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EQUIVALENT WEIGHTS AND STANDARD HOMOMORPHISMS FOR CONVOLUTION ALGEBRAS ON $\mathbb{R}^+$

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Abstract We take a second look at two basic topics in the study of weighted convolution algebras $L^1(\omega)$ on $\mathbb{R}^+$. An early result showed that one could replace the weight ω with a very well-behaved weight without changing the space $L^1(\omega)$ as long as $L^1(\omega)$ was an algebra. We prove the analogous result for measure algebras when $M(\omega)$ is an algebra. This allows us to preserve not only the norm topology but also the relative weak* topology on $L^1(\omega)$. A homomorphism between weighted convolution algebras is said to be standard if it preserves generators of dense principal ideals. The original proofs of standardness and its variants are all based on finding the generator of a particular strongly continuous convolution semigroup. In this paper we give much simpler direct proofs of these results. We also improve the statement and proof of the theorem, giving useful properties equivalent to the standardness of a homomorphism.

Keywords: convolution algebra; weight; standard homomorphism; semigroup

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1. Introduction

The modern study of weighted convolution algebras on $\mathbb{R}^+ = [0, \infty)$ began in the mid 1970s with the fundamental paper of Allan [**1**], which circulated for several years before its publication. In the present paper, we return to two early topics in the theory and give improvements of the basic results in the light of subsequent developments. In § 2 we study equivalence of weights, and in § 3 we study the most important question about homomorphisms.

At first only continuous weights were studied [**1**, **5**], but eventually the appropriate class of measurable weights was found [**10**, **11**, **14**]. For us, a *weight* is a positive Borel function $\omega(x)$ on $\mathbb{R}^+ = [0, \infty)$ for which both ω and $1/\omega$ are locally bounded. When $\omega(x)$ is a weight, then $L^1(\omega)$ is the Banach space of (equivalence classes of) locally integrable functions for which the norm $\|f\| = \|f\|_\omega = \int_0^\infty f(t)\omega(t)\,\mathrm{d}t$ is finite. We define $L^p(\omega)$ in a similar way, and we let $M(\omega)$ be the Banach space of locally finite complex measures μ for which the norm

$$\|\mu\| = \|\mu\|_\omega = \int_{\mathbb{R}^+} \omega(t)\,\mathrm{d}|\mu|$$

is finite. We usually identify the locally integrable function f with the measure $f(t)\,\mathrm{d}t$, so that $L^1(\omega) \subseteq M(\omega)$. We will also need the space $C_0(1/\omega)$, which is the subspace of $L^\infty(1/\omega)$ of continuous functions h for which

$$\lim_{x\to\infty} \frac{h(x)}{\omega(x)} = 0.$$

We are particularly interested in the case where $L^1(\omega)$ is an algebra under the usual convolution product

$$f * g(x) = \int_{\mathbb{R}^+} f(x-t)g(t)\,\mathrm{d}t.$$

In [**5**, § 1] Ghahramani shows that if $\omega(x)$ is continuous, submultiplicative (that is, $\omega(x+y) \leqslant \omega(x)\omega(y)$) and has $\omega(0) = 1$, then $M(\omega)$ is an algebra with the same relation to spaces of continuous functions and to multipliers that $M(G)$ has for a locally compact abelian group. Building on Willis's isomorphic theory [**14**, Lemma 1.2 and Theorem 1.3, pp. 303–306], we proved the same thing in the case when $\omega(x)$ is just right continuous [**11**, Theorem 2.2, p. 592]. More precisely, we say that the weight $\omega(x)$ is an *algebra weight* if it is submultiplicative, right continuous and has $\omega(0) = 1$ (these are the strongly algebraic weights of [**11**, Definition 1.1, p. 590.]). Then, as we showed in [**11**, Theorem 2.2, p. 592], the following holds.

Theorem 1.1. *If $\omega(x)$ is an algebra weight, then both $L^1(\omega)$ and $M(\omega)$ are Banach algebras and we have the following.*

(i) *$M(\omega)$ is isometric to the dual space of $C_0(1/\omega)$ when we identify the measure μ with the linear functional $\langle \mu, h\rangle = \int_{\mathbb{R}^+} h(x)\,\mathrm{d}\mu$.*

(ii) *$M(\omega)$ is isometrically isomorphic to the multiplier algebra of $L^1(\omega)$ when we identify the measure μ with the operator $f \mapsto \mu * f$ on $L^1(\omega)$.*

Theorem 1.1 is important because whenever $L^1(\omega)$ is an algebra we can construct an algebra weight $\omega'(x)$ with $L^1(\omega) = L^1(\omega')$ [**10**,**11**]. For a precise statement of this result, see Theorem 2.4. However, we cannot guarantee that $M(\omega) = M(\omega')$. As the theory developed from its early days, the algebra $M(\omega)$, and in particular the weak* topology on $M(\omega)$ and on $L^1(\omega) \subseteq M(\omega)$, became more important. In Theorem 2.5, we show that if $M(\omega)$ is an algebra, then there is an algebra weight ω' with $M(\omega) = M(\omega')$.

Starting with [**5**, **11**], one of the main parts of the study of weighted convolution algebras on $\mathbb{R}^+$ has been the study of homomorphisms between these algebras. Following [**9**], we say that the continuous homomorphism $\phi : L^1(\omega_1) \to L^1(\omega_2)$ is *standard* if $L^1(\omega_2) * \phi(f)$ is dense in $L^1(\omega_2)$ whenever $L^1(\omega_1) * f$ is dense in $L^1(\omega_1)$. This property is partly motivated by the problem of identifying dense principal ideals in radical $L^1(\omega)$, for which the key counter-example has recently been constructed [**13**]. It turns out that being standard is equivalent to a number of desirable properties of homomorphisms [**9**, Theorem 2.2, p. 280].

In [**9**, Theorem 3.4, p. 284], we showed that ϕ was standard whenever $\omega_2(x)$ was a regulated weight in the sense of Bade and Dales [**2**]. Various variants of this result were proved later. For instance, $L^1(\omega_2) * \phi(f)$ is always weakly* dense when $L^1(\omega) * f$ is norm dense [**12**]. The proof of the original result and its variants included proving that a particular semigroup in $M(\omega_2)$ was strongly continuous, and invoking some variant of [**11**, pp. 600–601] which identifies the domain of the generator of the semigroup with a particular principal ideal. In § 3 we give a simple direct proof of the original standardness theorem and its variants. We also give an improved version of the equivalence theorem.

2. Equivalent weights

We say that two weights $\omega_1(x)$ and $\omega_2(x)$ are *equivalent* if both of the ratios ω_1/ω_2 and ω_2/ω_1 are bounded on $\mathbb{R}^+$. We say that ω_1 and ω_2 are *essentially equivalent* if both ratios are essentially bounded. The following, essentially known, results make clear the importance of these concepts.

Lemma 2.1. *Suppose that ω_1 and ω_2 are two weights on $\mathbb{R}^+ = [0, \infty)$; then we have the following.*

(i) *The weights ω_1 and ω_2 are essentially equivalent if and only if $L^1(\omega_1) = L^1(\omega_2)$. Moreover, when $L^1(\omega_1) = L^1(\omega_2)$, the norms $\|f\|_{\omega_1}$ and $\|f\|_{\omega_2}$ are equivalent.*

(ii) *The weights ω_1 and ω_2 are equivalent if and only if $M(\omega_1) = M(\omega_2)$. Moreover, when $M(\omega_1) = M(\omega_2)$, the norms $\|\mu\|_{\omega_1}$ and $\|\mu\|_{\omega_2}$ are equivalent.*

Proof. Everything is straightforward and elementary except for the statements about equivalent norms. But for any weight ω, the Banach space $L^1(\omega)$ is continuously embedded in the Fréchet space $L^1_{\mathrm{loc}}(\mathbb{R}^+)$ and $M(\omega)$ is continuously embedded in $M_{\mathrm{loc}}(\mathbb{R}^+)$. Hence, the statements about equivalent norms follow easily from the closed graph theorem. □

Since we will allow changes to equivalent norms, the condition for $L^1(\omega)$ and $M(\omega)$ becoming algebras is weaker than the usual submultiplicative conditions. The sufficient conditions are given in the following definition.

Definition 2.2. Suppose that ω is a weight and K is a positive number. We say that ω is *K-submultiplicative* if $\omega(x + y) \leqslant K\omega(x)\omega(y)$ for all x and y in $\mathbb{R}^+$, and that ω is *essentially K-submultiplicative* if $\omega(x + y) \leqslant K\omega(x)\omega(y)$ for almost every (x, y) in the closed first quadrant of $\mathbb{R}^+$. When $K = 1$, we say ω is *submultiplicative* or *essentially submultiplicative*.

For right continuous weights, we have the following useful observation.

Lemma 2.3. *Suppose that ω_1 and ω_2 are right continuous weights. Then ω_1 is K-submultiplicative if and only if it is essentially K-submultiplicative. Similarly, ω_1 and ω_2 are equivalent if and only if they are essentially equivalent.*

For stronger versions of the two parts of the above lemma, which involve Lebesgue points, see [**10**, Lemma 2.4 (c), p. 534] and [**10**, pp. 533–534], respectively.

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We now determine when a given weight $\omega(x)$ is essentially equivalent to an algebra weight, and when it is equivalent. The result for essential equivalence extracts what now seem the most important points, from the more general results of [**10**, **11**].

Theorem 2.4. *If $\omega(x)$ is a weight on $\mathbb{R}^+$, then the following are equivalent:*

(i) *ω is essentially K-submultiplicative for some K;*

(ii) *$L^1(\omega)$ is an algebra;*

(iii) *there is an algebra weight ω' which is essentially equivalent to ω.*

Moreover, ω is essentially K-submultiplicative for a given K if and only if $\|f * g\|_\omega \leqslant K\|f\|_\omega\|g\|_\omega$ for all f and g in $L^1(\omega)$.

The equivalence of (i) and (ii) and the 'moreover' statement are in [**10**, Lemma 2.4, p. 534]. If (iii) holds, it follows from Theorem 1.1 and Lemma 2.1 (i) that $L^1(\omega) = L^1(\omega')$ is an algebra, so (ii) holds. We will outline a simplified proof that the equivalent conditions (i) and (ii) imply (iii) after we prove the following analogous result for $M(\omega)$.

Theorem 2.5. *If $\omega(x)$ is a weight on $\mathbb{R}^+$, then the following are equivalent:*

(i) *ω is K-submultiplicative for some K;*

(ii) *$M(\omega)$ is an algebra;*

(iii) *there is an algebra weight ω' which is equivalent to ω.*

Moreover, ω is K-submultiplicative for a given K if and only if $\|\mu * \nu\|_\omega \leqslant K\|\mu\|_\omega\|\nu\|_\omega$ for all μ and ν in $M(\omega)$.

The importance of the above theorem is that it tells us that as long as the weights are submultiplicative, or even just K-submultiplicative for some K, all the results proved for the special case of what we call algebra weights remain true isomorphically, though not isometrically. For instance, a continuous non-zero homomorphism $\phi : L^1(\omega_1) \to L^1(\omega_2)$ has a unique continuous extension to the corresponding measure algebras, but, unlike in the algebra weights case [**11**, Theorem 8.4, p. 596], the extension need not have the same norm. In particular, Theorems 1.1 and 2.5 give us the following result.

Corollary 2.6. *If $M(\omega)$ is an algebra (equivalently, if $\omega(x)$ is a K-submultiplicative weight), then $M(\omega)$ is naturally isomorphic to the dual space of $C_0(1/\omega)$ and to the multiplier algebra of $L^1(\omega)$.*

We will now prove Theorem 2.5. For (i) $\Rightarrow$ (ii), the standard proof for the submultiplicative case where $K = 1$ works for general K and shows that $\|\mu * \nu\|_\omega \leqslant K\|\mu\|_\omega\|\nu\|_\omega$. For (ii) $\Rightarrow$ (i), first recall that $M_{\mathrm{loc}}(\mathbb{R}^+)$ is a Fréchet algebra under the seminorms $|\mu|[0, a)$ for a positive. Since $M(\omega)$ is an algebra continuously embedded in $M_{\mathrm{loc}}(\mathbb{R}^+)$, it follows from the closed graph theorem that convolution by a fixed μ in $M(\omega)$ is a bounded linear operator on $M(\omega)$. Thus, multiplication in $M(\omega)$ is separately continuous. A standard argument, using the uniform boundedness theorem, then shows that

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convolution in $M(\omega)$ is a bounded bilinear operator. That is, there is a positive K with all $\|\mu * \nu\|_\omega \leqslant K\|\mu\|_\omega\|\nu\|_\omega$. Applying this to the point masses δ_x and δ_y yields $\omega(x+y) = \|\delta_{x+y}\|_\omega \leqslant K\|\delta_x\|_\omega\|\delta_y\|_\omega = K\omega(x)\omega(y)$. If (iii) holds, then $\omega'(x)$ is submultiplicative, so $M(\omega')$ is an algebra. But Lemma 2.1 shows that $M(\omega) = M(\omega')$, so (ii) must hold.

Finally, we assume (i) and (ii) and prove (iii). Since $\omega(x)$ is K-submultiplicative, $K\omega(x)$ is a submultiplicative weight, so $\lim_{x\to\infty} \omega(x)^{1/x} = \lim_{x\to\infty}(K\omega(x))^{1/x}$ exists and is finite. Hence, when r is large enough, $\omega(x)/\mathrm{e}^{rx}$ is bounded and K-submultiplicative; so we can assume without loss of generality that $\omega(x)$ is a bounded weight satisfying (i) and (ii).

We define $\omega_1(x) = \sup\{\omega(t) : t \geqslant x\}$. It is clear that $\omega(x)$ is (weakly) decreasing and right continuous. We now show that ω and ω_1 are equivalent weights. By our definition, $\omega(x) \leqslant \omega_1(x)$. For the reverse inequality, let M be an upper bound for $\omega(x)$. We fix x and consider $h \geqslant 0$. Then we have $\omega(x+h) \leqslant K\omega(h)\omega(x) \leqslant KM\omega(x)$. Hence, $\omega_1(x) = \sup_{h\geqslant 0}\omega(x+h) \leqslant KM\omega(x)$. So ω and ω_1 are equivalent weights, and hence $M(\omega_1) = M(\omega)$ is an algebra.

We conclude the proof with the following lemma, which we will also use in our discussion of Theorem 2.4.

Lemma 2.7. *Suppose that $\omega_1(x)$ is a right continuous decreasing weight. If $M(\omega_1)$ is an algebra, then ω_1 is equivalent to a decreasing algebra weight.*

Proof of lemma 2.7. Since $M(\omega_1)$ is an algebra, ω_1 must be K-submultiplicative for some K. Then $K\omega(x)$ is submultiplicative. Let $\omega'(x) = \min(1, K\omega(x))$. It is clear that $\omega'(x)$ is a decreasing, right continuous, submultiplicative weight equivalent to $\omega_1(x)$. By construction, $\omega_1(x) \leqslant 1$. Since $\omega'(x)$ is submultiplicative, we also have $\omega'(0) \geqslant 1$. Hence, $\omega'(0) = 1$, and therefore $\omega'(x)$ is an algebra weight equivalent to $\omega_1(x)$. This completes the proof of the lemma and of Theorem 2.5. □

We will now outline how to extract a proof that (i) and (ii) imply (iii) in Theorem 2.4 from the more general results of [**10**]. One first shows that $L^1(\omega)$ is invariant under right translations and that the right translations are bounded operators. This is [**10**, Lemma 3.3, p. 539] and the discussion in the following paragraph (or see [**4**, p. 390]). If we let $M_\omega(x)$ be the norm of right translation by x, we can show that

$$\operatorname*{ess\,lim}_{x\to\infty} \omega(x)^{1/x} \leqslant \lim_{x\to\infty} M_\omega(x)^{1/x} < \infty$$

(see [**10**, Lemma 2.8 (E), p. 537]). So we can find an $r > 0$ for which $\omega(x)/\mathrm{e}^{rx}$ is essentially bounded. Hence, one can assume without loss of generality that $\omega(x)$ is bounded. One then defines $\omega_1(x)$ as the essential supremum of $\{\omega(t) : t \geqslant x\}$. As in the proof of Theorem 2.5, we see that $\omega(x)$ is decreasing and right continuous and that ω_1 is essentially equivalent to ω (cf. the proof of Theorem 3.1 in [**10**, pp. 537–538]). Then $L^1(\omega) = L^1(\omega_1)$, which is an algebra, and hence ω_1 is essentially K-submultiplicative. Since ω_1 is also right continuous, it follows from Lemma 2.3 that ω_1 is actually K-submultiplicative so that $M(\omega_1)$ is an algebra. Finally, one uses Lemma 2.7 to find an algebra weight $\omega'(x)$ equivalent to $\omega_1(x)$, and hence essentially equivalent to $\omega(x)$.

3. Standard homomorphisms

Throughout this section, ω_1 and ω_2 are algebra weights and ϕ is a continuous non-zero homomorphism from $L^1(\omega_1)$ to $L^1(\omega_2)$. We also use ϕ to designate the unique extension of $\phi : L^1(\omega_1) \to L^1(\omega_2)$ to the corresponding measure algebras [**11**, Theorem 3.4, p. 596]. It will follow from Theorem 2.5 that our results remain valid as long as ω_1 and ω_2 are submultiplicative weights or, more generally, if $M(\omega_1)$ and $M(\omega_2)$ are algebras. Results that do not mention the measure algebras or the weak* topologies remain true in the more general case that $L^1(\omega_1)$ and $L^1(\omega_2)$ are algebras.

If f is a locally integrable function on $\mathbb{R}^+$, we let $\alpha(f)$ be the inf of the support of f, with $\alpha(0) = \infty$. If ω is a weight and $a \geqslant 0$, then $L^1(\omega)_a = \{f \in L^1(\omega) : \alpha(f) \geqslant a\}$. When $L^1(\omega)$ is an algebra, the spaces $L^1(\omega)_a$ are the *standard ideals* of $L^1(\omega)$. The function f in $L^1(\omega)$ is *standard* if the closure of $L^1(\omega) * f$ is a standard ideal. Thus, if $\alpha(f) = 0$, the f is standard if and only if the principal ideal $L^1(\omega) * f$ is dense in $L^1(\omega)$. Recall that the homomorphism $\phi : L^1(\omega_1) \to L^1(\omega_2)$ is *standard* if whenever $L^1(\omega_1) * f$ is dense in $L^1(\omega_1)$, then $L^1(\omega_2) * \phi(f)$ is dense in $L^1(\omega_2)$.

Following Bade and Dales (see [**2**, Definition 1.3, p. 81]), we say that the algebra weight ω is *regulated* at $a \geqslant 0$ if

$$\lim_{x\to\infty} \frac{\omega(x+b)}{\omega(x)} = 0 \quad \text{for all } b > a.$$

Bade and Dales characterize compactness of convolution in terms of regulated weights [**2**, Theorem 2.7, p. 90]. We will need the interpretation of their results in terms of convergence (see [**9**, Theorem 3.2, p. 284] and [**6**, Theorem 2.3, p. 509]). We now give simple direct proofs of the fundamental standard homomorphism results of [**9**, **12**].

Theorem 3.1. *Suppose that $\phi : L^1(\omega_1) \to L^1(\omega_2)$ is a continuous non-zero homomorphism. Suppose that f in $L^1(\omega_1)$ has $L^1(\omega_1) * f$ norm dense in $L^1(\omega_1)$. Then we have the following.*

(i) *$L^1(\omega_2) * \phi(f)$ is dense in the (relative) weak* topology on $L^1(\omega_2)$.*

(ii) *If ω_2 is regulated at some $a \geqslant 0$, then $L^1(\omega_2) * \phi(f)$ is norm dense (so that ϕ is standard).*

Notice that the hypothesis in (ii) just says that

$$\lim_{x\to\infty} \frac{\omega_2(x+b)}{\omega_2(x)} = 0 \quad \text{for some } b > 0.$$

We will need the following simple result from [**12**, Lemma 3.2, p. 1678].

Lemma 3.2. *If $L^1(\omega_1) * f$ is norm dense, then $\mathrm{cl}(L^1(\omega_2) * \phi(f))$ contains $L^1(\omega_2) * \phi(g)$ for all g in $L^1(\omega_1)$.*

Proof of Theorem 3.1. Let h be an arbitrary element of $L^1(\omega_2)$. Choose a bounded approximate identity $\{e_n\}$ in $L^1(\omega_1)$. Then $\lim \phi(e_n) * \phi(f) = \lim \phi(e_n * f) = \phi(f)$. Hence,

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$\phi(e_n) * \phi(f)$ converges weak* to $\phi(f) = \delta_0 * \phi(f) \neq 0$, where δ_0 is the point mass at 0 and the identity of $M(\omega_2)$. It then follows from [**11**, Lemma 3.2, p. 595] that $\phi(e_n) \to \delta_0$ weakly* in $M(\omega_2)$ and $\phi(e_n) * h$ converges weakly* to $\delta_0 * h = h$. But each $\phi(e_n) * h$ belongs to $\mathrm{cl}(L^1(\omega_2) * \phi(f))$ by Lemma 3.2. Thus, h belongs to the weak* closure of $L^1(\omega_2) * \phi(f)$, and (i) is proved.

Now suppose that ω_2 is regulated at a. Since $\phi(e_n)$ converges to δ_0, it follows from [**9**, Theorem 3.2, p. 284] that $\phi(e_n) * h$ converges in norm to h for all h in $L^1(\omega_2)_a$. If $a = 0$, this proves (ii). For the general case, let $J = \{g \in L^1(\omega_2) : \phi(e_n) * g \to g\}$, which is a closed ideal in $L^1(\omega_2)$. Since the closed ideal J contains a standard ideal, it must also be a standard ideal (see [**10**, Lemma 6.2, p. 548], [**3**, Theorem 4.7.66, p. 552] or [**1**]). Since $\phi(f)$ belongs to J and has $\alpha(\phi(f)) = 0$, it then follows that J is all of $L^1(\omega_2)$. This completes the proof of the theorem. □

Using essentially the same proof as in Theorem 3.1 (i), we can give a short proof of the L^p version of the standard homomorphism theorem [**7**, Theorem 3.1, p. 53].

Theorem 3.3. *Suppose that $\phi : L^1(\omega_1) \to L^1(\omega_2)$ is a continuous non-zero homomorphism and that $1 < p < \infty$. If $L^1(\omega_1) * f$ is norm dense in $L^1(\omega_1)$, then $L^p(\omega_2) * \phi(f)$ is norm dense in $L^p(\omega_2)$.*

Proof. Let h be an arbitrary element of $L^p(\omega_2)$, and choose an approximate identity $\{e_n\}$ in $L^1(\omega_1)$. As in the proof of Theorem 3.1, $\phi(e_n)$ converges weakly* to δ_0 in $M(\omega_2)$. Then, by [**7**, Lemma 2.1, p. 51], $\phi(e_n) * h$ converges weakly in $L^p(\omega_2)$ to $\delta_0 * h = h$. The L^p version of Lemma 3.2 holds, with essentially the same proof, so all $\phi(e_n) * h$ belong to the norm closure of $L^p(\omega_2) * \phi(f)$. Hence, $L^p(\omega_2) * \phi(f)$ is weakly dense in $L^p(\omega_2)$. But every weakly dense subspace is also norm dense, so this completes the proof of the theorem. □

There is another general standard homomorphism result [**8**, Theorem 5.12, p. 317]. Like the original proofs of Theorems 3.1 and 3.3, its proof rests on showing a particular semigroup to be strongly continuous. We give a direct proof of this result.

Suppose that $\omega_3(x)$ is a weight on $\mathbb{R}^+$. Following [**8**, definition 1.2, p. 305], we say that ω_3/ω_2 is a *convergence factor* for ω_2 provided that ω_3/ω_2 is bounded (so that $L^1(\omega_2) \subseteq L^1(\omega_3)$), and that $\lambda_n \to \lambda$ weakly* in $M(\omega_2)$ implies that for all h in $L^1(\omega_2)$ we have $\lambda_n * h \to \lambda * h$ in the norm of $L^1(\omega_3)$. For necessary and sufficient conditions for ω_3/ω_2 to be a convergence factor, see [**8**, Theorem 3.1, p. 310] and [**8**, Theorem 4.1, p. 312]. It is sufficient for ω_3/ω_2 to be integrable (see [**6**, Theorem 3.2, p. 512], [**8**, Theorem 4.1, p. 312]) or for

$$\lim_{x\to\infty} \frac{\omega_3(x+b)}{\omega_2(x)} = 0 \quad \text{for all } b > 0$$

(see [**8**, Theorem 3.1 (c), p. 310], [**8**, Theorem 3.2, p. 311]). With these preliminaries, we can state and prove our final standard homomorphism result, which is a variant of [**8**, Theorem 5.12, p. 317].

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Theorem 3.4. *Suppose that $\phi : L^1(\omega_1) \to L^1(\omega_2)$ is a continuous non-zero homomorphism. If ω_3/ω_2 is a convergence factor for ω_2, then for all f in $L^1(\omega_1)$ with $L^1(\omega_1) * f$ norm dense in $L^1(\omega_1)$, we have that $L^1(\omega_2) * \phi(f)$ is norm dense in $L^1(\omega_3)$.*

Proof. Choose an element h in $L^1(\omega_2)$ and a bounded approximate identity $\{e_n\}$ in $L^1(\omega_1)$. As in the proof of Theorem 3.1, $\phi(e_n) \to \delta_0$ weak* in $M(\omega_2)$, and all $\phi(e_n) * f$ belong to the $L^1(\omega_2)$ closure of $L^1(\omega_2) * \phi(f)$. Since ω_3/ω_2 is a convergence factor for ω_2, we have that $\phi(e_n) * h \to \delta_0 * h = h$ in the norm of $L^1(\omega_3)$. Hence, the closure of $L^1(\omega_2) * \phi(f)$ in $L^1(\omega_3)$ contains $L^1(\omega_2)$. Since $L^1(\omega_2)$ is dense in $L^1(\omega_3)$, this completes the proof of the theorem. □

When ω_3 is an algebra weight, Theorem 3.4 says that ϕ is standard as a homomorphism from $L^1(\omega_1)$ to $L^1(\omega_3)$. In this case it is enough for ω_3/ω_2 to be a convergence factor at some $a \geqslant 0$ [**8**, Definition 1.2, p. 305]. The proof for $a > 0$ is similar to the proof of Theorem 3.1 (ii) when $a > 0$ (cf. [**8**, Theorem 5.11, p. 317]).

One reason that standardness is a useful property for a homomorphism is that it is equivalent to many other desirable properties [**10**, Theorem 2.2, p. 280]. Because of Lemma 3.2, we can add to this list of equivalences that it is enough for $L^1(\omega_2) * \phi(f)$ to be dense in $L^1(\omega_2)$ for a single f in $L^1(\omega_1)$.

A stronger form of the equivalence theorem [**10**, Theorem 2.4, pp. 281–282] identifies a single subspace I of $L^1(\omega_1)$, with multiple descriptions, which is the 'largest' subspace of $L^1(\omega_2)$ on which ϕ acts like a standard homomorphism. Standardness then occurs precisely when $I = L^1(\omega_2)$, so that each of the different descriptions of I yields a different property equivalent to standardness. The following result is an improvement of the formulation in [**10**].

Theorem 3.5. *Suppose that $\phi : L^1(\omega_1) \to L^1(\omega_2)$ is a continuous non-zero homomorphism, where ω_1 and ω_2 are algebra weights, and that f is an element of $L^1(\omega_1)$ for which $L^1(\omega_1) * f$ is norm dense in $L^1(\omega_1)$. Let $\{e_n\}$ be a bounded approximate identity in $L^1(\omega_1)$. Let $I = \mathrm{cl}[L^1(\omega_2) * \phi(f)]$, and let $\mu_t = \phi(\delta_t)$ for each $t \geqslant 0$. Then I is a weak* dense ideal equal to each of the following subspaces of $L^1(\omega_2)$.*

(i) $I_A = L^1(\omega_2) * \phi(f')$, *where f' is any element of $L^1(\omega_2)$ for which $L^1(\omega_1) * f'$ is dense.*

(ii) $I_B = \{h \in L^1(\omega_2) : \phi(e_n) * h \to h\}$.

(iii) $I_C = \{h \in L^1(\omega_2) : \lim_{t\to 0^+} \mu_t * h = h\}$. *That is, I_C is the largest subspace on which $\{\mu_t\}$ acts as a strongly continuous semigroup.*

(iv) *I_D is the collection of all elements h in $L^1(\omega_2)$ for which $\phi(\lambda_n) * h \to \phi(\lambda) * h$ whenever $\{\lambda_n\}$ is a bounded sequence in $M(\omega_1)$ for which $\lambda_n \to \lambda$ in the strong operator topology; that is, $\lambda_n * g \to \lambda * g$ for all g in $L^1(\omega_1)$.*

(v) *I_E is the collection of all h in $L^1(\omega_1)$ for which $\phi(\lambda_n) * h \to \phi(\lambda) * h$ for all nets $\{\lambda_n\}$ in $M(\omega_1)$ which converge to λ in the strong operator topology.*

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(vi) $I_F = \{\phi(g) * h : g \in L^1(\omega_1) \text{ and } h \in L^1(\omega_2)\}$.

(vii) $I_G = \{\phi(g) * \mu : g \in L^1(\omega_1) \text{ and } \mu \in M(\omega_2)\}$.

Proof. It follows from Theorem 3.1 (i) that I is weak* dense, and it follows from Lemma 3.2 that $I = I_A$. Also, it is clear that I_D is a subset of I_B and also of I_C. We complete the proof by proving the following three assertions:

(a) each of I, I_D, I_E, I_F and I_G contains I_F and is contained in I_D;

(b) $I_B \subseteq I_F$;

(c) there is an f' with $L^1(\omega_1) * f'$ dense for which $v = \phi(f')$ satisfies $\mathrm{cl}(I_C * v) = I_C$, so that $I_C \subseteq \mathrm{cl}(L^1(\omega_2) * \phi(f')) = I$.

We now prove (a). It is clear that $I_F \subseteq I_G$ and $I_E \subseteq I_D$. Suppose that h belongs to I_G, and let $\{\lambda_n\}$ be a net in $M(\omega_1)$ which converges in the strong operator topology to λ in $M(\omega_1)$.

We let $h = \phi(g) * \mu$ and then have $\phi(\lambda_n) * h = \phi(\lambda_n) * \phi(g) * \mu = \phi(\lambda_n * g) * \mu \to \phi(\lambda * g) * \mu = \phi(\lambda) * h$. Hence, $I_G \subseteq I_E$.

Clearly, $L^1(\omega_2) * \phi(f) \subseteq I_F \subseteq I_D$. But for any bounded sequence $\{\nu_n\}$ and element ν in $M(\omega_2)$, the set $\{h \in L^1(\omega_2) : \nu_n * h \to \nu * h\}$ is a closed subspace. Hence, $I \subseteq I_D$. On the other hand, it follows from Lemma 3.2 that $I_F \subseteq I_D$. This completes the proof of (a).

The set I_B is a closed ideal in $L^1(\omega_2)$, so I_B is a $L^1(\omega_1)$ Banach module under the multiplication $g \cdot h = \phi(g) * h$. It then follows from the Cohen factorization theorem for modules that each element of I_B has an expression of the form $\phi(g) * h$ for some $g \in L^1(\omega_1)$, and, therefore, that $h \in I_B \subseteq L^1(\omega_2)$. Hence, $I_B \subseteq I_F$. This proves (b).

To prove (c), we adapt the proof in [**9**, pp. 281–282]. Notice that μ_t acts as a strongly continuous semigroup on I_C. We first consider the case when $u(t) \equiv 1$ belongs to $L^1(\omega_1)$, and we let $f' = u$. In this case we can show that convolution by $-v$, where $v = \phi(f')$, is the inverse of the generator of the semigroup μ_t on I_C (for the details, see [**9**] and its references to [**11**]). Since the generator of a strongly continuous semigroup has dense domain, this shows that $I_C * v$ is dense in I_C as required. In the general case, choose some e^{-rt} in $L^1(\omega_1)$. Since multiplication by e^{-rt} is an isometric isomorphism from $L^1(\mathrm{e}^{-rt}\omega_1(t))$ onto $L^1(\omega_1)$, we can apply the first case to $\psi : L^1(\mathrm{e}^{-rt}\omega_1(t)) \to L^1(\omega_2)$ defined by $\psi(g) = \phi(\mathrm{e}^{-rt}g)$. We then let $f' = \mathrm{e}^{-rt} = \mathrm{e}^{-rt}u(t)$, and let $v = \phi(f') = \psi(u)$, and use the first case on ψ. This completes the proof of (c), and therefore of the theorem. □

There are a number of other equivalent descriptions of I. For instance, it is easy to see that I is the smallest closed ideal containing the range of ϕ. Also, standard semigroup arguments show that $I = I_C = \mathrm{cl}[\bigcup_{t>0} \mu_t * L^1(\omega_2)]$ (see the proof of [**11**, corollary 3.14, p. 602]).

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