PART II

HISTORY AND PHILOSOPHY OF SCIENCE

https://doi.org/10.1086/psaprocbienmeetp.1986.1.193112 Published online by Cambridge University Press

Leibniz on Continuity[1]

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1. Introduction

Leibniz never tired of stressing the fundamental importance of the concept of continuity for philosophy, nor was he shy of attributing major importance to his own struggle through "the labyrinth of the continuum" for the subsequent development of his whole system of thought. Unfortunately, however, his own thought on the subject is something of a labyrinth itself, and from a modern point of view many of his pronouncements are apt to seem blatantly contradictory.

Certain quotations seem to commit him unambiguously to atomism. Thus to de Volder he writes: "Matter is not continuous, but discrete.... The same holds for changes, which are not truly continuous." (To de Volder, 11th October 1705: G.II.279).[2]

"Continuous quantity is something ideal, which pertains to possibles... . In Actuals there is only discrete Quantity...." (To de Volder, 19th January 1706 G.II.282). Here he appears to be saying that, in space and time, which are continuous but ideal, reality always consists in a strictly discrete ordering.

Yet juxtaposed with these remarks, often in the very same passages, are other remarks which appear to directly contradict them. Thus in the same breath Leibniz describes matter as discrete, he also describes it as being "actually divided to infinity", and thus dense, and further maintains that "no assignable part of [continuous] space is devoid of matter." (G.II.279). And although changes are likewise "not truly continuous", nevertheless according to Leibniz's much vaunted Law of Continuity, "no transition in nature ever occurs by a leap" (Initia Rerum Metaphysica Mathematicarum: GM.VII.25).

Russell's judgement on all this was characteristically extreme: "In spite of the law of continuity, Leibniz's philosohy may be described as a complete denial of the continuous." (Russell 1900, p. 111). I take the opposite view, and shall attempt to show how Leibniz's conception of continuity was not only non-contradictory, but actually quite profound. A measure of the depth of his conception, I shall maintain, is found in

PSA 1986, Volume 1, pp. 107-115

the fact that much the same approach to continuity was reinvented in the 20th century by the founders of modern Combinatorial Topology.

2. Leibniz's account of continuity

As a first step towards establishing this, I shall attempt to reconstruct the context in which Leibniz developed his views. I shall lay particular stress on the physics of fluids, because I think the problem of the continuity of the flow of matter in a plenum was a decisive factor in the development of Leibniz's views, and one that has been unduly overlooked.

There is a passage in Descartes' Principles (1644) that Leibniz alludes to again and again in his writings. It concerns the possibility of motion in a plenum. The problem is this. Suppose matter is flowing in a confined receptacle, but from a smaller space into a larger one; how is this possible without rarefaction, that is, without the creation of void spaces? The answer to this objection, according to Descartes, is easy: "We have only to notice the way that all the inequalities of place can be compensated by inequalities of velocity" (Principles, II, 33; 1644, p. 59) --in other words, the matter must travel faster through the narrower spaces, and vice versa. This is a rudimentary -- and as far as I know, the first formulation of what is now called the Law of Continuity of Matter: $\partial p/\partial t+\nabla j = 0$. But, as Descartes recognized, it is an immediate consequence of this that there must be at least some part of the fluid matter that adjusts its shape by infinitely gradual degrees: "and for this to happen, all imaginable parts of this piece of matter --in fact, innumerable parts-- must be to some degree displaced from their positions relative to one another; and this displacement is actual division."

There is, therefore, an "infinite or indefinite division of matter", so that however small a part we conceive "we must conceive it as undergoing actual division into still smaller parts".

"What Descartes says here", says Leibniz in his Critical Thoughts on the General Part of the Principles of Descartes of 1692, "is most beautiful and worthy of his genius... . Yet he does not seem to have weighed sufficiently the importance of this last conclusion." (G.IV.354-392: Loemker 1969, p. 393). But what is the importance of this conclusion of Descartes' that matter must be infinitely divided, why does Leibniz see it as being so significant? Because, as I shall try to show, in the context of his own thought on continuity it is precisely this actual division of matter at any given instant that precludes its being truly continuous: if matter is actually infinitely divided, then it cannot be constituted by a continuum of physical points.

As it stands, this seems a very enigmatic conclusion to draw from Descartes' argument. For at first sight the more obvious conclusion would appear to be that, if matter is indefinitely divided by the differing motions of its parts, it must be actually infinitely divided all the way down into indivisible points. In fact, this is the conclusion Leibniz himself originally adopts. In July of 1676, in his Paris notes, he writes: "It seems to follow from a solid in a liquid that perfectly fluid matter is nothing but a multitude of infinitely small points or bodies less than any assignable ones, or that there is

https://doi.org/10.1086/psaprocbienmeetp.1986.1.193112 Published online by Cambridge University Press

necessarily an interspersed metaphysical vacuum." (Loemker 1969, p. 158).

Here Leibniz appears to be espousing the theory of continuity that Galilei had proposed in his Two New Sciences according to which a continuous line is composed of an infinity of "unquantifiable parts" (parti non quante), separated by "unquantifiable voids" (Drake 1974, p. 33) - corresponding here to Leibniz's "bodies less than any assignable ones" separated by what he calls "inassignable metaphysical voids". But there is a subtle yet important difference, one that is seminal for the later development of Leibniz's views. This is Leibniz's denial that the physical plenum formed by these inassignably small bodies and voids does in fact constitute a continuum:

> A physical plenum is consistent with an inassignable, metaphysical vacuum... . No absurdity follows from this, for it would follow that a perfect fluid is not a continuum, but discrete or multitude of points. From this it does not further follow that the continuum is composed of points, since liquid matter will not be a true continuum, though space will be. Hence it is clear, further, how great the difference between space and matter. Matter alone can be explained by a plurality without continuity... . Matter therefore is discrete being, not continuous. It is merely contiguous and is united by motion or by some mind. (Loemker 1969, p. 158).

Let me draw your attention to the way Leibniz uses the word 'discrete' in this passage. Matter is said to be discrete because its parts--here, its infinitesimal parts--are actually discriminated from each other by their individual motions, and being discriminable these parts are therefore accountable. But since the voids that separate them are themselves "inassignable" or infinitesimally small, the parts of matter may be said to be next to each other; that is, matter is contiguous, rather than truly continuous.

We can already discern in this early position the basis for Leibniz's mature statements that matter is discrete--in the sense that it is divided into actual, discriminable parts, each contiguous to the next--and yet that no assignable part of space is devoid of matter. But the position depends on an interpretation of "inassignables" as actually infinitely small parts; that is, on an interpretation of the infinitesimals of his newly discovered differential calculus as infinitely small actuals.[3] But within the next three months of the same year, Leibniz was to change his mind on this interpretation. He realized that his infinitesimal differences are very different from the indivisibles of Galilei and Cavalieri. For hereas Cavalieri's indivisibles had a constant (though zero!) thickness, Leibniz's infinitesimal "differences" dx themselves varied with x, and thus also had infinitesimal differences (the second order differentials d^2x), and so on down to infinity. Moreover, the same finite line could be expressed by many different but equivalent infinite "progressions" of differences.[4] Thus these infinitely small elements of the continuum, being themselves infinitely divisible, could not be unities or first elements out of which the continuum could be constituted.[5] Indeed, the fact that it is possible to effect many different resolutions of the same line into different progressions of infinitesimals strongly suggests the fictional character of infinitesimals; and, in fact, this was Leibniz's intepretation of them for the rest of his career. But let us return to the problem of the physical continuum of matter.

In the Pacidius Philalethi, written on his voyage from England to Holland in October of the same year, Leibniz upholds his former distinction between merely contiguous matter and the continuous space that it fills. But now he rejects the idea of a "perfect fluid", matter that can be "resolved into a powder (so to speak) consisting of points".[6] Now the actually infinite division, instead of reaching a limit in indivisible points, is reinterpreted by him as an open-ended division, coming to no limit. Thus the particular motions of the infinity of parts of a body of matter at a given instant result in a particular infinite division (partition) of that body. At a subsequent instant, different instantaneous motions result in a different infinite partition.

But if matter were simply divided into an infinity of rigid, extended parts, motion in plenum would still be impossible. What Leibniz proposes instead is a conception of a matter that is "everywhere pliant", but which has "a certain unequal resistance to bending". Thus even though matter is divided to infinity at any given moment, its parts or cells resist such division with a kind of inherent elasticity--they are "merely formed for a while and then transformed"-- resulting in a different infinite partition at each assignable instant:

> If a perfectly fluid body is assumed, a finest division or division into minima cannot be denied; but a body that is indeed everywhere pliant, though not without a certain unequal resistance to bending, still has cohering parts, although these are variously opened up and folded together. Accordingly the division of the continuuum must not be considered to be like the division of sand into grains, but like that of a sheet of paper or tunic into folds, so that even if the folds are infinite in number, there are [still] some smaller than others, and for this reason a body is never dissolved into points or minima... . Although some folds are smaller than others down to infinity, bodies are always extended and points never become parts, but always remain mere extremities. (Leibniz 1676, p. 615).

I should mention in passing here that Leibniz derived further motivation for his reappraisal of the actual infinite from his work on infinite series. Any converging infinite series, such as 1/2 + 1/4 + 1/8 + 1/16 ..., is an example of a finite whole made up of an infinite number of finite parts. But for Leibniz the series is open-ended and comes to no limit: it has no limiting or infinitieth term. It is a distributive rather than a collective whole, whose unity is determined by the series' law. To say that it has a sum of 1 is to say that it can be made as close as desired to 1 by taking a sufficient number of (always finite) terms. Similarly, the physical continuum is also conceived by him as a distributively infinite whole, rather than an infinite collection of points. The same whole can of course be divided in different ways, just as unity is also the sum of the series 2/3 + 2/9 + 2/27 + ... Thus "The continuum is not divided into points, nor is it divided in all possible ways. Not into points, because points are not parts, but limits; not in all possible ways, because not all creatures are present in the same thing, but only a certain infinite progression of them--just as a man who supposes a straight line and any

https://doi.org/10.1086/psaprocbienmeetp.1986.1.193112 Published online by Cambridge University Press

bisected part of it is establishing other divisions than a man who supposes a trisected part." ("Primary Truths", Parkinson 1973, p. 91)

The fact that a continuum can be divided in all possible ways, however, is exactly what make it continuous according to Leibniz. He defines the abstract property of continuity as the potentiality for infinite division in arbitrary many ways: there is an arbitrary number of different possible infinite partitions of a continuous whole.

This construal of continuity as an abstract property is the key to understanding some of the apparently conflicting statements about continuity I quoted to begin this paper. For it helps explain a different way Leibniz has of discussing continuity, in which he apparently denies the very idea of a physical continuum. This is when Leibniz characterizes a continuous whole --one that comprises all possibilities of infinite division in its very concept-- as a merely abstract and thus ideal object: "Continuity is only an ideal thing. In ideal or continuous things, the whole is prior to the parts, just as arithmetical unity is prior to the fractions which divide it, and which can be assigned to it arbitrarily; the parts are only potential... ." (Letter to Remond, July 1714: G.II.622).

Any actual object, on the other hand, is divided in only one way, and is consequently "not truly continuous" in this sense, even though it may, of course, be approximated arbitrarily closely by continuous magnitudes: "But in real things, that is, bodies, the parts are not indefinite, ... but are actually assigned in a determinate way, in accordance with the divisions and subdivisions corresponding to the varieties of motions which nature has actually established, even though these divisions proceed to infinity... ." (Letter to de Volder, June 30th, 1704 G.II.268)

Such an actual thing, however, may nevertheless be regarded as continuous over time, in that it will successively take on a series of different possible infinite partitions. It is in this latter sense that Leibniz writes of the "physical continuum".

Let me now advance a couple of centuries in an effort to make good my claim that Leibniz's approach to continuity has a modern counterpart that is in many respects strikingly similar.

3. Combinatorial Topology

Topology, generally speaking, deals with the properties of geometrical objects that remain invariant under (bi-)continuous transformations. But, depending on how we construe continuity, we can get quite distinct approaches to the subject.[7] There are essentially two main ways of construing continuity, resulting in the two more or less distinct strains of topology recognized today, point set topology and combinatorial topology. The point set definition construes continuity in terms of one-one mappings between uncountably infinite sets of points. It is essentially a generalization of the ϵ-δ definition familiar from elementary calculus, so I shall not go into the details of that here. The other approach, pioneered particularly by Weyl, Brouwer, Veblen and Alexander, is unfamiliar to philosophers, and will require some elaboration.

Combinatorial topology (hereafter CT) had its beginnings in what is now acknowledged to be the oldest recognizably topological problem: the attempt in the theory of polyhedra to prove the empirical formula connecting the number of vertices, edges and faces of any closed convex polyhedron: V - E + F = 2.[8] Although Descartes had first discovered this law, it is named after Euler, since he was the first to publish it (in 1750), and the first to offer a proof of it (1751). But it is with Cauchy's proof (1813) that some of the essential features of modern CT first emerge. Cauchy proceeded by removing the interior of one face of a polyhedron, and stretching the remaining figure out in a plane, yielding a polyhedron for which V - E + F should equal 1. This he then proved by first decomposing the figure into constituent adjoining triangular regions, and then successively deleting triangular regions of the figure until there remains only one triangle. It is shown that the removal of each triangle leaves the number V - E + F unchanged, and the last triangle has V - E + F = 1. Now since this technique of triangulation could be applied to any plane figure with the same result, this shows that V - E + F, the Euler characteristic, is the same for all plane figures, including even those with curved edges, namely 1. Likewise any closed convex polyhedron --for instance, a sphere-- will have the same Euler characteristic as the cube, 2.[9] Thus the Euler characteristic is a topological invariant, discoverable by the purely combinatorial process of triangulation. Other invariance properties of importance can be obtained by similar purely combinatorial techniques.

This idea of triangulation plays a central role in the development of CT in the late 19th and early 20th centuries by Mobius, Riemann and Poincaré. As we have seen, to triangulate a figure or surface is to decompose it into, or cover it with, a finite number of triangles (or objects topologically identical to them), in such a way that each triangle is <u>connected</u> with another, that is, shares an edge or vertex with it. A generalization of this idea leads to the conception of a space (or manifold) as something determined by a <u>partition</u>, composed of a finite (or at least countable) number of connected n-dimensional <u>cells</u> or <u>n-simplexes</u>. Here an n-simplex is an n-dimensional analogue of the triangle (including boundary) a, containing n+1 vertices. Thus the triangle itself is a 2-simplex, whilst a 0-simplex is a vertex or point, a 1-simplex is a line with endpoints, a 3-simplex is a tetrahedral region with surfaces, edges and vertices, and so forth for higher dimensions. Now on the version of CT we have been discussing so far, <u>Flat CT</u>, all these lines, vertices etc. are embedded in Euclidean space. But it is possible to adopt a more abstract approach, that of <u>Symbolic CT</u>, where simplexes are treated as abstract objects and no Euclidean structure is presupposed.

If we now take any locally finite set of simplexes, not necessarily of the same dimension, and fit them together in such a way that any two intersect if at all in a common subsimplex or component, then the result is an <u>n-complex</u>, where n is the dimension of the greatest constituent simplex. Now a <u>space</u>, on this approach, is essentially a collection of connected complexes, which can be replaced by a single complex or <u>partition</u> in the same way that a collection of connected simplexes can be replaced by a complex.

<u>Continuity</u> is now spelled out in terms of transformations which preserve the <u>equivalence</u> of these partitions. Two different complexes

are equivalent if they are transformable one into the other by some finite sequence of simple transformations. Such a sequence of transformations is the purely algebraic counterpart of the "cut and paste" operations of flat CT, replacing cells by partitions and vice versa.

4. Conclusion

I hope that by now you can see something of the similarity between Leibniz's approach to continuity and that of CT. The basis of this similarity is the rejection in each case of the interpretation of space as somehow composed of its points. Instead there is a return to a more Euclidean, finitistic approach. In CT, space is partitioned into a possibly infinite number of simplexes or cells and continuity is represented in terms of equivalence between partitions under certain elementary operations. Likewise, Leibniz regards matter as divided into an actual (though merely countable) infinity of contiguous cells at any instant; he too characterized its continuity in terms of transitions from one such partition to the next in such a way that the cells --his "natural machines" or "organic bodies"-- fill the same space. In both cases, moreover, the infinite partitioning subdivision is conceived to be continued without any limit.[10] Points, so far as they occur, are always boundaries of lines (or the abstract equivalent thereof), but never the end results of some limit process.

There is a good deal more I would like to say here about how all of this relates to Leibniz's theories of space and time, and also his introduction of monads. But now I am out of space, so I shall leave these remarks to use in my replies to comments.

Notes

[1]The latter part of this paper is based on joint research on the relation between Leibniz's work and Combinatorial Topology, which I have been undertaking with Graham Solomon (Philosophy, University of Western Ontario) over the last several months.

[2]My abbreviations are G.II.279 for Gerhardt (1875-90), vol. II, p. 279; and GM. for Gerhardt (1849-55).

[3]Leibniz writes: "there must be infinite number if a liquid is really divided into parts infinite in number. But if this is impossible, it will follow too that a (perfect) liquid is impossible. Since we see that the hypothesis of infinites and infinitesimals turns out to be consistent in geometry, this also increases the probability that it is true." (Loemker 1969, p. 159).

[4]As H.J.M. Bos has explained in his excellent (1974/5), this was at once both the weakness and the strength of his calculus; although from a modern standpoint the idea of second order differentials is regarded as a mistake, it was the fact that Leibniz's differentials themselves varied in magnitude with x that allowed him to correctly derive his rule for transforming integrals (what we now call integration by substitution).

[5]For Leibniz it is axiomatic that for anything to qualify as a simple substance--i.e., one of the building blocks of the world it must, "like Gassendi's atoms or St. Thomas' souls", be indivisible. (See, e.g., "New System", Parkinson 1973, p. 117.)

[6]Quotations from this dialogue on continuity, the Pacidius Philalethi (Leibniz 1676), are taken from my projected forthcoming book, The Labyrinth of the Continuum, which is a compilation of translations of Leibniz's views on the continuum, together with introduction and commentary.

[7]Cf. Alexander (1932), p. 249. The following discussion is based mainly on this paper of Alexander's, and to a lesser extent on the same author's (1930), on M.H.A. Newman (1962), and on Oswald Veblen (1925).

[8]A succinct account of the origins of topology, both point set and combinatorial, is given by Morris Kline (1972, pp. 1158-1181).

[9]Cauchy gave his proof in his (1813). As Kline points out (1972, p. 1163), this proof is not generally valid, since it assumes that any close convex polyhedron is homeomorphic with a sphere.

[10]Cf. Veblen (1925), p. 137: "Such a treatment would put in evidence the properties of space which have to do with continuous deformations and indefinite subdivision, without carrying the process to any limit."

https://doi.org/10.1086/psaprocbienmeetp.1986.1.193112 Published online by Cambridge University Press

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https://doi.org/10.1086/psaprocbienmeetp.1986.1.193112 Published online by Cambridge University Press