \lambda y), \lambda \in \Gamma_\mathfrak{j}$.*

Proof. We shall first check that the action is well-defined. This follows from the relations $f(\lambda z) = (N(\lambda x), \bar{x}\bar{\lambda}\lambda y, N(\lambda y)) = N\lambda(Nx, \bar{x}y, Ny) = f(z) = \sigma$

and

$$d_\sigma(\lambda z) = \lambda x\mathfrak{o} + \lambda y\mathfrak{o} = \lambda d_\sigma(z) = \lambda\mathfrak{j} = \lambda\alpha\mathfrak{o} = \alpha\varepsilon\mathfrak{o} = \alpha\mathfrak{o} = \mathfrak{j}\ ,$$

where $\varepsilon \in \mathfrak{o}^\times$. Next, clearly, the isotropy group is trivial everywhere. Finally, let $z = (x, y), z' = (x', y')$ be any two points of $d_\sigma^{-1}(\mathfrak{j})$. Assume, for the moment, that both of x, y are $\neq 0$. Then, from the relation $f(z) = (Nx, \bar{x}y, Ny) = f(z') = (Nx', \bar{x}'y', Ny')$, we can find λ, $\mu \in K$ with $N\lambda = N\mu = 1$ such that $x' = \lambda x$ and $y' = \mu y$. Substituting these in the relation $\bar{x}'y' = \bar{x}y$, we get $\bar{\lambda}\mu = 1$ and hence $\lambda = \mu$. In case where one of x or y, say $y = 0$, then $y' = 0$ automatically, and we have $x' = \lambda x$, $y' = \lambda y$, $N\lambda = 1$, again. In any case, we claim that this λ belongs to $\Gamma_\mathfrak{j}$. In fact, the assumption $d_\sigma(z) = d_\sigma(z') = \mathfrak{j}$ implies that $\mathfrak{j} = \alpha\mathfrak{o} = x\mathfrak{o} + y\mathfrak{o} = x'\mathfrak{o} + y'\mathfrak{o} = \lambda\alpha\mathfrak{o}$ and so $\lambda\alpha = \alpha\varepsilon$ for some $\varepsilon \in \mathfrak{o}$. However, since $N\lambda = 1$, we must have $\varepsilon \in \mathfrak{o}^\times$. Thus, $\lambda = \alpha\varepsilon\alpha^{-1} \in \Gamma_\mathfrak{j}$, q.e.d.

Combining (3.3) and (3.4), we obtain the following relation of cardinalities:

$$\text{(3.5)} \qquad \mathrm{Card}\,(X(\sigma)) = \sum_{\mathfrak{j}} \mathrm{Card}\,(\Gamma_\mathfrak{j}) = 24\,\mathrm{Card}\,(I_K^+(n_\sigma)) = 24 s_0(n_\sigma)\ .$$

Our formula (1.8) is a translation of (3.5) through the bijection $g_{t,Z}$ in the diagram (2.3).

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