Canad. Math. Bull. Vol. **54** (2), 2011 pp. 370–380
doi:10.4153/CMB-2010-103-5

Manifold-Valued Holomorphic Approximation

Edgar Lee Stout

Abstract. This note considers the problem of approximating continuous maps from sets in complex spaces into complex manifolds by holomorphic maps.

1 Introduction

A compact subset X of a complex space $\mathscr{X}$ is said to *admit holomorphic approximation* if each continuous $\mathbb{C}$-valued function on X can be approximated uniformly on X by functions that are holomorphic on varying neighborhoods of X. If $\mathscr{Y}$ is a second complex space, the set X is said to *admit $\mathscr{Y}$-valued holomorphic approximation* if $\mathscr{Y}$-valued continuous maps defined on X can be approximated uniformly on X by holomorphic $\mathscr{Y}$-valued maps defined on varying neighborhoods of X in $\mathscr{X}$. Throughout this note, complex spaces are reduced and paracompact.

We shall see below that for trivial reasons, $\mathscr{Y}$-valued holomorphic approximation cannot occur without essential restrictions on the space $\mathscr{Y}$. For this reason, we only consider the case of $\mathscr{M}$-valued approximation in which $\mathscr{M}$ is a complex manifold.

The notion of uniform convergence for sequences of functions with values in metrizable spaces can be formulated as follows. If X and Y are metrizable spaces with X compact, a sequence $\{f_j\}_{j=1,\dots}$ of Y-valued functions on X converges uniformly to f if for every neighborhood U of the graph of f in $X \times Y$, the graph of f_j lies in U for all large j.

The main result of the present note is the following theorem about manifold-valued approximation.

Theorem 1.1 *Let $\mathscr{X}$ be a complex space, and let $\mathscr{M}$ be a complex manifold. If the compact subset X of $\mathscr{X}$ admits holomorphic approximation, then X admits holomorphic $\mathscr{M}$-valued approximation.*

A particular case is that of rectifiable arcs in $\mathbb{C}^N$, which are known to admit holomorphic approximation from the work of Alexander [1, 10].

Corollary 1.2 *If λ is a rectifiable arc in $\mathbb{C}^N$, and if f is a continuous map from λ to a complex manifold $\mathscr{M}$, then f can be approximated uniformly on λ by maps to $\mathscr{M}$ that are holomorphic on neighborhoods of λ.*

Manifold-valued holomorphic approximation has been considered earlier in the papers [2–4, 6, 7, 11].

Received by the editors May 15, 2008; revised January 19, 2009.
Published electronically August 26, 2010.
AMS subject classification: **32E20**.

2 Preliminaries

In what follows, we shall often use the expression X admits holomorphic approximation without reference to the ambient complex space.

The simplest nontrivial case of the theorem is that in which the target manifold $\mathscr{M}$ is a Stein manifold. This result, which was noted by Gauthier and Zeron in [6] and which will be used in the proof of Theorem 1.1, is a direct consequence of the embedding theorem for Stein manifolds and the existence, due to Docquier and Grauert, of holomorphic retractions onto submanifolds of Stein manifolds. (For the latter point, see [8, p. 257, Thm. 8].)

When X admits holomorphic approximation (or holomorphic $\mathfrak{Y}$-valued approximation) we shall often write that $\mathscr{O}(X)$ is dense in $\mathscr{C}(X)$ (or that $\mathscr{O}(X,\mathfrak{Y})$ is dense in $\mathscr{C}(X,\mathfrak{Y})$), understanding by $\mathscr{O}(X)$ the algebra of germs of holomorphic functions on X, and by $\mathscr{O}(X,\mathfrak{Y})$ the space of germs of holomorphic $\mathfrak{Y}$-valued functions on X. Properly speaking, this is an abuse of notation: Germs are not functions.

There is the following simple observation: If $\mathfrak{X}, \mathfrak{Y}$, and $\mathfrak{Z}$ are complex spaces, if $X \subset \mathfrak{X}$ is a compact set such that $\mathscr{O}(X,\mathfrak{Y})$ is dense in $\mathscr{C}(X,\mathfrak{Y})$, and if $\eta\colon \mathfrak{Y} \to \mathfrak{Z}$ is a holomorphic map, then every continuous $\mathfrak{Z}$-valued map defined on X that is of form $\eta \circ g$ for a function $g \in \mathscr{C}(X,\mathfrak{Y})$ is uniformly approximable by elements of $\mathscr{O}(X,\mathfrak{Z})$.

An easy example of this phenomenon follows.

Example 2.1 Let X be a compact set that admits holomorphic approximation. Consider the map $\varphi\colon \mathbb{C} \to \mathbb{C}^2$ given by $\varphi(\zeta) = (\zeta^2, \zeta^3)$.

The map φ is a holomorphic homeomorphism onto its image, the variety $V = \{(z_1, z_2) : z_1^3 = z_2^2\}$ in $\mathbb{C}^2$, but it is not biholomorphic: The variety V has a singularity at the origin. Because φ is a homeomorphism, continuous maps from X to V factor through $\mathbb{C}$, and consequently continuous maps from X to V can be approximated by holomorphic ones.

Example 2.2 Trivial examples show that for the general subvariety Y of $\mathbb{C}^N$ there cannot be holomorphic Y-valued approximation. Let

$$Y = \{(z_1, z_2) \in \mathbb{C}^2 : z_1 z_2 = 0\}.$$

If $f\colon [-1,1] \to Y$ is given by $f(t) = (t,0)$ for $t \in [-1,0]$ and $f(t) = (0,t)$ for $t \in [0,1]$, then f is a continuous Y-valued function that cannot be approximated on $[-1,1]$ by holomorphic Y-valued functions defined near $[-1,1]$.

Perhaps if we restrict our attention to *normal* varieties as targets, there is an approximation theorem of the kind we are considering.

3 Proof of Theorem 1.1

The proof uses the standard result that a totally real submanifold of a complex manifold has Stein neighborhoods. The details of this result are given in [10, p. 278, Thm. 6.1.2].

The proof of the theorem depends on a lemma.

Lemma 3.1 *If $\mathscr{M}$ is a complex manifold and $h\colon \mathscr{M} \to \mathbb{R}^p$ is an immersion of class $\mathscr{C}^1$, then the graph of h is a totally real submanifold of $\mathscr{M} \times \mathbb{C}^p$.*

The condition that h be an immersion is the condition that at each point of $\mathscr{M}$, the differential of h have trivial kernel. Thus, $p \geq \dim_{\mathbb{R}} \mathscr{M}$ if $\dim_{\mathbb{R}} \mathscr{M}$ is the dimension of $\mathscr{M}$ as a real manifold. It is not required that h be injective, *i.e.,* h need not be an embedding.

Proof Denote by Γ the graph $\{(x, h(x)) : x \in \mathscr{M}\}$ of h, which is a $\mathscr{C}^1$ submanifold of $\mathscr{M} \times \mathbb{R}^p$ and, *a fortiori*, of $\mathscr{M} \times \mathbb{C}^p$. If ξ is a vector tangent to Γ at the point $z = (x, h(x))$, then ξ is of the form $(\xi', dh_x(\xi'))$ for some vector ξ' tangent at x to $\mathscr{M}$. If in addition $i\xi$ is tangent to Γ, then $i\xi = (\xi'', dh_x(\xi''))$ for some other vector ξ'' tangent at x to $\mathscr{M}$. We have then that $idh_x(\xi') = d_xh(\xi'')$. As both $d_xh(\xi')$ and $d_xh(\xi'')$ lie in $\mathbb{R}^p$, both must be zero. The injectivity of dh_x implies that $\xi' = 0$ whence $\xi = 0$. Thus, as claimed, Γ is totally real. ■

Proof of the Theorem With $\mathscr{X}$ a complex space and $\mathscr{M}$ a complex manifold, let X be a compact subset of $\mathscr{X}$ that admits holomorphic approximation, and let $f\colon X \to \mathscr{M}$ be a continuous function.

Fix a metric ϱ, *i.e.,* a distance function, not a Hermitian metric, on $\mathscr{M}$ that defines the topology of $\mathscr{M}$.

Fix $\varepsilon > 0$.

Apply the lemma to the manifold $\mathscr{M}$: if $h\colon \mathscr{M} \to \mathbb{R}^p$ is a smooth embedding, then the graph $\Gamma = \{(z, h(z)) : z \in \mathscr{M}\}$ is a totally real submanifold of $\mathscr{M} \times \mathbb{C}^p$.

Let ϱ' be the metric on $\mathscr{M} \times \mathbb{C}^p$ defined by

$$\varrho'\big((x, y), (x', y')\big) = \max\{\varrho(x, x'), \|y - y'\|\}.$$

Let $\mathscr{V}$ be a Stein neighborhood of Γ in $\mathscr{M} \times \mathbb{C}^p$; $\mathscr{V}$ is a Stein *manifold*. The map $f_1\colon X \to \mathscr{M} \times \mathbb{C}^p$ defined by $f_1(x) = (f(x), h(f(x)))$ is continuous and $\mathscr{V}$-valued. Accordingly, there is a holomorphic $\mathscr{V}$-valued map φ defined on a neighborhood of X in $\mathscr{X}$ that satisfies $\varrho'(f_1(x), \varphi(x)) < \varepsilon$ for all $x \in X$.

If $\pi\colon \mathscr{M} \times \mathbb{C}^p \to \mathscr{M}$ is the natural projection, which is holomorphic, then $\pi \circ \varphi$ is an $\mathscr{M}$-valued map holomorphic on a neighborhood of X in $\mathscr{X}$. Since $\varrho(f(x), \pi(\varphi(x)) < \varepsilon$ for all $x \in X$, the theorem is proved. ■

The method used above to prove Theorem 1.1 can be used equally well to deal with tangential approximation in the following sense.

Definition 3.2 A closed subset X of a complex space $\mathscr{X}$ is said to *admit holomorphic tangential approximation* if for each positive continuous function ε defined on X and for each $\mathbb{C}$-valued continuous function f on X, there is a function φ defined and holomorphic on a neighborhood of X in $\mathscr{X}$ such that for all $x \in X$, $|f(x) - \varphi(x)| < \varepsilon(x)$. Similarly, if $\mathscr{M}$ is a complex manifold, X is said to *admit holomorphic tangential $\mathscr{M}$-valued approximation* if for each metric ϱ on $\mathscr{M}$, for each $\mathscr{M}$-valued continuous function f on X, and for each positive continuous function ε on X, there is a holomorphic $\mathscr{M}$-valued function φ defined on a neighborhood of X in $\mathscr{X}$ such that for all $x \in X$, $\varrho(f(x), \varphi(x)) < \varepsilon(x)$.

In the second part of Definition 3.2, it is equivalent to require the approximation with respect not to all metrics but only with respect to some fixed metric. If the condition is satisfied for some metric, it is satisfied for all.

In the case that X is compact, this notion of tangential approximation coincides with uniform convergence, so the following result contains Theorem 1.1 as a special case.

Theorem 3.3 *Let $\mathscr{X}$ be a complex space, and let $\mathscr{M}$ be a complex manifold. If the closed subset X of $\mathscr{X}$ admits holomorphic tangential approximation, then it admits holomorphic tangential $\mathscr{M}$-valued approximation.*

Proof The case that $\mathscr{M}$ is a Stein manifold follows from the embedding theorem for Stein manifolds and the theorem of Docquier and Grauert on holomorphic retractions, as was noted in [7].The case of general $\mathscr{M}$ follows then by approximating $\mathscr{M}$ by a totally real manifold as in the proof of Theorem 1.1 and replacing the small constant ε of that theorem by the positive function ε of the definition of holomorphic tangential approximation.

Questions of holomorphic tangential $\mathscr{M}$-valued approximation have been considered previously in [6, 7].[1]

4 Sections of Vector Bundles over Complex Manifolds

There is an analogue of the approximation theorem above for sections of vector bundles, at least when the domain space is a manifold. To formulate it properly, we need to use Hermitian metrics. Recall that given a complex vector bundle $\mathscr{V} \xrightarrow{\eta} \mathscr{M}$, a Hermitian metric on $\mathscr{V}$ is an assignment of a Hermitian inner product $\langle \cdot, \cdot \rangle_x$ on each fiber $\mathscr{V}_x = \eta^{-1}(x)$ in such a way that if s and s' are smooth sections of $\mathscr{V}$, then $\langle s(x), s'(x) \rangle_x$ depends smoothly on $x \in \mathscr{M}$. Associated with a Hermitian metric on $\mathscr{V}$ is the corresponding family of norms $\| \cdot \|_x$ on the fibers $\mathscr{V}_x = \eta^{-1}(x)$ given by $\|v\|_x = \sqrt{\langle v, v \rangle_x}$ for all $v \in \mathscr{V}_x$.

Suppose now that $\mathscr{V} \xrightarrow{\eta} \mathscr{M}$ is a holomorphic vector bundle.

Definition 4.1 The closed subset X of $\mathscr{M}$ *admits asymptotic holomorphic $\mathscr{V}$-valued bundle approximation* if for every Hermitian metric on $\mathscr{V}$, for every positive continuous function ε defined on X, and for every continuous section $s\colon X \to \mathscr{V}$ of $\mathscr{V}$ over X, there is a holomorphic section φ of $\mathscr{V}$ defined on some neighborhood of X in $\mathscr{M}$ such that for all $x \in X$

$$\|s(x) - \varphi(x)\|_x < \varepsilon(x).$$

In the particular case that the set X is compact, this notion of approximation is simply the notion of uniform approximation of continuous sections of $\mathscr{V}$ over X by sections holomorphic on a neighborhood of X.

[1] I am indebeted to Paul Gauthier for drawing the paper [7]to my attention.

Theorem 4.2 *Let $\mathscr{M}$ be a complex manifold, and let X be a closed subset of $\mathscr{M}$ that admits asymptotic holomorphic approximation. If $\mathscr{V} \xrightarrow{\eta} \mathscr{M}$ is a holomorphic vector bundle over $\mathscr{M}$, then X admits asymptotic holomorphic $\mathscr{V}$-valued bundle approximation.*

Whether there is an analogous result in the case that the complex manifold $\mathscr{M}$ is replaced by a complex space is not evident. There is also the open problem of establishing a result of this kind with the vector bundles of the theorem replaced by more general holomorphic bundles.

Given what we have in hand, there is an obvious approach to this theorem *in the case that $\mathscr{M}$ is a Stein manifold and X is compact*. The vector bundle $\mathscr{V}$ is a Stein manifold [8, p. 258, Thm. 9], so a continuous section $s\colon X \to \mathscr{V}$ can be approximated uniformly by maps $f\colon X \to \mathscr{V}$ that are holomorphic on a neighborhood of X. However, this does not prove the theorem in the restricted case under consideration. We are not guaranteed that the approximating holomorphic functions f are sections of $\mathscr{V}$, *i.e.*, that they satisfy $\eta \circ f = \mathrm{id}$ on X.

Proof of Theorem 4.2 Assume to begin with that $\mathscr{M}$ is a Stein manifold. By [8, p. 256, Thm. 7], there is a split exact sequence

$$0 \to \mathscr{V}'' \xrightarrow{\iota} \mathscr{V}' \xrightarrow{\tau} \mathscr{V} \to 0$$

of holomorphic vector bundles over $\mathscr{M}$ with $\mathscr{V}'$ the trivial bundle $\mathscr{M} \times \mathbb{C}^q$ for some q. Let $\sigma\colon \mathscr{V} \to \mathscr{V}'$ be a spliting map so that $\tau \circ \sigma = \mathrm{id}$ on $\mathscr{V}$. Let $\mathscr{V}$ and $\mathscr{V}'$ be endowed with Hermitian metrics, and let the associated norms be $\|\cdot\|$ and $\|\cdot\|'$. We assume the metric on $\mathscr{V}'$ to be chosen so that the norm $\|\cdot\|'$ is given by the condition that for $(x,z) \in \mathscr{M} \times \mathbb{C}^q$, $\|(x,z)\|'$ is the Euclidean norm $|z|$. Let $c\colon \mathscr{M} \to (0,\infty)$ be a continuous function that satisfies $\|\tau(v)\|_x < c(x)\|v\|'_x$ for all $x \in \mathscr{M}$ and all $v \in \mathscr{V}'_x$.

Let $s\colon X \to \mathscr{V}$ be a section of $\mathscr{V}$ over X, and let ε be a positive continuous function on X. The composition $\sigma \circ s$ is a section of $\mathscr{M} \times \mathbb{C}^q$ over X. That is to say, $\sigma \circ s(x) = (x, f(x))$ for a continuous $\mathbb{C}^q$-valued function f on X. By hypothesis, there exists a holomorphic $\mathbb{C}^q$-valued map g defined on a neighborhood of X such that, with $|\cdot|$ the norm on $\mathbb{C}^q$,

$$|f(x) - g(x)| < \varepsilon(x)/c(x)$$

when $x \in X$.

The map $\tilde{g}$ defined on a neighborhood of X with values in $\mathscr{M} \times \mathbb{C}^q$ given by $\tilde{g}(x) = (x, g(x))$ is a holomorphic section of $\mathscr{M} \times \mathbb{C}^q$ on a neighborhood of X. The map $\tau \circ \tilde{g}\colon X \to \mathscr{V}$ is a holomorphic section of $\mathscr{V}$ on a neighborhood of X that satisfies

$$\|\tau \circ \tilde{g}(x) - s(x)\| = \|\tau \circ \tilde{g}(x) - \tau \circ \sigma \circ s(x)\| \le c(x)\|\tilde{g}(x) - \sigma \circ s(x)\|' < \varepsilon(x).$$

The theorem is proved under the supplementary hypothesis that $\mathscr{M}$ is a Stein manifold.

We now deduce the general case of the theorem from that just established.

Again fix a holomorphic vector bundle $\mathscr{V} \xrightarrow{\eta} \mathscr{M}$, fix a Hermitian metric H on $\mathscr{V}$ with associated norm function $\|\cdot\|$, fix a continuous section $s\colon X \to \mathscr{V}$, and fix a positive continuous function ε on X. The section s can be extended to a continuous section of $\mathscr{V}$ defined on a neighborhood of X. We fix such an extension and denote it also by s. Similarly, we assume the function ε to be defined, continuous, and positive on all of $\mathscr{M}$.

Let $h\colon \mathscr{M} \to \mathbb{R}^p$ be a smooth embedding, and let $\Gamma \subset \mathscr{M} \times \mathbb{C}^p$ be the graph of h. Thus, by Lemma 3.1, Γ is a totally real submanifold of $\mathscr{M} \times \mathbb{C}^p$. Let $\mathscr{W}$ be a Stein neighborhood of Γ in $\mathscr{M} \times \mathbb{C}^p$. Put $X' = \{(x, h(x)) \in \mathscr{M} \times \mathbb{C}^p : x \in X\}$.

Lemma 4.3 *The set X' admits asymptotic holomorphic approximation.*

Proof Let $g\colon X' \to \mathbb{C}$ be continuous, and let ε be a positive continuous function on X'. The function ε^* defined on X by $\varepsilon^*(x) = \varepsilon(x, h(x))$ is a continuous positive function on X, so there is a holomorphic $\mathbb{C}$-valued function τ defined on a neighborhood of X in $\mathscr{M}$ such that

$$|\tau(x) - g(x, h(x))| < \varepsilon^*(x).$$

If $\tilde{\tau}(x, z) = \tau(x)$, then $\tilde{\tau}$ is a holomorphic function defined on a neighborhood of X' that satisfies $|\tilde{\tau}(x, z) - g(x, z)| < \varepsilon(x)$ for all $x \in X'$. The lemma is proved.

On $\mathscr{M} \times \mathbb{C}^p$ we take coordinates (x, z) with $x \in \mathscr{M}$, $z \in \mathbb{C}^p$. Let $\pi\colon \mathscr{M} \times \mathbb{C}^p \to \mathscr{M}$ be the projection given by $\pi(x, z) = x$. There is the holomorphic vector bundle $\pi^*\mathscr{V}$ on $\mathscr{W}$ induced by π (properly by $\pi|\mathscr{W}$) that is defined by

$$\pi^*\mathscr{V} = \{((x, z), v) \in \mathscr{W} \times \mathscr{V} : \pi(x, z) = \eta(v)\}$$

with projection $\eta'\colon \pi^*\mathscr{V} \to \mathscr{W}$ the map given by $\eta'((x, z), v) = (x, z)$. For every point $(x, z) \in \mathscr{W}$, the fiber $(\pi^*\mathscr{V})_{(x,z)}$ of $\pi^*\mathscr{V}$ over (x, z) is the collection of points $((x, z), v)$ with v in the fiber $\mathscr{V}_x$ of $\mathscr{V}$ over x. The Hermitian metric H on $\mathscr{V}$ gives rise to a Hermitian metric H^* on $\pi^*\mathscr{V}$ with associated norm $\|\cdot\|^*$ for which if $((x, z), v) \in \pi^*\mathscr{V}_{(x,z)}$, we have $\|((x, z), v)\|^*_{(x,z)} = \|v\|_x$.

The map $s'\colon X' \to \pi^*\mathscr{V}$ given by $s'(x, z) = ((x, z), s(x))$ is a continuous section of $\pi^*\mathscr{V}$ on X', so by the case of the theorem that we have already established, there is a holomorphic section t of $\pi^*\mathscr{V}$ defined on a neighborhood U of the set X' that approximates s' on X'. We can choose t so that $\|t(x, z) - s'(x, z)\|^* < \varepsilon(x)$. The section t is of the form $t(x, z) = ((x, z), \tau(x, z))$ for a holomorphic map $\tau\colon U \to \mathscr{V}$ that satisfies $\pi(x, z) = \eta(\tau(x, z))$, *i.e.*, $x = \eta(\tau(x, z))$, and

$$\|s \circ \pi(x, z) - \tau(x, z)\|^* < \varepsilon(x). \tag{4.1}$$

The hypothesis that X admits asymptotic holomorphic approximation implies that the embedding h can be approximated asymptotically on X by a map h' that is holomorphic on a neighborhood of X. Define a map φ holomorphic on a neighborhood of X and taking values in $\mathscr{M} \times \mathbb{C}^p$ by $\varphi(x) = (x, h'(x))$. We have $\pi(\varphi(x)) = x$ for x near X.

The composition $\tau \circ \varphi$ is a holomorphic map from a neighborhood of X into $\mathscr{V}$; it is a section of $\mathscr{V}$. The composition $s \circ \pi \circ \varphi$ is defined and coincides with the section s on a neighborhood of X, provided h' is a sufficiently good approximation to h. As

$$\tau \circ \varphi - s = \tau \circ \varphi - s \circ \pi \circ \varphi = (\tau - s \circ \pi) \circ \varphi$$

it follows from (4.1) that $\|\tau \circ \varphi(x) - s(x)\| \leq \varepsilon(x)$. The theorem is proved. ∎

5 Rational Approximation

For a compact set X in $\mathbb{C}^N$ we use the standard notation that $\mathscr{R}(X)$ is the subspace of $\mathscr{C}(X)$ composed of the functions that can be approximated uniformly on X by rational functions without poles on X. Gauthier and Zeron [6, 11] have proved the following two results.

Theorem 5.1 *If X is a compact subset of $\mathbb{C}^N$ such that $\mathscr{R}(X) = \mathscr{C}(X)$, then each continuous null-homotopic map from X to $\mathbb{P}^N$ can be approximated uniformly by rational maps whose critical set does not meet X.*

A rational map is a map f of the form $f = \pi \circ P$ in which $\pi \colon \mathbb{C}^{N+1} \setminus \{0\} \to \mathbb{P}^N$ is the usual projection and $P = (p_1, \dots, p_{N+1}) \colon \mathbb{C}^N \to \mathbb{C}^{N+1}$ is a polynomial map. The critical set of f is the set $P^{-1}(0)$.

There is a partial converse.

Theorem 5.2 *If X is a compact subset of $\mathbb{C}^N$ that is a CW-complex of real dimension not more than $2m$, and if $f \colon X \to \mathbb{P}^m$ is a continuous map that can be approximated uniformly by rational maps whose critical sets miss X, then f is null-homotopic.*

Gauthier and Zeron [6] exhibit the particular example of

$$\mathbb{T}^2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| = |z_2| = 1\}$$

that satisfies $\mathscr{R}(\mathbb{T}^2) = \mathscr{C}(\mathbb{T}^2)$ but for which not all continuous maps $f \colon \mathbb{T}^2 \to \mathbb{P}^1$ can be approximated by rational maps. To be sure, such an f can be approximated by maps holomorphic on neighborhoods of $\mathbb{T}^2$ in $\mathbb{C}^2$.

Our first object in the present section is to show that in the Theorem 5.2, the condition that X be a CW-complex is unduly restrictive.

Theorem 5.3 *If X is a compact subset of $\mathbb{C}^N$ that has dimension not more than $2m$, and if $f \colon X \to \mathbb{P}^m$ is a continuous map that can be approximated uniformly by rational maps, then f is null-homotopic.*

Here we understand *dimension* to be topological dimension. For dimension theory one should consult the classical treatment given by Hurewicz and Wallman in [9].

The proof of this theorem depends on the following fact from dimension theory.

Lemma 5.4 *If X is a separable metric space of topological dimension not more than $2n$, then every continuous map from X to $\mathbb{C}^{n+1} \setminus \{0\}$ is homotopic to a constant.*

Proof If $\varphi\colon X \to \mathbb{C}^{n+1} \setminus \{0\}$ is a continuous map, it is homotopic to the map $\varphi/|\varphi|$ from X to the sphere $\mathbb{S}^{2n+1}$. According to [9, p. 88, Thm. VI.6], $\varphi/|\varphi|$ is homotopic to a constant because X is of dimension not more than $2n$, whence φ itself is homotopic to a constant. ■

Proof of Theorem 5.2 The proof of this theorem follows precisely the lines of that given by Gauthier and Zeron, except that we use Lemma 5.4. By that lemma, each rational map from the set X of the theorem is homotopic, on X, to a constant, so every continuous map from X that can be approximated uniformly by rational maps holomorphic on X must be homotopic to a constant on X.

As Zeron [11] noted, Theorem 5.2 implies that even for a polynomially convex set, X in $\mathbb{C}^N$, the condition that $\mathscr{P}(X) = \mathscr{C}(X)$ does not imply that every holomorphic $\mathbb{P}^1$-valued map defined on a neighborhood of X can be approximated by rational maps. Zeron gave an explicit example, which is the stereographic projection from the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3 \subset \mathbb{C}^3$ to the Riemann sphere. This map is of degree ± 1 and thus is not homotopic to a constant.

Theorems 5.3 and 5.1 lead naturally to the problem: When does $f\colon X \to \mathbb{P}^N$ lift to $\tilde{f}\colon X \to \mathbb{C} \setminus \{0\}$? This is an entirely classical problem in *lifting theory* . A simple answer, which is surely well known, is the following.

Lemma 5.5 *If X is a paracompact space, if $f\colon X \to \mathbb{P}^n$ is continuous, and if the induced map $f^*\colon \check{H}^2(\mathbb{P}^N;\mathbb{Z}) \to \check{H}^2(X;\mathbb{Z})$ is the zero map, then f lifts to $\tilde{f}\colon X \to \mathbb{C}^{n+1} \setminus \{0\}$. That is, there is a continuous map $\tilde{f}\colon X \to \mathbb{C}^{n+1} \setminus \{0\}$ such that $f = \pi \circ \tilde{f}$. Conversely, if f lifts, the map induced from $\check{H}^2(\mathbb{P}^N;\mathbb{Z})$ to $\check{H}^2(X;\mathbb{Z})$ is the zero map.*

In this and below, $\check{H}^*$ denotes Čech cohomology.

A simple example is this: The bundle $\mathbb{C}^2 \setminus \{0\} \to \mathbb{P}^1$ is not trivial. There does not exist a lifting of the identity map $\mathbb{P}^1$ to itself. We have that $\check{H}^2(\mathbb{P}^1;\mathbb{Z}) \neq 0$, and, in fact, the identity map is a generator of $\pi_2(\mathbb{P}^1)$.

Proof of the Lemma The exact sequence of sheaves

$$0 \to \mathbb{Z} \to \boldsymbol{\mathscr{C}}_X \to \boldsymbol{\mathscr{C}}_X^* \to 0$$

with $\boldsymbol{\mathscr{C}}_X$ the sheaf of germs of continuous $\mathbb{C}$-valued functions on X and $\boldsymbol{\mathscr{C}}_X^*$ the sheaf of germs of continuous, zero-free $\mathbb{C}$-valued functions on X yields the isomorphism $\check{H}^2(X;\mathbb{Z}) \simeq \check{H}^1(X;\boldsymbol{\mathscr{C}}_X^*)$, for the sheaf $\boldsymbol{\mathscr{C}}_X$ is fine and thus has vanishing cohomology.

Let $\mathfrak{U} = \{U_\alpha\}_{\alpha\in A}$ be an open cover, which can be chosen to be finite, of $\mathbb{P}^N$ such that for each α there is a continuous map $\mu_\alpha\colon U_\alpha \to \mathbb{C}^N \setminus \{0\}$ with $\pi \circ \mu_\alpha$ the identity map on U_α. If we set $W_\alpha = f^{-1}(U_\alpha)$, then $\mathfrak{W} = \{W_\alpha\}_{\alpha\in A}$ is an open cover of X. Thus, $\mu_\alpha \circ f$ is a lift of f over the set W_α. Set $U_{\alpha,\beta} = U_\alpha \cap U_\beta$ and $W_{\alpha,\beta} = W_\alpha \cap W_\beta$.

The fibers $\pi^{-1}(p)$ for $p \in \mathbb{P}^N$ are copies of $\mathbb{C} \setminus \{0\}$, so there are continuous maps $\mu_{\alpha,\beta}\colon U_{\alpha,\beta} \to \mathbb{C} \setminus \{0\}$ such that on $U_{\alpha,\beta}$ we have $\mu_\alpha = \mu_{\alpha,\beta}\mu_\beta$. The family $\{(U_{\alpha,\beta}), \mu_{\alpha,\beta}\}_{\alpha,\beta\in A}$ determines an element of $\check{H}^2(\mathbb{P}^N;\mathscr{C}^*) \simeq \check{H}^2(\mathbb{P}^N;\mathbb{Z})$. By hypothesis, the induced map $f^*\colon \check{H}^2(\mathbb{P}^N;\mathbb{Z}) \to \check{H}^2(X;\mathbb{Z})$ is the zero map, so the cohomology class in $\check{H}^2(X;\mathscr{C}^*)$ determined by $\{(W_{\alpha,\beta}, \mu_{\alpha,\beta} \circ f)\}_{\alpha,\beta\in A}$ is trivial. There exist zero-free continuous functions g_α on W_α such that $\mu_{\alpha,\beta} \circ f = g_\alpha/g_\beta$ on $W_{\alpha,\beta}$.

If we set $h_\alpha = g_\alpha \mu_\alpha \circ f$ on W_α, then $h_\alpha = h_\beta$ on $W_{\alpha,\beta}$. Accordingly, the h_α taken together yield a lift $\tilde{f}\colon X \to \mathbb{C}^{n+1} \setminus \{0\}$.

Conversely, suppose $f\colon X \to \mathbb{P}^N$ to lift to $\tilde{f}\colon X \to \mathbb{C}^{N+1} \setminus \{0\}$. At the cohomology level we have $f^* = \tilde{f}^* \circ \pi^*$. The group $\check{H}^2(\mathbb{C}^{N+1} \setminus \{0\})$ vanishes, so f^* is the zero map. The lemma is proved. ■

Corollary 5.6 *If X is a paracompact space with $\check{H}^2(X;\mathbb{Z}) = 0$, and if $f\colon X \to \mathbb{P}^N$ is a continuous map, then there is a continuous map $\tilde{f}\colon X \to \mathbb{C}^{n+1} \setminus \{0\}$ that satisfies $\pi \circ \tilde{f} = f$.*

Corollary 5.7 ([11]) *If $X \subset \mathbb{C}^n$ is a rationally convex set that admits holomorphic approximation, and if $\check{H}^2(X;\mathbb{Z}) = 0$, then every continuous map $f\colon X \to \mathbb{P}^N$ can be approximated uniformly on X by a rational map.*

In these corollaries, no restriction is placed on the dimensions n and N.

There is an extension of Corollary 5.7.

Corollary 5.8 *If the compact set X admits holomorphic approximation, then every continuous map $f\colon X \to \mathbb{P}^N$ for which the induced map $f^*\colon \check{H}^2(\mathbb{P}^N;\mathbb{Z}) \to \check{H}^2(X;\mathbb{Z})$ is the zero map can be approximated uniformly on X by maps of the form $\pi \circ f$ with π the projection from $\mathbb{C}^{N+1} \setminus \{0\}$ to $\mathbb{P}^N$ and with $f = (f_1, \dots, f_{N+1})$ a holomorphic map from a neighborhood of X to $\mathbb{C}^{N+1} \setminus \{0\}$.*

6 On the Size of Sets that Admit Holomorphic Approximation

We have been concerned with compact sets that admit holomorphic approximation. There are certain obvious restrictions on such sets. Trivially, no such set can have interior points. For subsets of $\mathbb{C}^N$ a much stronger condition can be established.

Theorem 6.1 *If the compact subset X of $\mathbb{C}^N$ admits holomorphic approximation, then* $\dim X \le N + 1$. *If $\mathscr{P}(X) = \mathscr{C}(X)$, then* $\dim X \le N$.

In particular, if $\mathscr{R}(X) = \mathscr{C}(X)$, then $\dim X \le N + 1$. Simple examples show that the bound in the case of polynomial approximation is sharp. If X is any compact subset of $\mathbb{R}^N \subset \mathbb{C}^N$, then $\mathscr{P}(X) = \mathscr{C}(X)$. It is unknown whether the bound of $N + 1$ in the cases of rational and holomorphic approximation are sharp. Does there exist in $\mathbb{C}^N$ a set of dimension $N + 1$ that admits holomorphic approximation or that satisfies $\mathscr{R}(X) = \mathscr{C}(X)$? When $N = 1$, the answers are no. The general case does not seem to be evident.

Proof (See [5, p. 62, Lem. 5.4].) Consider first the case that $\mathscr{P}(X) = \mathscr{C}(X)$. This condition implies that X and all its closed subsets are polynomially convex and therefore that if E is a closed subset of X, then [10, p. 96, Cor. 2.3.6], $\check{H}^N(E;\mathbb{Z}) = 0$. This implies that $\dim X \le N$. See [9, p. 151, Thm. VIII 3]. The case of holomorphic approximation follows precisely the same lines but depends on the vanishing of the groups $\check{H}^{N+1}(X;\mathbb{Z})$ for holomorphically convex subsets of $\mathbb{C}^N$. See [10, Th. 6.2.13, p. 298]. ■

One can go somewhat further in the direction of the preceding result:

https://doi.org/10.4153/CMB-2010-103-5 Published online by Cambridge University Press

If $\mathscr{X}$ is a complex space of dimension N and if the compact subset X of $\mathscr{X}$ admits holomorphic approximation, then the topological dimension of X does not exceed $N + 1$.

The proof of this assertion is by induction on the dimension of $\mathscr{X}$. If $\dim \mathscr{X} = 1$, then the result is plainly so. No subset of $\mathscr{X}$ has topological dimension greater than two. Assume the result to be correct for all complex spaces of dimension not more than N. Consider a complex space $\mathscr{X}$ of dimension $N + 1$ and in it a compact subset X that admits holomorphic approximation. Two cases can occur. It could be that X is contained in some closed subspace $\mathscr{Y}$ of $\mathscr{X}$ that has dimension not more than N, *e.g.*, X might be contained in the singular variety $\mathscr{X}_{\text{sing}}$ of $\mathscr{X}$. In this case the topological dimension of X does not exceed $N + 1$ because of the induction hypothesis, for the assumption that X, viewed as a subset of $\mathscr{X}$, admits holomorphic approximation implies, by restricting the approximating functions, that X admits holomorphic approximation when considered as a subset of the space $\mathscr{Y}$.[2] In the case that X is contained in no closed subspace of $\mathscr{X}$ of dimension less than $N + 1$, let us denote by $\mathscr{X}_{\max}$ the union of the $(N + 1)$-dimensional branches of $\mathscr{X}$. We will be done if we can show that the topological dimension of the set $X \cap \mathscr{X}_{\max}$ is not more than $N + 2$. For this, it suffices to show that if $x \in X \cap \mathscr{X}_{\max}$ is a regular point of $\mathscr{X}$, then a neighborhood of x in $\mathscr{X}_{\max}$ has dimension not more than $N + 2$. This, however, is clear. Given such a point x, there is a neighborhood, say V, of it that is contained in a coordinate neighborhood in the manifold of regular points of $\mathscr{X}_{\max}$. This neighborhood is biholomorphically equivalent to a domain in $\mathbb{C}^{N+1}$, say under the biholomorphism ψ. If E is any compact subset of V, then $\psi(E)$ is a compact subset of $\mathbb{C}^{N+1}$ that admits holomorphic approximation (as a subset of $\mathbb{C}^{N+1}$), and thus has dimension not more than $N + 2$ by the theorem given above. Our assertion follows from the Sum Theorem of dimension theory. See [9, p. 30, Thm. III.2].

Acknowledgments I am indebted to Franc Forstnerič for asking a question that aroused my interest in manifold-valued approximation and subsequently for a useful comment in connection with this paper. The paper was motivated in good measure by a reading of the papers of Gauthier and Zeron [6, 11].

References

[1] H. Alexander, *Polynomial approximation and hulls in sets of finite linear measure in* $\mathbb{C}^n$. Amer. J. Math. **93**(1971), 65–74. doi:10.2307/2373448

[2] D. Chakrabarti, *Coordinate neighborhoods of arcs and the approximation of maps into (almost) complex manifolds.* Michigan Math. J. **55**(2007), no. 2, 299–333. doi:10.1307/mmj/1187646996

[3] ———, *Sets of approximation and interpolation in* $\mathbb{C}$ *for manifold-valued maps.* J. Geom. Anal. **18**(2008), no. 3, 720–739. doi:10.1007/s12220-008-9030-2

[4] B. Drinovec-Drnovšek and F. Forstnerič, *Approximation of holomorphic mappings on strongly pseudoconvex domains.* Forum Math. **20**(2008), no. 5, 817–840. doi:10.1515/FORUM.2008.039

[5] T. Duchamp and E. L. Stout, *Maximum modulus sets.* Ann. Inst. Fourier (Grenoble) **31**(1981), no. 3, v, 37–69.

[2] It is by no means obvious that if we are given a (closed) subspace $\mathscr{Z}$ of another complex space $\mathscr{W}$ and a compact subset Z of $\mathscr{Z}$ that admits holomorphic approximation when viewed as a subset of $\mathscr{Z}$, the set Z admits holomorphic approximation when viewed as a subset of $\mathscr{W}$. It is a question of whether functions defined and holomorphic on neighborhods of Z in $\mathscr{Z}$ can be approximated on Z by functions defined and holomorphic on a neighborhood of X in $\mathscr{W}$.

[6] P. M. Gauthier and E. S. Zeron, *Approximation by rational mappings, via homotopy theory*, Canad. Math. Bull. **49**(2006), no. 2, 237–246.

[7] ———, *Embedding Stein manifolds and tangential approximation.* Complex Var. Elliptic Equ. **51**(2006), no. 8–11, 953–958. doi:10.1080/17476930600673005

[8] R. C. Gunning and H. Rossi, *Analytic functions of several complex variables.* Prentice-Hall, Inc., Englewood Cliffs, NJ, 1965.

[9] W. Hurewicz and H. Wallman, *Dimension theory.* Princeton Mathematical Series, 4, Princeton University Press, Princeton, NJ, 1941.

[10] E. L. Stout, *Polynomial convexity.* Progress in Mathematics, 261, Birkhäuser Boston Inc., Boston, MA, 2007.

[11] E. S. Zeron, *Approximation and the topology of rationally convex sets.* Canad. Math. Bull. **49**(2006), no. 4, 628–636.

Department of Mathematics, The University of Washington, Seattle, WA 98195, U.S.A.
e-mail: stout@math.washington.edu